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ELECTRICAL MEASUREMENTS AND MEASURING INSTRUMENTS

BY

E. W. GOLDING
O.B.E., M.Sc.Tech., M.I.E.E., Mem. A.I.E.E.
SOMETIME LECTURER IN ELECTRICAL ENGINEERING AT
UNIVERSITY COLLEGE, NOTTINGHAM

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PREFACE

TO THE FOURTH EDITION

In dealing with such an extensive range of subjects as those included under the broad title of this book it would perhaps be too optimistic to hope that all the many important developments which have taken place since its last revision have been dealt with adequately. A strenuous effort has been made to ensure that this aim should be accomplished as far as possible, but the task has been formidable and the author must ask the indulgence of the reader who may yet find a few gaps in the subject matter.

As for the revision itself, in the first place the symbols and nomenclature have been changed throughout to conform to the most recent recommendations of the responsible committees even though the resulting formulae now seem to have a strangely unfamiliar appearance. The subjects of M.K.S. units and rationalization have been covered much more fully than in the last edition and the basic equations derived in Chapter I have been given in both unrationalized and rationalized forms. The author feels, however, that the present is a transitional period between that when C.G.S. units held sway and the future period when the M.K.S. rationalized system may be universally adopted. He has therefore attempted to carry along both systems except where this would obviously cause confusion—the older system has then been adhered to. Chapter II has been altered considerably for this reason and also because of the abandonment of the International units in 1948. Recent advances in the study of dielectrics have necessitated some changes and the author is indebted to his colleague Mr. C. G. Garton for kindly reading the text dealing with this subject and making several helpful suggestions. Applications of resistance measurements in the determination of non-electrical quantities are now numerous and several of the more important of these have been included in Chapter VII. Developments in illumination, high-voltage measurements, and cathode-ray oscillography also called for modifications to and extension of the text. In the chapter on transients a short introduction to the use of Laplace transformations has been given.

In the chapters concerned with measuring instruments there have been many changes to include newly-developed instruments and to illustrate the latest forms of older ones but, bearing in mind the book's function as a text for students, descriptions of a few instruments now little used have been retained when their operating principles have been considered sufficiently important. Many manufacturers have made available data and illustrations for their instruments and their kind co-operation is gratefully acknowledged.

The introduction of new measuring techniques based on recent developments in electronics has had a very important influence on the whole subject of electrical measurements and a new chapter on "Electronics and Electronic Measuring Devices" has therefore been included. For the writing of this chapter and for valuable assistance in the revision of the book, the author is greatly indebted to Mr. H. H. Rosenbrock.

In conclusion the author would like to express his most sincere appreciation of the encouragement which he has received from the kind interest of readers in various parts of the world and his gratitude for their helpful suggestions which he has endeavoured to follow.

E. W. G.

LONDON,
OCTOBER, 1954

PREFACE TO FIRST EDITION

When this book was first contemplated, it was the author's intention to produce a short text-book covering the syllabuses of the B.Sc. (Eng.), City and Guilds (Final) and I.E.E. examinations. As the work progressed, however, it was realized that the groundwork for these examinations could not be adequately covered in a book of the size originally planned. Extension of the text being thus necessitated, the author felt that the scope and method of treatment could with advantage be modified to suit, also, the requirements of electrical engineers in general, for whom, owing to the rapid advances taking place in the electrical industry and in the interconnection of power systems, the subject of electrical measurements is assuming an ever-increasing importance. In particular, it was thought that engineers engaged in the standardizing and metering side of the industry could be catered for.

Some of the subjects dealt with—e.g. Transient Phenomena—are essentially mathematical, and cannot well be treated otherwise, but the mathematics throughout the book has been kept as simple as possible and should be followed easily by most readers. The theory of most of the methods of measurement has been given in full, but in many cases smaller type has been used for such theoretical discussions, so that the reader may omit them, if desired, and consider only the resulting expressions.

References and bibliographies are given at the ends of the chapters, their inclusion being both an acknowledgment of the sources of the author's information—in certain cases—and an augmentation of the text, which can, at the best, give only a general outline of so large a subject in the space of a single volume.

A number of worked examples have been given as illustrations, and a selection of examination questions (with answers) is included at the end of the book. In this connection the author's thanks are due to the Senate of the University of London, the City and Guilds of London Institute, and the Institution of Electrical Engineers for permission to use questions from their examination papers.

Most of the illustrations have been specially made for this book, but the author is glad to acknowledge his indebtedness to the many firms and authorities who have very kindly supplied him with information, drawings, and photographs relating to their apparatus. Individual acknowledgments are made in the text. Messrs. Elliott Bros., Ltd., had drawings specially made from photographs of their instruments. The author is indebted also to Messrs. A. T. Dover, B. Hague, and P. Dunsheath for permission to reproduce five illustrations from their books; to Messrs. B. G. Churcher and C. Dannatt, Prof. W. M. Thornton, and Prof. S. P. Smith for their co-operation, and to Mr. N. A. Allen for information regarding cable tests with high-voltage direct current.

The author gratefully acknowledges the help which he has received from advanced students in the Electrical Engineering Department of University College, Nottingham, in the shape of proof reading. He would like, also, to tender his sincere thanks to Dr. H. Cotton for his continued and lively interest in the book during its preparation; to Mr. T. M. E. Ward for kindly reading the proofs of Chapters XX and XXI; and to his wife for her unfailing help and consideration while the book was being prepared.

Acknowledgment should also be made to the British Standards Institution for permission to make extracts from several of their instrument specifications. Copies of these specifications can be obtained from the Institution at British Standards House, 2 Park Street, London, W.1.

E. W. GOLDING.

ERSITY COLLEGE,
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CHAPTER I

ELECTROSTATIC AND ELECTROMAGNETIC THEORY

ELECTROSTATICS

Coulomb's Law. The earliest recorded facts in connection with the subject of electricity were obtained as a result of experiments carried out by the ancient Greek philosopher Thales of Miletus, about 600 B.C., and related to the forces of repulsion and attraction between bodies charged with static electricity. Those facts were qualitative only, and it was left for Coulomb, many centuries later, to state them in a quantitative form by his Inverse Square Law, which is the most fundamental law of electrostatics—

$$F = \frac{Q_1 Q_2}{\kappa r^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where F is the force between two small bodies charged respectively with Q_1 and Q_2 units of electricity, their centres being a distance r apart. κ is a constant depending upon the medium, or "dielectric," in which the bodies are situated, and is called the "permittivity," or "dielectric constant," of the medium.

The units in which both these, and other quantities mentioned in this chapter, are measured will be discussed more fully in the next chapter, but it must be noted here that this law forms the foundation for the electrostatic C.G.S. system of units. In this system the e.s. unit of electricity is such that two infinitely small bodies, each having unit charge and being 1 cm. apart in air (or, more strictly, in a vacuum), when $\kappa = 1$, experience a force of 1 dyne, it being assumed that they are infinitely distant from all other charged bodies.

The M.K.S. (metre-kilogramme-second) or Giorgi system of units (see p. 57) is now recognized as having many advantages and, in this system, the force F is in "newtons," r is in metres, Q_1 and Q_2 in coulombs and $\kappa = 1{\cdot}113 \times 10^{-10}$. (One "newton" is the force which produces an acceleration of 1 metre per second per second in a mass of 1 kilogramme.)

To avoid confusion, the units for the expressions derived in the rest of this chapter will be omitted but will be dealt with in Chapter II.

Electric Field Round Charged Conductors. If unit positive charge of electricity be placed in the neighbourhood of a charged body it will experience a force of attraction or repulsion according as the charged body is negatively or positively charged. If this unit charge be allowed to move freely, it will trace out a "*line of electric force.*" For all points on this line the resultant force on the unit charge will be in a direction tangential to the line at the given point. The electric field in the neighbourhood of any charged conductor or system of charged conductors can be represented by such lines of force, arrow heads placed upon them giving the direction in which unit positive charge would move along the line.

The magnitude of the force upon unit positive charge placed at any point, is a measure of the "electric force" or "intensity" at that point, it being assumed that the introduction of the unit charge does not affect the distribution of charge upon the conductors to which the field is due.

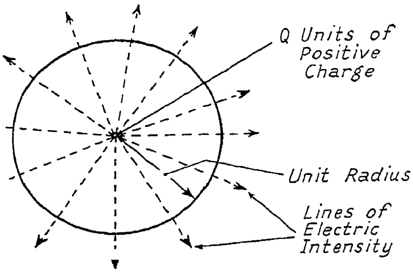


FIG. 1. LINES OF ELECTRIC INTENSITY ROUND A POSITIVE CHARGE

Lines of Electric Intensity. In the preceding paragraph lines of force are spoken of as giving the *direction* of the electric field at any point. If defined as below, they can be made to express the *intensity* at a point as well as the *direction* of the field there. To distinguish them from lines which, without regard to intensity, merely show the direction of the field, such lines will be called *lines of electric intensity.* These lines may be defined as such that a positive charge of Q units will radiate $4\pi Q$ of them, uniformly distributed in the space surrounding it (Fig. 1), this number being the same whatever the medium surrounding the charge. Thus, if the charge Q be situated at the centre of an imaginary sphere of unit radius, then the number of such lines per unit of surface of the sphere will be $\frac{4\pi Q}{4\pi} = Q$ units. Now, if the surrounding medium has a permittivity κ, then the intensity E normal to the sphere's surface is $\frac{Q \times 1}{\kappa \times 1^2} = \frac{Q}{\kappa}$.

In this case the intensity $= \frac{\text{No. of lines per unit of surface}}{\kappa}$

In general, therefore,

$$E = \frac{D}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $D =$ lines of electric intensity per unit of cross-section, the cross-section being measured perpendicular to the direction of the field at the point.

If $\kappa = 1$, obviously, $D = E$.*

Electric Flux. A number of lines of electric intensity are spoken of collectively as *electric flux*. It follows from the above definition of such lines that

$$\psi = 4\pi Q \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ψ is the electric flux radiating from a charge of Q units. "D" may thus be referred to as "*electric flux density.*"

The Rationalized System. Together with the M.K.S. system of units, "rationalization" of units, first stated in principle by Heaviside in 1892, is now finding favour (*see also* p. 59). The rationalized system follows on definitions of electrostatic and magnetic flux in which the 4π is omitted so that the definition is then $\psi = Q$ instead of $\psi = 4\pi Q$. Rationalization has not yet been widely adopted however and so the definition of equation (3) will be followed in this chapter. The object here is to derive certain basic formulae which can be modified, if desired, in accordance with the rationalized definition taking this definition as the basis.

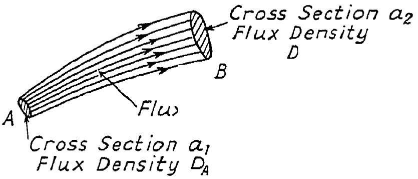


FIG. 2. TUBE OF FLUX

All the numbered formulae in this chapter are repeated in rationalized form in Table I at the end of the chapter (p. 47).

Tubes of Flux. Fig. 2 represents a number of lines of electric intensity forming a "tube of flux." If A and B are two points in an electric field in air such that the intensity at A is greater than that at B, then, from Equation (2), the intensity at A

$$E_A = D_A$$

where D_A = lines per unit surface of cross-section of the tube at A.

Similarly, intensity at B

$$E_B = D_B$$

If ψ is the electric flux in the tube, and a_1 and a_2 are the areas of cross-section of the tube at A and B, these areas being measured perpendicular to the direction of the field at the points, then

$$\frac{\psi}{a_2}$$

---

* Note that this law is similar to the magnetic law—
$B = \mu H$ where B = lines of magnetic force per unit surface
H = magnetic intensity
μ = magnetic permeability of the medium.

Electrical Intensity Inside a Charged Spherical Conductor. Imagine a hollow sphere of conducting material which has been given a charge of Q positive units of electricity. If its area of surface be S the density of charge on the surface (which will be uniform) is $\frac{Q}{S}$ per unit of surface. The electric intensity at the surface will be at all points normal to the surface, since the sphere is of conducting material. This follows from a consideration of the fact that, if it were not so, the intensity would have a tangential component which would produce a movement of charge until the direction of the intensity became normal. Consider a point P inside the sphere (Fig. 3) at which small areas of surface a_1 and a_2 subtend a solid angle $\delta\theta$ as shown. Points A_1 and A_2 are mid-points of the areas a_1 and a_2. Angle OA_1P = angle $OA_2P = \alpha$.

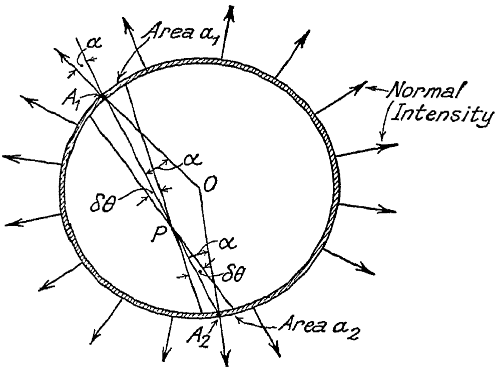


Fig. 3. Intensity Inside a Spherical Conductor

Let $A_1P = d_1$, $A_2P = d_2$

Let κ = permittivity of the medium inside the sphere.

Then, Charge on area $a_1 = \frac{Q}{S} \cdot a_1$

„ „ „ $a_2 = \frac{}{S} \cdot a_2$

Since the intensity is everywhere normal to the surface, intensity at P due to charge on a_1

$$= \frac{Qa_1}{S} \frac{\cos \alpha}{\kappa\, d_1{}^2} \text{ in direction } A_1P$$

Similarly, intensity at P due to charge on a_2

$$= \frac{Qa_2}{S} \cdot \frac{\cos \alpha}{\kappa \, d_2^2} \text{ in direction } A_2P$$

directly opposite to direction A_1P.

Now, the solid angle subtended at the centre of a sphere of radius R by any area A on its surface is $\frac{A}{R^2}$.

Hence,

$$\text{Solid angle } \delta\theta = \frac{a_1 \cos \alpha}{d_1^2} = \frac{a_2 \cos \alpha}{d_2^2}$$

Thus, the intensities at P due to charges on a_1 and a_2 are opposite and are each equal to $\frac{Q}{S\kappa} \cdot \delta\theta$, giving a resultant intensity due to these two charges of zero.

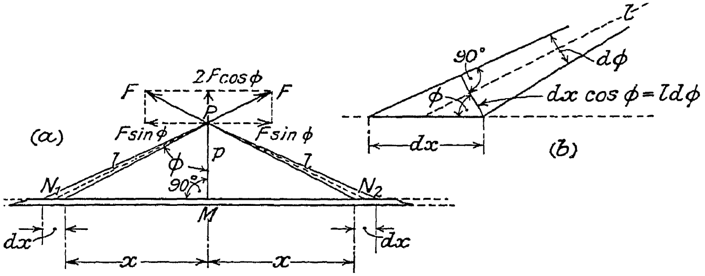


FIG. 4. ELECTROSTATIC FIELD NEAR A STRAIGHT CONDUCTOR

As the same is true for all similar pairs of areas such as a_1 and a_2, the *total intensity at any point inside a charged spherical conductor is zero.*

Intensity in the Neighbourhood of a Charged Straight Conductor. Fig. 4 (*a*) represents a long, thin, straight conductor which carries a uniform charge of Q per unit length. P is a point whose perpendicular distance from the conductor is p where p is small compared with the length of the wire. Consider two elements of the conductor each of length dx, as shown at N_1 and N_2, the elements being equidistant from P.

Let $\qquad N_1P =$

Then, if the elements dx are so small that the charges on them can be considered as concentrated at N_1 and N_2, the forces (F) upon unit positive charge placed at P will be each equal to $\frac{Qdx}{\kappa l^2}$ Equation (1), where κ is the permittivity of the medium.

The directions of these forces will, as shown, each make an angle of $(90 - \phi)$ with the direction of the conductor, and will together be equivalent to one force of $2F \cos \phi$ in direction MP, the horizontal components neutralizing one another.

The same applies to all such pairs of elements as those shown, so that the total force upon unit charge at P—i.e. the intensity at P—due to the whole length of the wire, will be in the direction MP, and is given by

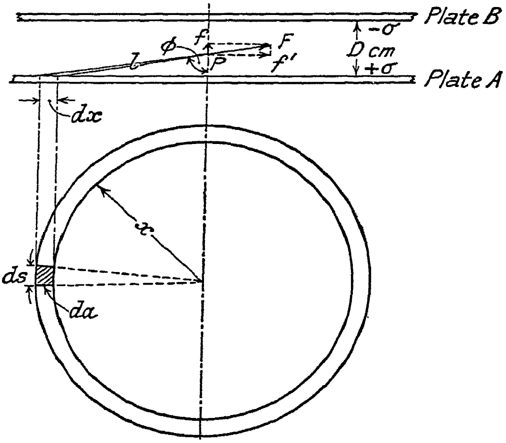


FIG. 5. ELECTROSTATIC FIELD BETWEEN TWO CHARGED PLATES

$$E_p = \int_{x=0}^{x=\infty} \frac{2Q \cos \phi}{\kappa l^2} \, dx$$

where E_p = total intensity at P, if the distance p is small compared with the length of the wire.

From Fig. 4 (b), it can be seen that, if dx is very small, then

$$l d\phi = dx \cos \phi = dx \, . \, \frac{p}{l}$$

$$\therefore \ \frac{dx}{l^2} = \frac{d\phi}{p} \text{ where } d\phi \text{ is the angle subtended at } P \text{ by } dx$$

$$\therefore \ E_p = \int_{\phi=0}^{\phi=+\frac{\pi}{2}} \frac{2Q \cos \phi}{\kappa p} \, d\phi = \frac{2Q}{\kappa p} \text{ in the direction } MP \ . \qquad (4)$$

Intensity in the Space Between Two Charged Parallel Conducting Plates. Fig. 5 represents the two conducting plates, which are close together. Their extent is supposed to be so great as compared with their distance apart that the electrostatic field on or near their common axis is unaffected by the fringing field at the edges of the plates. Let plate A be charged positively and B charged negatively. Neglecting the edge effects the distribution of charge will be uniform.

Let charge density on $A = + \sigma$ units per unit of surface
,, ,, $B = - \sigma$ units per unit of surface

P is a point between the plates on or near their common axis. Let their distance apart be D.

A similar method to that set out in the preceding paragraph can be followed, except that the intensities at P due to *elemental rings* must now be considered instead of elements of length of conductor as previously considered.

Since the charge on area da is $+ \sigma da = + \sigma ds \,.\, dx$, the force (F) upon a unit positive charge at P due to this area is $+ \dfrac{\sigma \,.\, ds \,.\, dx}{\kappa \,.\, l^2}$, κ being the permittivity of the medium between the plates. This force F may be split up into two components, f and f', perpendicular to and parallel with the plates as shown. Since the components of all such forces as F due to the whole of the elemental ring in a direction parallel to the plates will neutralize one another, the total force at P due to the whole of the elemental ring will be the sum of all such components as f perpendicular to the plates. Calling this total perpendicular force due to the ring f_T, then we have

$$f_T = \sum_{s=0}^{s=2\pi x} f$$

$$\text{Now } f = \text{F} \cos \phi = \frac{\sigma \,.\, ds \,.\, dx}{\kappa \,.\, l^2} \cos \phi$$

$$\therefore f_T = \int_{s=0}^{s=2\pi x} \frac{\sigma \,.\, dx}{\kappa \,.\, l^2} \cos \phi \,.\, ds = \frac{\sigma \,.\, dx}{\kappa \,.\, l^2} \cos \phi \times 2\pi x$$

If P = total force at P in a direction perpendicular to the plates due to all such elemental rings. then

$$P = \int_{x=0}^{x=\infty} \frac{\sigma \cos \phi}{\kappa \,.\, l^2} \times 2\pi x \,.\, dx$$

As in the previous paragraph

$$l d\phi = dx \,.\, \cos \phi$$

$$\text{i.e.} \qquad \cos \phi = \frac{l d\phi}{dx}$$

$$\therefore P = \int_{x=0}^{x=\infty} \frac{\sigma l}{\kappa l^2} 2\pi x \, dx \,.\, \frac{d\phi}{dx}$$

$$= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{\sigma 2\pi}{\kappa l} x \,.\, d\phi$$

$$= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{\sigma \,.\, 2\pi}{\kappa} \sin\phi \,.\, d\phi$$

$$P = \frac{2\pi\sigma}{\kappa}$$

This force will be one of repulsion if unit positive charge is placed at P. There will also be an equal force attracting the unit charge to plate B. Thus the total force on unit positive charge at P—i.e. the intensity at P—

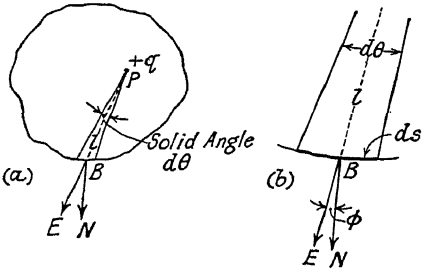


FIG. 6. ILLUSTRATING GAUSS'S THEOREM

$$E_p = \frac{4\pi\sigma}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Gauss's Theorem. Briefly this theorem states that the total electric flux traversing a surface which completely encloses a charge of Q units is $4\pi Q$. This holds true whatever the shape of the surrounding surface, and for any dielectric. Consider a small element of surface ds upon the surface surrounding a charge of $+ Q$ (Fig. 6 (*a*)). Let this element subtend a solid angle $d\theta$ at P and let the angle between the intensity E at B, due to the charge, and the normal N at B be ϕ. Let the distance $PB = l$.

Then, electric flux crossing element $ds = D \cos\phi \,.\, ds$

$$= E\kappa \,.\, \cos\phi \,.\, ds$$

$$= \frac{Q}{l^2} \,.\, \kappa \,.\, \cos\phi \,.\, ds$$

$$\frac{Q}{l^2} \cos\phi \,.$$

Thus, the total flux crossing the whole surface

$$\psi = \Sigma \frac{Q}{l^2} \cos \phi \,.\, ds$$

or, since $\frac{ds \,.\, \cos \phi}{l^2}$ = solid angle $d\theta$

$$\psi = \Sigma Q d\theta = 4\pi Q \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

If there are a number of charges inside the surface, some positive and some negative, then, if the charges are Q_1, Q_2, etc.,

$$\psi \qquad \pi\,(Q_1 \pm Q_2 \pm Q_3 \pm \cdot \cdot \cdot) \quad . \quad . \qquad (7)$$

the flux in the outward direction being considered positive.

Coulomb's Theorem. This theorem states, in effect, that *the electric intensity at the surface of a conductor, charged to a surface density of* σ *units per unit of surface, is* $\frac{4\pi\sigma}{\kappa}$, where κ is the dielectric constant of the medium outside the conductor.

This follows from Gauss's Theorem. Consider an element of surface of the conductor ds. This element carries a charge of σds units. From Gauss's Theorem the flux radiating from this charge is $4\pi\sigma \,.\, ds$, and, since no flux exists inside the conductor, the whole of this flux passes outwards normally.

Thus, electric flux density at the surface $D = \qquad .\, ds \quad = 4\pi\sigma.$

Hence, the intensity

$$E = \frac{D}{\kappa} = \frac{4\pi\sigma}{\kappa} \quad . \quad . \quad . \quad . \qquad (8)$$

its direction being normal to the surface.

Potential. If unit positive charge is moved towards a positively charged body, work is done in overcoming the force of repulsion acting on the charge. If this movement of the unit charge is from a point P_1 to some point P_2 nearer to the positively charged body, then the point P_2 is said to be at a higher electric potential than point P_1 and the difference of potential between the two points is defined as the quantity of work required to move unit positive charge from the point at the lower potential to the point at the higher potential.

In general, the potential of any point in the vicinity of a system of charged bodies is defined as the work required to move unit positive charge from an infinite distance to the point considered, assuming that the distribution of the charges on the bodies is unaffected by the approach of the unit charge.

The *electrostatic C.G.S. unit of potential difference* is defined as the potential difference between two points such that 1 erg of work is done in moving unit positive charge (as defined on p. 1) from the point at the lower potential to the point at the higher potential, the potential being assumed unaltered by the presence of the unit charge.

If two points at a very small distance ds apart have a difference of potential dV units, then the work done in moving unit positive charge from one point to the other up the gradient of potential is Eds where E is the average force on the unit charge during the movement.

Thus
$$dV = -Eds$$
or
$$E = -\frac{dV}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where E is the intensity at any point in an electric field, $\frac{dV}{ds}$ being the potential gradient at the point, the positive direction of s being *down* the gradient of potential. Again, the potential difference V between any two points A and B is given by

$$V_{AB} = \int_{A}^{B} E \, . \, ds \quad . \quad . \quad . \quad . \qquad (10)$$

Potential at a Point Due to a Number of Charges. The potential at any point P distance d from a single charge of Q units is equal to the work done in bringing unit charge from an infinite distance up to the point P, i.e.

$$\text{Potential at } P, \quad V_P = \int_{d}^{\infty} E \, . \, ds = \int_{d}^{\infty} \frac{Q}{\kappa s^2} \, ds$$

$$\therefore \; V_P = \frac{Q}{\kappa d} \text{ units} \quad . \quad . \quad . \quad . \qquad (11)$$

Similarly, the potential at a point P due to a number of charges Q_1, Q_2, etc., distant d_1, d_2, etc., respectively, from P is given by

$$V_P = \frac{Q_1}{\kappa d_1} + \frac{Q_2}{\kappa d_2} + \frac{Q_3}{\kappa d_3} + \text{etc.} \quad . \quad . \quad . \quad . \qquad (12)$$

κ being the permittivity of the medium.

If a number of conductors have charges $Q_1, Q_2 \, . \, . \, . \, Q_n$, then their potentials V_1, V_2, . . . V_n will be given by the expressions

$$V_1 = P_{11}Q_1 + P_{21}Q_2 + \; . \; . \; . \; P_{n1}Q_n, \text{ etc.}$$

If all but (say) the second conductor are uncharged and it has unit charge then by putting $Q_1 = Q_3 = Q_4 = \; . \; . \; . \; Q_n = 0$, and $Q_2 = 1$, we find that P_{21}, for example, is the potential of the first body produced by unit charge on the second. The value of P_{21} is evidently obtained by considering all the contributions $\frac{\delta Q}{\kappa d}$ which the elementary charges on the second conductor produce at the first. The coefficients P are called the "potential coefficients" of the conductors and depend only on the geometry of the system.

Green's Reciprocation Theorem enables us to assert that $P_{12} = P_{21}$, etc., so that in general $P_{mn} = P_{nm}$. Thus, the potential produced at conductor n by unit charge on conductor m (all other charges being zero) is equal to the potential of m produced by unit charge on n.

Equipotential Surfaces. An equipotential surface is a surface such that all points on it are at the same potential. Obviously the potential gradient $\frac{dV}{ds}$ for such a surface is zero, and from Equation (9) it follows that the intensity along such a surface is also zero. Thus the lines of electric intensity of the field in which the equipotential surface is situated have no component along the surface, i.e. they cut such a surface at right angles.

Capacitance. Consider, again, a number of conductors having charges Q_1, Q_2, Q_3, etc. and potentials V_1, V_2, V_3, etc. If the series of equations

$$V_1 = P_{11}Q_1 + P_{21}Q_2 + \ . \ . \ . + P_{n1}Q$$
$$V_2 = P_{12}Q_1 + P_{22}Q_2 + \ . \ . \ . + P_{n2}Q_n, \text{ etc.}$$

is solved for the values of Q, we obtain

$$Q_1 = C_{11}V_1 + C_{21}V_2 + \ . \ . \ . + C_{n1}V_n$$
$$Q_2 = C_{12}V_1 + C_{22}V_2 + \ . \ . \ . + C_{n2}V_n, \text{ etc.}$$

Thus, from the theory of linear simultaneous equations

$$C_{21} = \frac{-\begin{vmatrix} P_{21} & P_{31} & \dots & P_{n1} \\ P_{23} & P_{33} & \dots & P_{n3} \\ \dots & \dots & \dots & \dots \\ P_{2n} & P_{3n} & \dots & P_{nn} \end{vmatrix}}{\begin{vmatrix} P_{11} & P_{21} & \dots & P_{n1} \\ P_{12} & P_{22} & \dots & P_{n2} \\ \dots & \dots & \dots & \dots \\ P_{1n} & P_{2n} & \dots & P_{nn} \end{vmatrix}}$$

and the other C values can be found similarly. From the solution $P_{mn} = P_{nm}$ it readily follows that $C_{mn} = C_{nm}$.

The coefficients C are termed "coefficients of capacitance." C_{11} is the "earth capacitance" of the first conductor, and, by putting $V_1 = 1$, $V_2 = V_3 = \ . \ . \ . = V_n = 0$ it is easily seen that C_{11} is the charge required to produce unit potential on the first conductor, all the other conductors having zero potential. C_{12} (or C_{21}) is the inter-capacitance between the first and second conductors. By putting $V_1 = V_3 = V_4 = \ . \ . \ . = V_n = 0$ and $V_2 = 1$ it is seen that C_{21}

is the charge produced on the first conductor by unit potential on the second. Similarly C_{12} is the charge produced on the second conductor by unit potential on the first and, as shown above, this is equal to C_{21}.

In the case of a capacitor consisting of two metal plates separated by an insulator, or "dielectric," the equations will reduce to

$$Q_1 = C_{11}V_1 + C_{21}V_2$$
$$Q_2 = C_{12}V_1 + C_{22}V_2$$

where $C_{21} = C_{12}$.

It is usual to refer to C_{21} as *the* capacitance of the capacitor, and to remind ourselves of the coefficients C_{11} and C_{22} we redraw the simple diagram (*a*) in Fig. 7 as shown in diagram (*b*).

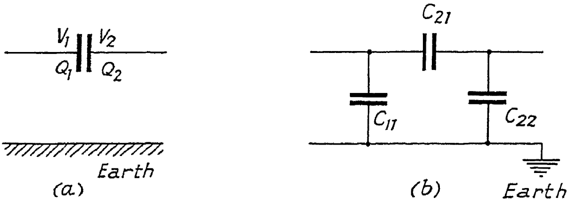


FIG. 7

In the conventional diagram (*b*) the conductors of all the capacitors are understood to have no earth capacitance. Obviously the second circuit gives exactly the same result as the first. Generally C_{21} is much greater than C_{11} or C_{22}.

If a single conductor is very remote from all other conductors, the coefficient C_{11} takes a value depending only on the geometry of the conductor. The value of C_{11} under these conditions is known as the "self-capacitance" of the conductor.

The coefficients of capacitance are calculated, in any practical case, by finding the electric field strength at each point due to a given distribution of charges on the conductors and integrating along lines between the conductors to find the potential differences. This gives the potential coefficients P and, by solving the equations, the coefficients of capacitance C can be found.

In the simple case of a capacitor formed by two conductors with negligible earth capacitances the capacitance C is given by

$$C = \frac{Q}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If Q is the charge, in electrostatic C.G.S. units, producing a potential difference of V electrostatic C.G.S. units between the conductors, then C will be given in electrostatic C.G.S. units of capacitance. This last unit is defined as the capacitance of a field such that one e.s. unit of charge causes a potential difference between the conductors, between which the field exists, of one e.s. unit of potential.

A capacitor is most commonly thought of as an arrangement of two conductors placed comparatively close together so that a strong electric field exists between them.

Energy Stored in an Electric Field. If two conductors X and Y are charged so as to have a potential difference of V units, then an electrostatic field will exist between them, and this field will represent a quantity of stored energy, since, from the definition of potential, work must be done to produce a potential difference between two points. If the charges on the two conductors X and Y are $+Q$ and $-Q$ units respectively, and the conductors be considered as originally uncharged, then the potential difference V may be considered as produced by the transference of Q units of charge from Y to X.

Since the potential difference V is, from Equation (13), proportional to the charge at any time, thus the average potential difference due to transference of the Q units is $\frac{1}{2}V$, and the work done during the transference is, from the definition of potential, $\frac{1}{2}QV$. This is obviously equal to the energy stored. Hence—

Energy stored in the field between the conductors

$$= \tfrac{1}{2}QV = \tfrac{1}{2}(VC)V$$

$$= \tfrac{1}{2}CV^2 \text{ ergs} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where C is the capacitance of the field.

Energy Stored per Unit Volume of Dielectric in an Electric Field. It can be shown experimentally that the energy stored in a capacitor is actually stored in the dielectric between the conductors bounding the field. The energy stored per unit volume of dielectric

$$= \frac{\kappa E^2}{8\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

as shown below, if the intensity is E and the permittivity is κ.

Consider two charged conductors having surface densities of charge σ and σ_1, whose potentials are V and V_1, the charges on the conductors being $+Q$ and $-Q$ (Fig. 8). The tube of flux shown starts on area a, and finishes on a_1. It contains a flux ψ, where

If κ is the permittivity of the medium between the conductors, the intensity at any point P in the tube of flux, distant x from end a, is given by

$$E_x = \frac{\psi}{a_x \kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where a_x is the area of cross-section of the tube at P.

Now, on the assumption that the energy stored per unit volume of the dielectric is $\frac{\kappa E^2}{8\pi}$, then the energy stored in an element of the tube of length dx at P is

$$\frac{\kappa \,.\, E_x^2}{8\pi} a_x \,.\, dx$$

and the total energy stored in the tube

$$= \int_0^l \frac{\kappa \,.\, E_x^2}{8\pi} \,.\, a_x \,.\, dx$$

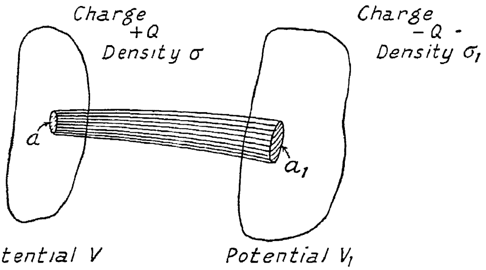


FIG. 8. TUBE OF FLUX BETWEEN TWO CHARGED CONDUCTORS

$$= \frac{\kappa}{8\pi} \int_0^l E_x^2 a_x \, dx$$

where l is the length of the tube,

$$= \frac{\kappa E_x}{8\pi} a_x \int_0^l F_x \,.\, dx$$

$$= \frac{\psi}{8\pi} \int_0^l E_x \,.\, dx \quad \text{from (16)}$$

$$= \frac{4\pi\sigma a}{8\pi} \int_0^l E_x \,.\, dx$$

$$= \frac{\sigma a}{2} \int_0^l E_x \,.\, dx$$

$$= \frac{\sigma a}{2} (V - V_1), \text{ since } \int_0^l E_x \,.\, dx$$

is the work done in moving unit charge from a_1 to a, i.e. the potential difference $(V - V_1)$.

But, since the charge on a is σa the expression $\frac{\sigma a}{2}(V - V_1)$ gives the energy stored in the small capacitor, whose plates are a and a_1 from (14), which is, of course, the same as the energy of the tube of flux considered.

Thus, the assumption that the energy stored per unit volume of dielectric is $\frac{\kappa \cdot E^2}{8\pi}$ is correct for this tube of flux and, since the same reasoning applies to all such tubes of flux, the assumption is true generally.

In general, the energy stored in a dielectric the intensity in which is a variable quantity, as above, is given by

$$\iiint \frac{\kappa \cdot E^2}{8\pi}\, dv$$

where E is the intensity at any point and dv is an element of volume of the field at this point.*

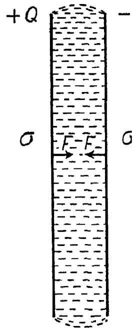


FIG. 9. STATIC FIELD BETWEEN TWO CHARGED PLATES

Force of Attraction Between Oppositely Charged Parallel Conducting Surfaces. Fig. 9 represents two parallel conducting surfaces of equal area and close together, possessing charges of $+Q$ and $-Q$ units. Since their areas are equal, the surface density of charge (σ) is the same on both surfaces and the intensity in between them is at all points $\frac{4\pi\sigma}{\kappa}$ where κ is the permittivity of the medium between them. It is assumed that the effect of the fringing field at the edges of the surfaces is negligible.

Let F be the force of attraction between the surfaces. Then, if one surface is moved away from the other by an infinitely small distance δx, the work done is $F\delta x$, it being assumed that the movement is so small that the intensity is unaffected by the movement. If the area of each surface is A, then the increase in the energy stored, owing to the increase in volume of the dielectric, is $\frac{A\delta x \cdot \kappa E^2}{8\pi}$, which is, of course, equal to the work done.

---

* Note that the expression obtained for the energy stored per unit volume of dielectric, viz. $\frac{\kappa E^2}{8\pi}$, is similar to the expression for energy stored in a magnetic field, i.e. $\frac{\mu H^2}{8\pi}$ per unit volume, where

H = intensity of the magnetic field
μ = the permeability of the medium (see page 45)

$$\therefore \quad \mathrm{F}\delta x = \frac{A\delta x\kappa E^2}{8\pi}$$

$$\mathrm{F} = \frac{A\kappa E^2}{8\pi}$$

$$= \frac{A \,.\, \kappa \,.}{8\pi}\left(\frac{4\pi\sigma}{\kappa}\right)^2$$

$$= \frac{2A\pi\sigma^2}{\kappa}$$

$$\therefore \mathrm{F} = \frac{2\pi\sigma \,.\, Q}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

since $Q = A\sigma$.

Magnetism

Coulomb's Law. Coulomb's Inverse Square Law is true for magnetic quantities as well as for electrostatic quantities. Substituting magnetic quantities for electrostatic, we have

$$\mathrm{F} = \frac{m_1 m_2}{\mu r^2}$$

where F is the force between two magnetic poles of pole strengths m_1 and m_2 units, r being their distance apart and μ being the "permeability" of the medium separating them (see page 20).

In the electromagnetic C.G.S. system of units, unit magnetic pole is defined as "a pole such that if placed at a distance of 1 cm. from an exactly similar pole, in air, (for which the permeability μ is taken as unity) it will be repelled with a force of 1 dyne."

Thus, using units in this system,

$$\mathrm{F} = \frac{m_1 m_2}{r^2} \text{ dynes} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The force between the poles is one of repulsion or attraction, according to whether they are of like or unlike polarity respectively.

Magnetic poles are considered as being concentrated at a point.

Lines of Force and Intensity of Magnetic Field. The magnetic field existing in the neighbourhood of a magnetic pole can be represented by lines of force similar to the lines of electric force which are used to represent an electrostatic field, the arrowheads upon the lines of force indicating the direction in which a unit *north* pole would move if placed upon the line of force. The "intensity" of a magnetic field at any point is expressed by the "force which would be exerted on a north pole of unit strength placed at the point," it being assumed that the introduction of the unit pole does not affect the field. Thus, the intensity at a point, distant r from a pole of strength m units, in air is given by

$$H = \frac{m}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Lines of Magnetic Intensity. As in electrostatics, lines of force can be made to express intensity of field by the number of such lines crossing unit area at the point considered, the area being taken perpendicular to the direction of the field at the point and the medium having unit permeability. Thus, in air, if the intensity at a point were H, then H lines of magnetic intensity would cross unit area in a direction perpendicular to the direction of the field at the point. In a medium whose permeability is μ the intensity at a point is given by $\frac{\text{No. of lines of intensity per unit area}}{\mu}$, the area being, of course, perpendicular to the direction of the field.

Magnetic Flux and Flux Density. A number of lines of magnetic intensity are spoken of, collectively, as "magnetic flux," and the number per unit of cross-section as the "flux density." The symbols used to represent these quantities are Φ and B respectively.*

If Φ lines of force cross an area A in a direction perpendicular to the area it follows from the above definitions of flux density and magnetic intensity that

$$\Phi = B . A \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$H = \frac{B}{\mu} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where μ is the permeability of the medium in which the field exists.

Magnetic Flux Radiating from a Pole of Strength *m* Units. (Gauss's Theorem.) It can be proved by an exactly similar method to that used for the proof of Gauss's Theorem in electrostatics that the total magnetic flux which traverses, in a normal direction, a surface completely surrounding a pole of strength m units is $4\pi m$ lines, being independent of the permeability of the medium.

Magnetic Moment and Intensity of Magnetization of a Magnet. A magnet having poles of strength m units distant l apart is said to have a "magnetic moment" of ml units. This term arises from the fact that, if the magnet were placed on a magnetic field of unit intensity so that its magnetic axis (i.e. the line joining its poles) is perpendicular to the direction of the field, it would be acted on by two forces each of m units, as shown in Fig. 10, these forces forming a couple whose turning moment is ml. The symbol M is used to express magnetic moment, hence

$$M = ml \quad . \quad . \quad . \quad . \quad . \quad (22)$$

---

* Although these lines are spoken of above as lines of "intensity," to distinguish them from lines which merely indicate the direction of the field, they are so generally referred to as "lines of force" that they will be referred to hereafter as such.

The intensity of magnetization (J) of a magnet is expressed by the ratio $\frac{\text{pole strength}}{\text{cross-sectional area}}$ the magnet being assumed to be uniformly magnetized.

Thus $$J = \frac{m}{A} \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

In the case of a bar magnet, of uniform cross-section A, and of length l, the poles being considered at the extreme ends, the magnetic moment is ml and the volume of the magnet (V) is lA.

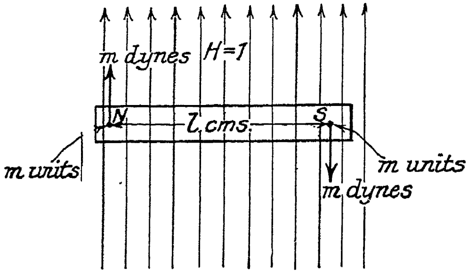


FIG. 10. BAR MAGNET SITUATED IN A MAGNETIC FIELD

Then $$J = \frac{m}{A} = \frac{ml}{Al} = \frac{M}{V}$$

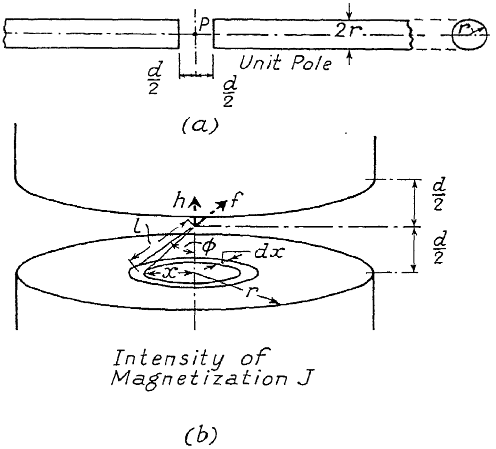
and thus the intensity of magnetization may be expressed as the magnetic moment per unit volume, the magnetization being uniform.

If the magnetization is not uniform, J varies at different points in the magnet, and is expressed by $\frac{\delta M}{\delta V}$ where δM is the magnetic moment of an element of volume δV taken at the point considered.

Relations Between Intensity of Magnetization, Flux Density, and Magnetic Intensity. In order to determine the connection between the intensity of magnetization in a magnetized body and the flux density in the body, consider a long, thin rod of some magnetic substance (say iron) which has been uniformly magnetized. The demagnetizing effect of the ends of the rod (see Chap. IX) may be neglected if it is very long and thin. Now imagine a very narrow air gap of length d in the rod, with a unit magnetic pole placed in the air gap, on the axis of the rod, and equidistant from the bounding faces of the air gap, as shown in Fig. 11 (*a*). This unit pole will be repelled from one face and attracted to the other with equal forces, since it is equidistant from them, and the direction of these forces will, of course, be along the axis of the rod.

Let the rod be of circular cross-section, radius r, and let its intensity of magnetization be J. Then, as shown in Fig. 11(*b*), the intensity of the field in the air gap at P, due to the intensity of magnetization J in the iron rod, can be considered as being produced by a number of elemental rings of radius x and width dx, having a pole strength per unit area of J (i.e. intensity of magnetization of J).

If Fig. 11 is compared with Fig. 5, it will be observed that the present determination of intensity at P is almost exactly similar to the determination of the electric intensity at a point P in between two oppositely-charged surfaces with surface densities of charge σ units per unit of surface. The differences between the two cases are that instead of σ units of charge per unit of surface we now have J units of pole strength per unit of surface, and that the integration for the total effect at P of the elemental rings must be performed between the limits $\phi = \tan^{-1} \frac{r}{d/2}$ and $\phi = 0$, instead of the limits $\phi = \frac{\pi}{2}$ and $\phi = 0$.



(a)

(b)

FIG. 11. FIELD INTENSITY IN THE AIR GAP OF A MAGNETIZED BAR

The permittivity κ also is replaced by the permeability, which is here taken as unity. Thus, making these necessary modifications, we have for the total field intensity at P due to *one* circular bounding surface

$$h = \int_{\phi=0}^{\phi=\tan^{-}} J \,.\, 2\pi \sin\phi \,.\, d\phi$$

i.e.

$$h = \Big[-2\pi J \cos\phi \Big]_{\phi = 0}^{\phi = \tan^{-1} \frac{r}{d/2}}$$

$$= -2\pi J \left[\frac{\frac{d}{2}}{\sqrt{\left(\frac{d}{2}\right)^2 + r_2}} - 1 \right]$$

If the air gap is very small, so that $\frac{d}{2}$ is negligible compared with the radius r, then

$$h = 2\pi J \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This is the intensity in the gap due to *one* bounding surface only. If this is a force of repulsion upon the unit pole at P, then the other surface, which is of opposite polarity, will attract the unit pole at P with an equal force. Thus, the total field intensity at P is $4\pi J$ and, since the introduction of the very narrow air gap does not affect the intensity of magnetization in the iron, the same would be true in the iron if the air gap did not exist. The flux density, also, in the air gap is $4\pi J$, which, again, is the flux density which will exist in the iron due to the intensity of magnetization J.

If this iron rod is placed in a magnetic field of intensity H, the direction of this field being along the axis of the rod and in the same direction as the intensity $4\pi J$, then the total flux density B in the iron is $H + 4\pi J'$ where J' is the intensity of magnetization in the iron when the rod is situated in the field of intensity H and will, of course, be different from the intensity of magnetization J, existing before the rod was placed in the magnetizing field.

If the iron rod has no magnetization before being placed in the magnetizing field of intensity H, and if J is the intensity of magnetization *produced by the field,* then

$$B = H + 4\pi J \quad . \quad . \quad . \quad . \quad (25)$$

The term $4\pi J$, which is the flux density which exists due to the magnetization of the iron itself, is sometimes called the "ferric induction." "Ferric induction" is, therefore, the flux density which exists in excess of that produced in air by the same magnetizing force, and the intensity of magnetization is $\frac{\text{ferric induction}}{4\pi}$ or $\frac{B - H}{4\pi}$

Equation (25) can be written

$$B = \mu H \quad . \quad . \quad . \quad . \quad . \quad . \quad (26$$

where μ is the "magnetic permeability" of the iron, which is thus defined as the ratio of the flux density produced by a magnetizing field of strength H to the magnetizing intensity H,

i.e. $$\mu = \frac{B}{H}$$

If the iron has some residual magnetism before being placed in the magnetic field, then the ratio $\frac{B}{H}$, B being the total flux density in the iron, is not the correct value of the permeability.

If J were directly proportional to H ($= kH$ say) for all values of H, then Equation (25) could be written

$$B = H(1 + k) = \mu H$$

and the permeability μ would be a constant factor. This, however, is not so, J having a maximum value depending upon the iron or other magnetic material considered, and μ is thus a variable factor depending upon the value of H (or B).

When, during the magnetization of a specimen of magnetic material, the maximum value of J is attained, further increase in the magnetizing force H only increases the flux density B by increasing H (Equation (25)), and the material is said to be "saturated."

Fig. 12 shows a typical magnetization (B–H) curve for cast steel. It can be seen that, above a value of B of about 16,000 lines per sq. cm., the increase in flux density as H is increased is very small, which means that the intensity of magnetization J of the iron is being increased only very slightly by increase of H.

The second curve in Fig. 12 shows the variation of permeability with the magnetizing force H.

Magnetic Susceptibility. The ratio

$$\frac{\text{intensity of magnetization}}{\text{magnetizing force producing this intensity of magnetization}}$$

is called the "magnetic susceptibility" of the material and is given by

$$\text{Magnetic susceptibility} = \frac{J}{H} \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

$$= \frac{B-H}{4\pi H} \text{ from equation (25)}$$

$$\therefore \text{ Magnetic susceptibility} = \frac{\mu - 1}{4\pi} \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

Obviously, since $\mu = 1$ for non-magnetic substances, the magnetic susceptibility of such substances is zero.

Fröhlich-Kennelly Equation. An attempt to obtain an equation which would represent with fair accuracy the law of the magnetization curve of a magnetic material was made by Fröhlich, who stated that "the permeability is proportional to the magnetizability." Expressed as an equation, this means that

$$B = \frac{H}{m + nH} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where m and n are constants for any material, having the values

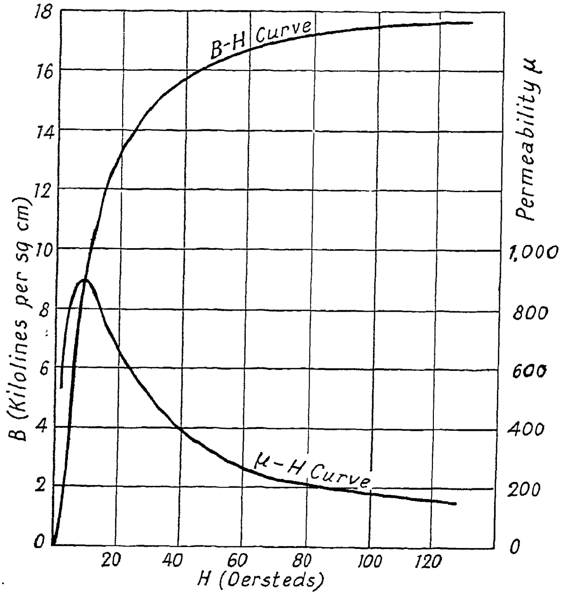


FIG. 12. TYPICAL B–H AND μ–H CURVES FOR CAST STEEL

$n = \frac{1}{B_s}$ and $m = \frac{1}{kB_s}$. B_s is the value of the flux density at saturation point in the material, and k is another constant depending upon the material. This equation can be otherwise expressed as

$$\mu = k\,(B_s - B) \quad . \quad . \quad . \quad . \quad . \quad (30)$$

which is derived from the above by simple algebra.

If H is small compared with $4\pi J$, so that $B \doteqdot 4\pi J$

$$4\pi\, J = \frac{H}{m + nH}$$

$$J = \frac{\frac{1}{4\pi}H}{m + nH} = \frac{AH}{m + nH} = \frac{\frac{A}{m}H}{1 + \frac{n}{m}H}, \text{ where } A = \frac{1}{4\pi}$$

$$\therefore J = \frac{pH}{1 + kH} \quad . \quad . \quad . \quad . \quad (31)$$

where $p = \frac{kB_s}{4\pi}$

Thus, the ratio $\frac{p}{k} = \frac{B_s}{4\pi} = J_{max}$ which is the saturation value of the intensity of magnetization J. The equations hold fairly closely for moderate values of H, but it should be noted that the hysteresis effect (see page 39) is not taken into account by them.

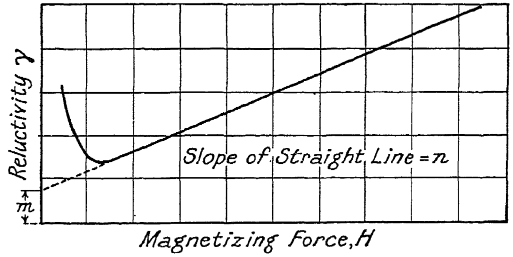


FIG. 13. RELUCTIVITY CURVE

Kennelly (Ref. (5)) modified Fröhlich's equation by the introduction of the term "reluctivity" (γ) to express the reciprocal of the permeability.

Thus $$\gamma = \frac{1}{\mu} = \frac{H}{B}$$

and from Equation (29)

$$\gamma = m + nH \quad . \quad . \quad . \quad . \quad (32)$$

m is called the "magnetic hardness coefficient" and n (which, as previously defined, equals $\frac{1}{B_s}$) is called the "saturation coefficient."

The general shape of the reluctivity curve for a magnetic material is shown in Fig. 13. For low values of H the graph is not a straight line, but turns upwards as shown. The value of H above which the graph becomes a straight line depends upon the magnetic material, varying considerably with the percentage of carbon in the case of steel.

Magnetic Potential. This can be defined in a similar way to that used in the definition of electrostatic potential. The work required to move unit north pole from an infinite distance to a given point is considered to be the magnetic potential of the point.

Thus, in the electromagnetic C.G.S. system, if H_x is the intensity of field at a distance x cm. from a pole of strength m units, the force upon unit pole situated at this distance away is H_x dynes, and the work done in moving the unit pole from infinite distance to a point in the neighbourhood of the pole is $\int_x^0 H_x \, . \, dx$ ergs. Also

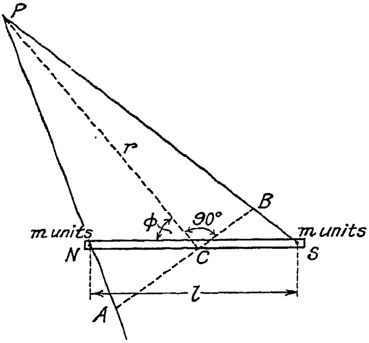


FIG. 14. POTENTIAL NEAR A BAR MAGNET

$H_x = \frac{m}{x^2}$, so that the potential at a point distant r cm. from a pole of strength m units is given by

$$V = \int_r^\infty \frac{m}{x^2} \, dx = \frac{m}{r} \, . \qquad . \quad . \quad . \quad . \quad . \quad (33)$$

Potential Near a Short Bar Magnet. Fig. 14 represents a bar magnet whose length l is small compared with the distance r from its centre to the point (P) whose magnetic potential is being considered.

Let the magnet have a pole strength of m units and centre C as shown, and let the line ACB, perpendicular to PC, cut PS and PN produced, in B and A respectively. Since PC is great compared with the length of the magnet, $PA = PB = r$ very nearly, and $N\hat{A}C = S\hat{B}C = 90°$ very nearly.

$$\text{Thus} \qquad PN = PA - AN = r - \frac{l}{2} \cos \phi$$

$$PS = PB + BS = r + \frac{l}{2} \cos \phi$$

$\therefore$ Potential at P is

$$V_p = \frac{m}{r - \frac{l}{2}\cos\phi} - \frac{m}{r + \frac{l}{2}\cos\phi}$$

the negative sign being due to the opposite polarities of the poles of strength m units.

$$V_p = \frac{m\left(r + \frac{l}{2}\cos\phi\right) - m\left(r - \frac{l}{2}\cos\phi\right)}{r^2 - \frac{l^2}{4}\cos^2\phi}$$

$$= \frac{ml\cos\phi}{r^2} \text{ if } r \text{ is great compared with } l$$

Thus, $$V_p = \frac{M\cos\phi}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where M is the magnetic moment of the bar magnet.

Force of Attraction Between Oppositely-magnetized Surfaces. Fig. 15 represents the ends of two magnetized bars, each of cross-section A, the ends having opposite polarity. Let B be the flux density (considered uniform) between them. If μ is the permeability of the medium between the poles, the intensity H in the field is $\frac{B}{\mu}$. Let F be the force of attraction existing between them. Then if one of the bars is moved an infinitesimal distance dx farther away from the other, the work done is Fdx. Now, the energy stored per unit volume of a magnetic field is $\frac{\mu H^2}{8\pi}$ (see page 45). Thus, assuming that the intensity is unaltered by the infinitesimal movement, the increase in the energy stored in the field s $Adx\,\frac{\mu H^2}{8\pi}$, which is, of course, equal to the work done in overcoming the force of attraction F.

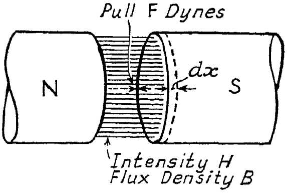


FIG. 15. ATTRACTION MAGNETIZED SURFACES

$$\therefore \mathrm{F}dx = Adx\,\frac{\mu H^2}{8\pi}$$

$$\mathrm{F} = \mu\,\frac{AH^2}{8\pi}$$

$$= \frac{AB^2}{8\pi\mu}$$

or, if the field is in air,

$$\mathrm{F} = AB^2$$

This force is in dynes if A is in sq. cm. and B in lines per sq. cm.

Magnetic Shells. A thin iron sheet, magnetized as in Fig. 16, may be thought of as consisting of an infinite number of small bar magnets, with all the north poles on one side of the sheet and all the south poles on the other. This constitutes what is known as a "magnetic" shell, and the idea is useful in considering various problems in electromagnetism. The "strength of the shell" is defined as "magnetic moment per unit area." Thus, if m is the pole strength per unit area and the thickness of the shell is t, the "strength of the shell" is mt.

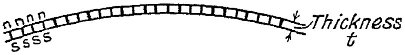


FIG. 16. MAGNETIC SHELL

ELECTROMAGNETISM

The study of electromagnetism originated with Oersted's discovery that a pivoted magnetic needle in the neighbourhood of a conducting wire is deflected when a current of electricity flows in the wire. This means that a magnetic field exists around a wire which carries current, and as a development of Oersted's discovery we have the definition of the "absolute" or "*C.G.S. electromagnetic*" *unit of current.*

This unit of current is defined as such that, if flowing in a circular wire of 1 cm. radius, a force will be exerted upon unit magnetic pole placed at the centre of the circle of 2π dynes (i.e. magnetic intensity at the centre $= 2\pi$).

In general, if i units of current flow in the circular wire, the intensity at the centre

$$H = 2\pi i \qquad . \quad . \quad . \quad . \quad . \quad (36)$$

(Since the reasoning in the remainder of this chapter is based upon this definition, the units of current are e.m. c.g.s. units unless otherwise stated.)

Ampère's Theorem. Ampère showed that the magnetic effect of a current i units, flowing in a small closed circuit, is the same as that of a small bar magnet placed with its axis perpendicular to the plane of the circuit, provided that the magnetic moment of such a bar magnet is equal to idA where dA = area of small circuit.

By considering a number of such small circuits placed together as shown in Fig. 17(a), with currents of i units flowing in each, it can be shown that, *for any closed circuit* carrying a current of i units, the magnetic effect is the same as that of a magnetic shell occupying the space enclosed by the circuit, provided the "strength" of such a shell is equal to the current i. This constitutes what is known as Ampère's Theorem.

Weber showed experimentally that the potential at any point P distant r from a small closed circuit of area dA carrying i units of current

$$= \frac{dA\ .\ i \cos \phi}{} \qquad (37)$$

where ϕ is the angle between the normal to the plane (whose dimensions are small compared with the distance r) and the distance r.

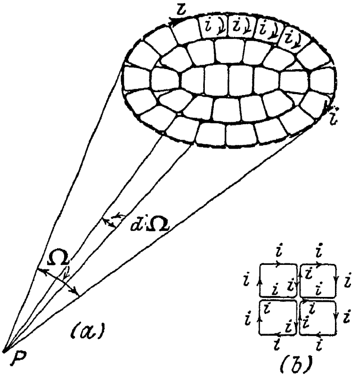


FIG. 17. POTENTIAL DUE TO CURRENT IN A CLOSED CIRCUIT

From Equation (34), the potential at a distance r from a short bar magnet is

$$V_p = \frac{M \cos \phi}{r^2}$$

Thus, if the small magnet which is equivalent to the closed circuit has magnetic moment M we can write

$$\frac{dA\ .\ i\ .\ \cos \phi}{r^2} = \frac{M \cos \phi}{r^2}$$

$$\therefore dA\ .\ i = M$$

Again, from Equation (37),

$$V_p = i\ .\ d\Omega$$

where $d\Omega$ is the solid angle subtended at P by the small closed circuit.

The potential at P due to a large number of small circuits (Fig. 17)

$$V_n = \Sigma i d\Omega = i\Omega \qquad . \quad (38)$$

where Ω is the total solid angle subtended at P by all the small circuits.

It can be seen also, that the currents i in the small circuits neutralize one another at all parts except the outside (as in Fig. 17 (*b*)), so that the whole is equivalent to one circuit lying along the perimeter of the group of small circuits and carrying a current i. Thus it is shown that for any circuit carrying i units of current, the potential at a point P is $i\Omega$ where Ω is the solid angle subtended at the point by the circuit. Again, if each of the small circuits of Fig. 17 (*a*), each of area dA, be replaced by a small magnet, of moment M, then

$$i = \frac{M}{dA}$$

Now, obviously, $\frac{M}{dA}$ is the strength of the magnetic shell formed by such a replacement of the small circuits by bar magnets of moment M, and thus the circuit is equivalent to a magnetic shell having a strength equal to i, this strength being expressed with the centimetre as the unit of length, and the unit magnetic pole as the unit of pole strength, the current i being in e.m.c.g.s. units.

Potential Energy of a Current and a Magnetic Flux. From the preceding paragraph it follows that the potential energy of a magnetic pole of strength m units, at a point P in the neighbourhood of a closed circuit in which a current i flows, is $mi\Omega$, where Ω is the solid angle subtended at P by the circuit. This expression represents energy, since, from the definition of potential difference as the work done in moving unit pole from one point to another, the work done in moving m units is $mV_p = mi\Omega$.

Now, in the unrationalized system, a magnetic pole of strength m radiates $4\pi m$ lines of force distributed uniformly in all directions. Thus, the magnetic flux threading the current-carrying circuit is $\frac{\Omega}{4\pi} \times 4\pi m = \Omega m$.

Hence, the potential energy of the current and magnetic flux

$$= i\Phi \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where Φ is the flux threading through the current-carrying circuit.

[Note that if the flux Φ changes with time t, then $i\frac{d\Phi}{dt}$ represents rate of change of energy, i.e. power.]

Forces Due to and Acting Upon Current in a Long Straight Conductor. Biot and Savart were the first to examine, by means of a compass needle, the intensity of the magnetic field, at different distances from a straight conductor, which is carrying current. They showed that the intensity of the field was "inversely proportional to the distance from the conductor in which the current flows." This is known as Biot-Savart's Law.

Fig. 18 (*a*) shows a small element of a conductor of length dl carrying a current i and situated at a distance r from a magnetic pole of strength m units in a direction making an angle of θ with the element of conductor.

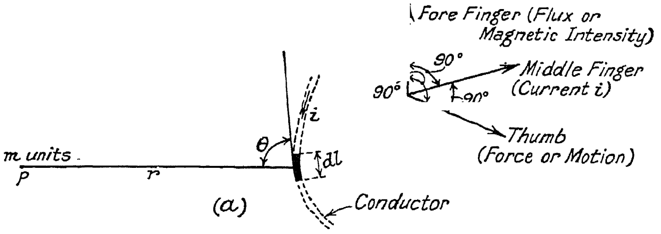

Fore Finger (Flux or Magnetic Intensity)
90°
Middle Finger (Current i)
90°
90°
Thumb (Force or Motion)
m units
P
r
θ
i
dl
(a)
Conductor

FIG. 18. MAGNETIC INTENSITY DUE TO CURRENT IN A CONDUCTOR

Laplace established the equation, related to the above case, that the force upon the element of conductor is

$$f = \frac{m \,.\, i \,.\, dl \sin \theta}{r^2} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Now, the intensity of field at the conductor due to the pole is, by Equation (19), $\frac{m}{r^2}$. Hence, the force

$$f = Hidl \sin \theta$$

If the conductor is straight and is situated in a uniform field of intensity H whose direction is perpendicular to the conductor, then

$$\mathrm{F} = Hil \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where F is the total force on the conductor. F is in dynes if l is in centimetres and i in e.m.c.g.s. units.

Incidentally, this gives another means of defining the e.m.c.g.s. unit of current (essentially the same as the previous method of definition), i being a current such that $\mathrm{F} = 1$ dyne when $H = 1$ and $l = 1$ cm., the conductor and field being perpendicular.

If in Fig. 18(*a*) the pole m has north polarity, the force f is, by the *left-hand rule* (Fig. 18(*b*)), perpendicular to the plane of the paper *outwards*.

The *Left-hand Rule* states that if the thumb, forefinger, and middle finger of the left hand are placed mutually at right angles, then the corresponding directions of magnetic intensity, current and force (or motion) are given as shown in Fig. 18(*b*).

The force upon the pole, due to the current in the conductor, is equal in magnitude, but opposite in direction, to that of the pole upon the conductor; force upon the pole is $\frac{midl}{r^2}\sin\theta$ in a direction perpendicular to the plane of the paper *inwards.*

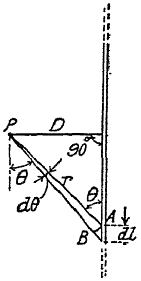


Fig. 19
Magnetic Intensity near a Straight Conductor

The force upon *unit* pole at P (i.e. the intensity at P) due to the current is

$$dH = \frac{idl\sin\theta}{r^2} \quad . \quad . \quad . \quad (42)$$

if the conductor is straight and carries i units of current, the intensity at a point P, which is perpendicularly distant D from the conductor, can be found as follows: Consider an element of conductor dl as in Fig. 19. The intensity due to it is, as above, $dH = \frac{idl\sin\theta}{r^2}$.

The length $AB = dl\sin\theta = rd\theta$, and $\sin\theta = \frac{D}{r}$

Thus, $$dH = i\frac{\sin\theta}{D}d\theta$$

and the total intensity at P is

$$H = 2\int_{\theta=0}^{\theta=+\frac{\pi}{2}} i\frac{\sin\theta}{D}d\theta$$

if D is small compared with the length of the conductor.

$$\therefore H = \frac{2i}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

It follows from this equation that the work done in carrying unit north pole through a circular path surrounding the conductor is

$$H \times 2\pi D = \frac{2i}{D} \times 2\pi D$$

$$\therefore \text{Work done} = 4\pi i \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Force Between Two Parallel Current-carrying Conductors. The forces of attraction or repulsion between two parallel conductors A, B carrying currents i_1 and i_2 respectively and distant D apart, as in Fig. 20, can be calculated from the equations obtained in the previous paragraph. If the currents are in the same direction (as shown), the force between the conductors is one of attraction, and if in opposite directions the force is one of repulsion.

Consider the attractive force of unit length of conductor *A* upon unit length of conductor *B* adjacent and parallel to it as shown. Conductor *B* is situated in a magnetic field, due to *A*, of intensity of $\frac{2i_1}{D}$ (from Equation (43)). Hence, from Equation (41), the force per unit length of conductor is

$$F = \frac{2i_1}{D} i_2$$

The force of attraction on conductor *A*, due to conductor *B*, is, by similar reasoning, the same.

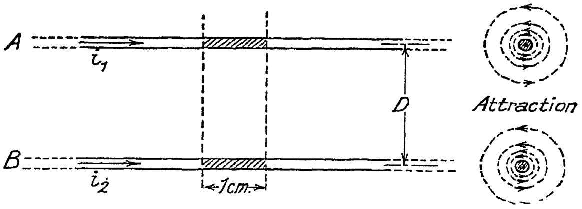


Fig. 20. Force Between Two Parallel Current-carrying Conductors

Thus, the force per unit length between the conductor is

$$F = \frac{2i_1i_2}{D}$$

or, for a length l

$$F = \frac{2i_1i_2}{D} l \quad . \quad . \quad . \quad . \quad (45)$$

The force is in dynes if i_1 and i_2 are in e.m.c.g.s. units and D and l are in centimetres. It is assumed that the distance D is small compared with the lengths of the conductors.

The dots, placed on the conductor sections to the right of Fig. 20, indicate that the current in them flows in an outward direction. A cross so placed indicates inward direction.

Magnetic Field Due to Current in a Circular Conductor. Fig. 21(*a*) shows a circular conductor carrying i units of current. Consider the intensity (dH) at a point P, on the axis of the circle, due to the current in an element dl of the conductor. Then

$$dH = \frac{idl}{R^2}$$

in a direction perpendicular to the line joining the element to P.

This intensity may be split up into two components, one in direction OP produced, namely $dH \sin \theta$, and one perpendicular to OP, namely $dH \cos \theta$. Considering all such elements of the circular conductor it is seen that the components perpendicular to OP neutralize one another, leaving, as the intensity at P, only the sum of all components in a direction OP produced. Thus the total intensity at P due to the current is in direction OP, and is

$$H = \Sigma dH \sin\theta = \Sigma \frac{idl}{R^2}\sin\theta$$

$$= \frac{i \times 2\pi r}{R^2}\sin\theta$$

$$H = \frac{i \times 2\pi r^2}{R^3} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

At the centre O of the circular conductor $R = r$,

$$\therefore H = \frac{2\pi i}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Magnetic Field Produced by Two Parallel Coils Carrying the Same Current. Consider two similar circular coils A and B, each of N turns, and each carrying a current of i units in such directions that their magnetic effects are in the same direction, placed coaxially with their mean planes parallel as in Fig. 21(b). The magnetic intensity at a point P on their common axis and midway between them is, from the previous paragraph (Equation (46)),

$$H = \frac{2Ni \times 2\pi r^2}{R^3}$$

r being the radius of the coils and R the distance from P to the mean circumference of either coil, it being assumed that the cross-sections of the coils are small compared with the other dimensions involved.

If l cm. is the distance between them

$$R^2 = r^2 + \left(\frac{l}{2}\right)^2 = r^2 + \frac{l^2}{4}$$

Thus H at $P = \dfrac{2Ni \, . \, 2\pi r^2}{\left(r^2 + \dfrac{l^2}{4}\right)^{\frac{3}{2}}}$

For a point Q on the axis distant x from coil A,

Intensity at Q due to coil $A = \dfrac{Ni \, . \, 2\pi r^2}{(r^2 + x^2)^{\frac{3}{2}}}$

" " " $B = \dfrac{Ni \, . \, 2\pi r^2}{[r^2 + (l - x)^2]^{\frac{3}{2}}}$

Thus, resultant intensity at Q

$$= \frac{Ni \, . \, 2\pi r^2}{(r^2 + x^2)^{\frac{3}{2}}} + \frac{Ni \, . \, 2\pi r^2}{[r^2 + (l - x)^2]^{\frac{3}{2}}}$$

$$= 2\pi r^2 Ni \left[\frac{1}{(r^2 + x^2)^{\frac{3}{2}}} + \frac{1}{(r^2 + (l - x)^2)^{\frac{3}{2}}}\right] \quad . \quad (48)$$

If the ordinates representing the intensity of field at various points along the axis are drawn, the result gives curves as shown in Fig. 21(*b*). The curve marked (*a*) gives the intensity due to coil *A*, and that marked (*b*) the intensity due to coil *B*. The curve marked (*t*) gives the resultant intensity, which is seen to fall slightly towards the midway position *P*.

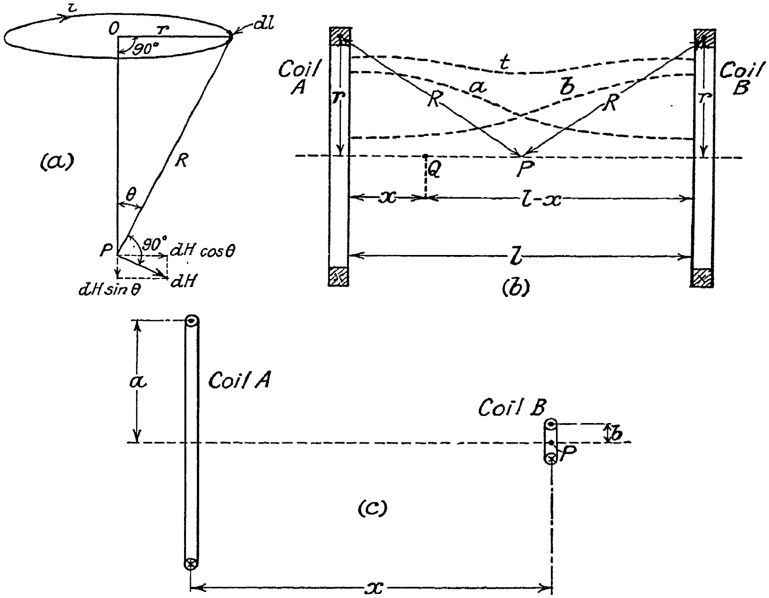


FIG. 21. MAGNETIC FIELD INTENSITY NEAR CIRCULAR, CURRENT-CARRYING COILS

If the distance l between the coils is made equal to the radius r of the coils, Equation (48) becomes

$$H_Q = 2\pi r^2 Ni\left[\frac{1}{(r^2+x^2)^{\frac{3}{2}}} + \frac{1}{[r^2+(r-x)^2]^{\frac{3}{2}}}\right]$$

Now $\quad [r^2+(r-x)^2]^{\frac{3}{2}} = [r^2+x^2+r^2-2rx]^{\frac{3}{2}}$

Expanding we have

$$[r^2+x^2+r^2-2rx]^{\frac{3}{2}} = (r^2+x^2)^{\frac{3}{2}} + \frac{3}{2}\cdot(r^2+x^2)^{\frac{1}{2}}(r^2-2rx)$$

$$+\frac{3}{2}\cdot\frac{1}{2}\,\frac{(r^2+x^2)^{-\frac{1}{2}}\,(r^2-2rx)^2}{2\,.\,1} + \quad \cdots$$

If $x = \frac{r}{2}$ all terms but the first disappear. If x has any value which is of the same order as $\frac{r}{2}$ the succeeding terms are small compared with the first. Thus, the resultant intensity

$$H_Q \doteqdot 2\pi r^2 Ni \left[\frac{2}{(r^2 + x^2)^{\frac{3}{2}}}\right]$$

This arrangement of the two coils produces a field of almost uniform intensity between the two coils for a considerable distance on either side of the mid-point P and was used by von Helmholtz in a special form of tangent galvanometer for use in the absolute measurement of current.*

Force Between Two Parallel Coaxial Circles in which Currents are Flowing. Consider the two circles A and B shown in Fig. 21(*c*). They are coaxial, and their planes are parallel. Let circle A have radius a, and carry a current i_A, while circle B has radius b, and carries a current i_B. Assume also that the radius b is very small.

From Equation (46) the intensity at the centre P of circle B due to the current in circle A is $\frac{2\pi i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}}$ where x is the distance between the circles. The flux density (considered uniform, since the radius b is very small) inside circle B is therefore $\frac{2\pi i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}}$ and thus the total flux threading through circle B is $\frac{\pi b^2 \,.\, 2\pi i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}}$.

From Equation (39) the potential energy of the coils is

$$\frac{\pi b^2 \,.\, 2\pi i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}} \times i_B = \frac{2\pi^2 i_A i_B \,.\, a^2 b^2}{(a^2 + x^2)^{\frac{3}{2}}} = \text{P.E.}$$

The force between the circles (of attraction or repulsion, depending upon the directions of the currents i_A and i_B) is from the law $\text{F} = \frac{dV}{dx}$ equal to $\frac{d(\text{P.E.})}{dx}$

i.e.
$$\text{F} = -\frac{6\pi^2 i_A i_B \,.\, a^2 b^2 x}{(a^2 + x^2)^{\frac{5}{2}}} \quad . \quad . \quad . \quad . \quad (49)$$

The force (in dynes if the e.m.c.g.s. system of units is used) is one of attraction if the currents are in the same direction, and of repulsion if the currents are in opposite directions. It can be shown by differentiation that F is maximum when $x = \frac{a}{2}$.

---

* A diagram of the field between two coils so placed is given in Maxwell's *Electricity and Magnetism*, Vol. II, and the complete theory is given in Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II, Part I.

The complete theory in connection with the force between two parallel current-carrying coaxial coils, when the radius of neither may be considered very small, and when they each have a number of turns giving a finite cross-section, is necessary for the purpose of measuring current in absolute units by means of a current balance, and has been given by Gray (Ref. (10)) and by J. V. Jones (Ref. (11)).

Magnetic Field of a Solenoid. Consider the solenoid shown in Fig. 22 to be made up of a large number of circular conductors such as that considered above.

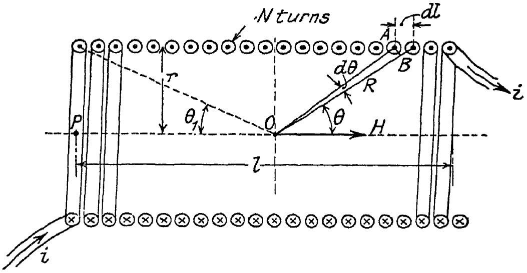


FIG. 22. MAGNETIC FIELD OF A SOLENOID

Let N = total number of turns on solenoid
i = current in the solenoid in e.m.c.g.s. units
l = length of solenoid in centimetres
r = radius of solenoid in centimetres
c = number of turns per centimetre length $= \frac{N}{l}$

Then, intensity at the centre O of the solenoid due to a length dl of the solenoid is

$$dH = c.i.dl \times \frac{2\pi r^2}{R^3} \text{ (from Equation (46))}$$

in the direction of the axis of the solenoid.

The length $AB = Rd\theta = dl \sin\theta$

$$\therefore \quad dl = \frac{Rd\theta}{\sin\theta}$$

$$dH = ci \times \frac{2\pi r^2}{R^3} \cdot \frac{Rd\theta}{\sin\theta}$$

Also $\sin\theta = \frac{r}{R}$

Substituting for $\frac{r}{R}$ in the equation for dH

$$\therefore dH = 2\pi ci \sin\theta \, d\theta$$

The expression for the total intensity H at O due to the whole solenoid is, therefore,

$$H = \int_{\theta_1}^{\pi - \theta_1} 2\pi ci \sin\theta \, d\theta$$

where θ_1 is the limiting value of θ and is $\tan^{-1} \frac{2r}{l}$

$$\therefore H = 4\pi ci \cos\theta_1$$

$$H = \frac{4\pi Ni}{l} \cos\theta_1 \quad . \quad . \quad . \quad . \quad . \quad (50)$$

or, if the solenoid is very long compared with its radius r,

$$H = \frac{4\pi Ni}{l} \text{ since } \cos\theta_1 \doteqdot 1$$

The intensity at one end of the solenoid (at P) is obtained by integrating between the limits π and 0 (if the assumption is made that l is very great compared with r).

Thus, intensity at one end

$$H_P = \int_0^{\frac{\pi}{2}} 2\pi ci \sin\theta d\theta$$

$$= 2\pi ci$$

$$H_P = 2\pi \frac{N}{l} i \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

i.e. the intensity at one end is half that at the centre.

Induction of Electromotive Force. Faraday, in 1831, showed that *whenever the number of lines of magnetic flux linking with an electric circuit is changed, an electromotive force is induced in the circuit, the magnitude of which is proportional to the rate of change of flux.* Thus, if e is the e.m.f. induced by a rate of change of flux of $\frac{d\Phi}{dt}$ lines per second,

$$e \propto \frac{d\Phi}{dt}$$

The e.m.f. is also proportional to the number of turns of wire, N, in the circuit in which the e.m.f. is induced.

Thus $$e \propto N \frac{d\Phi}{dt}$$

If e is expressed in "*electromagnetic C.G.S. units of e.m.f.*" then

$$e = N \frac{d\Phi}{dt}$$

this *unit of e.m.f.* being defined as that induced in a circuit of one turn when the interlinking flux is changed at the rate of one line per second.

A law giving the direction of the induced e.m.f. in general terms was stated, very shortly after Faraday's discovery, by Lenz.

Lenz's Law states that "*the direction of the induced e.m.f. is such as to tend to oppose the change in the inducing flux.*" The mathematical expression of this law, in conjunction with Faraday's Law, introduces a negative sign, and gives

$$e = -N\frac{d\Phi}{dt} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Statically-induced e.m.f.s. An e.m.f. induced in a stationary electric circuit, by a change in the number of magnetic lines linking with it, is referred to as a "statically induced" e.m.f. as distinct from a "dynamically induced" e.m.f., which occurs when an electric conductor cuts through lines of force which are stationary.

The simplest method of producing a statically-induced e.m.f. is by inserting one pole of a bar magnet in the space enclosed by a coil of wire. If the coil forms a closed electric circuit then, upon inserting the magnetic pole, the statically induced e.m.f. produces a current in such a direction that its magnetic effect *opposes* the magnetic field due to the pole. Upon withdrawing the pole, the e.m.f. is in the opposite direction. The current produced by it then has a magnetic effect in the *same direction* as that of the magnetic pole. In each case, therefore, the direction of induced e.m.f. is such as to oppose the change in the number of interlinking lines of force.

Statically induced e.m.f.s are more often produced in a circuit by alternating current which is flowing in an adjacent circuit, the latter being so placed that some of the lines of force produced by the current in it thread through the other circuit. The magnitude of the induced e.m.f. at any instant depends upon the rate of change of current in the inducing circuit, and upon the relative positions of the two circuits.

Dynamically-induced e.m.f.s. Whenever an electric conductor *cuts through* lines of magnetic force an e.m.f. is "dynamically induced" in the conductor. This e.m.f. is proportional to "the rate of cutting flux." There is no difference, as regards the phenomena involved, between statically and dynamically induced e.m.f.s. It can be seen that if the conductor in Fig. 23(*a*) forms part of a closed circuit (shown dotted), and moves left to right, the direction of the induced e.m.f. is from X to Y, since the flux linking with the circuit is increased. If the conductor moves right to left the direction of e.m.f. is Y to X, since the interlinking flux is then reduced.

The direction of the e.m.f. is given quite simply by the right-hand rule. By this rule, the corresponding directions of motion, flux, and induced e.m.f., are given by the thumb, forefinger, and middle finger respectively of the right hand, these being so placed as to be mutually perpendicular (see Fig. 23(*b*)).

The magnitude of the induced e.m.f. in a conductor which moves in a plane *perpendicular to the lines of force*, as in Fig. 23(*a*), is

$$\begin{aligned} e &= \text{flux cut per second} \\ &= B \times \text{area swept out by the conductor per second} \\ \therefore \quad e &= Blv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53) \end{aligned}$$

If l is the length of the conductor XY in centimetres, and v its

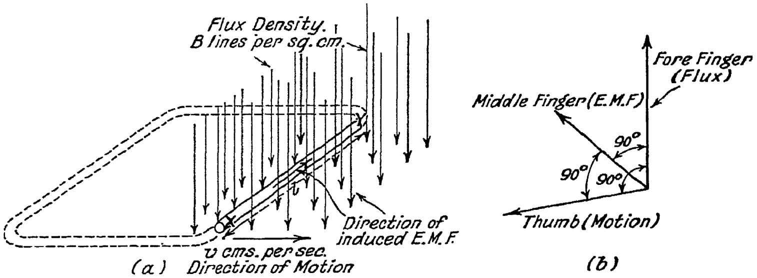


FIG. 23. DYNAMICALLY INDUCED E.M.F.

velocity perpendicular to the lines of force in centimetres per second e is expressed in electromagnetic c.g.s. units of e.m.f.

If a conductor cuts through magnetic flux whilst moving in a direction making an angle θ with that of the lines of force, then the component of its velocity in a direction perpendicular to the lines of force is $v \sin \theta$, and the induced e.m.f. is then

$$e = Blv \sin \theta \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

It is assumed in the above that the magnetic field exists in air, so that $B = H$. Equation (53) can thus be written also as

$$e = Hlv.$$

Energy in an Electric Circuit. If the conductor in which the e.m.f. is induced forms part of a closed circuit, and if a current of i units flows in this circuit, then there will be a force opposing the motion of the conductor through the magnetic field, which force is given by Equation (41) as Hil. Thus, the work done in moving the conductor a distance x through the field

$$= Hilx$$

If the conductor takes a time t sec. to pass through distance x when moving with velocity v, then

$$\begin{aligned}\text{work done} &= Hil\,.\,vt\\ &= eit\\ &= \text{energy given to the electric circuit}\end{aligned}$$

$$\therefore \text{Energy given to the electric circuit} = eit \quad . \quad . \quad . \quad (55)$$

Magnetic Hysteresis. This phenomenon is observed when the current flowing in a solenoid, for the purpose of magnetizing a bar or ring of iron, or other magnetic material upon which the solenoid is wound, is reduced. It is found that the flux density in the iron corresponding to this value of the current is higher than the flux density, corresponding to the *same value of the current,* which was produced when the current was being increased from zero (say) to some maximum value—i.e. the magnetism lags behind the magnetizing force producing it. This effect is known as the "hysteresis effect."

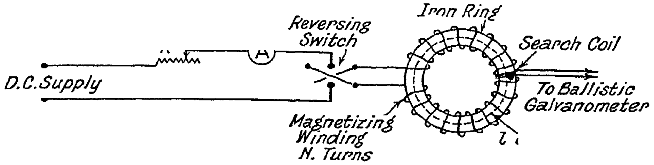


FIG. 24. MAGNETIZATION OF AN IRON RING

Consider an iron ring upon which is wound, uniformly, a magnetizing winding through which a current can be passed in either direction, as in Fig. 24. Suppose the ring has, initially, no magnetism. If the current in the magnetizing winding is increased from zero to some reasonably high value, the magnetizing force acting upon the ring is also increased, since $H = \frac{4\pi NI}{l}$, N being the number of turns on the magnetizing winding and l cm. being the length of the magnetic path in the ring.

If the flux density in the ring is measured (by means of a search coil and ballistic galvanometer as described in Chapter IX) for various values of H, and the B-H curve plotted, it will be found that its shape is as shown by the portion OA in Fig. 25(a). If the current is reduced gradually to zero again, after some flux density, B_{max} has been attained, the curve for descending values of H takes the form AC, this curve, owing to the hysteresis effect, being above the ascending curve OA. The flux density B_r remaining in the iron when the H is again zero is referred to as the "residual magnetism." This value depends upon the magnitude of B_{max} and upon the material of the specimen, being high for permanent magnet steels, and very low for such material as silicon sheet steel. Since H is now zero, $B_r = 4\pi J_r$ where J_r is the intensity of magnetization left in the iron when the magnetizing force is removed.

The residual flux density, after magnetization up to saturation point, is referred to as the "remanence" of the iron or steel concerned.

If the direction of the magnetizing current is now reversed, and gradually increased in this reverse direction until a flux density B_{max} is again obtained (but in the opposite direction), the curve follows the line CDE. The negative value of H required to reduce the residual flux density to zero, namely OD, is called the "coercive force" H_c, and will of course vary with the material and with B_{max}. The value of the coercive force after previous magnetization up to saturation point is the "coercivity" of the iron. If the "cycle of magnetization" is completed by first reducing H to zero and then increasing it in the opposite direction to produce $+B_{max}$ again, the curve follows the line $EFGA$. The loop so formed is called the "hysteresis loop."

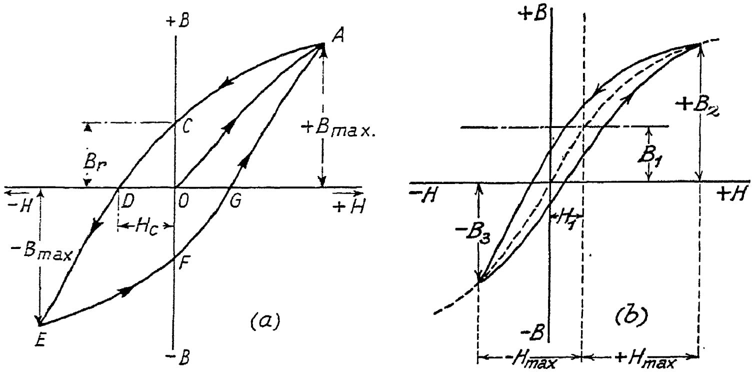


FIG. 25. HYSTERESIS LOOPS

In the above, the maximum negative flux density was taken as being equal to the maximum positive flux density. A similar effect is observed, of course, if this is not so, but in this case the loop is unsymmetrical. Symmetrical hysteresis loops, or cycles of magnetization, are most usual in practice, but when alternating magnetism is superimposed upon unidirectional magnetism, unsymmetrical hysteresis loops will occur. An example of this is found in the core of a radio transformer, when one of the windings often carries a direct current in addition to an alternating current. Such an unsymmetrical loop is shown in Fig. 25(b), where a symmetrical alternating magnetic field ($+ H_{max}$ to $- H_{max}$) is superimposed upon a steady field H_1, with its corresponding flux density B_1, giving $+ B_2$ and $- B_3$ for the limits of flux density.

It should be noted that, if the steady values of H_1 and B_1 are sufficiently high for the application of $+ H_{max}$ to produce saturation in the iron or magnetic material then the hysteresis loop is considerably distorted as shown. In such a case $B_2 - B_1$ is not equal to $B_1 + B_3$, although these quantities would be approximately equal if the saturation point of the material is not approached, as when $+ H_{max}$ and $- H_{max}$ are small. In this latter case the hysteresis loop would be very nearly the same in shape as the normal symmetrical loop but would, of course, be displaced relative to the axes of co-ordinates by H_1 and B_1. The space available will not permit further consideration of this question, but references to works on the subject are given in the bibliography (Ref. (4), (7), (8)).

Incremental Permeability. The term "incremental permeability," introduced by Dr. Thomas Spooner,* refers to a type of permeability which has become increasingly important with the advance of radio communication, since it relates to the case of superimposed direct and alternating magnetizations. Spooner defines it as "the ratio of ΔB to ΔH for any position on a magnetization curve, or hysteresis loop, where ΔB and ΔH may be of any magnitude, but ΔH must be in the reverse direction from the immediately preceding change."

Referring to Fig. 26, the incremental permeability at various points on the major hysteresis loop is given by $\Delta B_1/\Delta H_1$, $\Delta B_2/\Delta H_2$ etc., these being the slopes of the minor hysteresis loops corresponding to increments of H as shown.

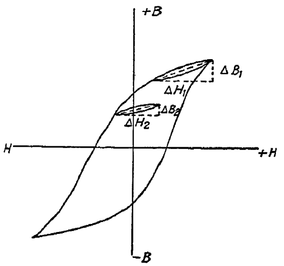


FIG. 26

L. G. A. Sims, discussed and summarized this question in a valuable paper before the British Association in September, 1937 (Ref. (13)).

Hysteresis Loss. Although no energy is required merely to maintain a magnetic field, it is found that energy is required to bring about a cycle of magnetization in a magnetic material. Energy is required to build up a magnetic field owing to the opposing e.m.f., which is induced in the magnetizing circuit when the flux in the magnetic material is increased. This energy is stored in the magnetic field, but it is found that the quantity of energy returned to the magnetizing circuit, when the current is reduced, is less than the quantity supplied when the field was built up. The difference is due to "molecular magnetic friction"—as it has been called by Steinmetz—and the energy absorbed is converted into heat. The energy absorbed when a magnetic material is passed through one cycle of magnetization can be shown to be proportional to the area of the hysteresis loop as below.

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* *Trans. A.I.E.E.*, Vol. XLII, p. 42.

Relation Between Hysteresis Loss and the Area of the Hysteresis Loop. Consider the magnetization of a ring specimen of iron by means of a magnetizing winding as shown in Fig. 24.

If the length of magnetic path in the ring is l cm., and its cross-section is a sq. cm., if the number of magnetizing turns is N, and the current in the magnetizing circuit at any instant is i, then the induced e.m.f. (in c.g.s. units) at any instant

$$e = \frac{Nd\,(Ba)}{dt}$$

where $B =$ flux density in the ring in lines per square centimetre.

Thus the power supplied at any instant to overcome this back e.m.f. (i.e. to build up the magnetic field in the ring) is

$$ei = i \,.\, N \frac{d\,(Ba)}{dt}$$

and the energy supplied in order to build up the magnetic field in time t sec. is

$$\int_0^t eidt = \int_0^t aiN \,.\, \frac{dB}{dt} \,.\, dt = a \int_{-B_r}^{B_{max}} iN \,.\, dB$$

since, when $t = 0$, $B = B_r$ as shown in Fig. 27, in which $OF = OE = B_r$.

Now, the magnetizing force acting upon the ring is, at any instant, given by

$$H = 4\pi \frac{Ni}{l}$$

from which $$Ni = \frac{l}{4\pi} H$$

and energy supplied $$= \frac{la}{4\pi} \int_{-B_r}^{B_{max}} HdB$$

since $la =$ volume of ring in cubic centimetres. It follows that the energy in ergs supplied per cubic centimetre to build up the field

$$= \frac{1}{4\pi} \int_{-B_r}^{B_{max}} HdB$$

This energy is stored in the magnetic field, and is represented in Fig. 27 by the area $FACDF$.

Upon reducing the current (and hence the flux) the induced e.m.f. is in the *same direction* as the applied e.m.f., so that energy is now returned to the magnetizing circuit as the flux is reduced. From the above reasoning the energy returned during the reduction of the magnetizing force from H_{max} to O is $\frac{1}{4\pi}\int_{B_r}^{B} HdB$ ergs per c.c. where B_r is the residual flux density.

This energy is represented in the figure by the area ECD. Thus the energy *absorbed by the specimen due to hysteresis* is the difference between energy put in, and energy returned to the magnetizing circuit, and is represented by the shaded area $FACEF$.

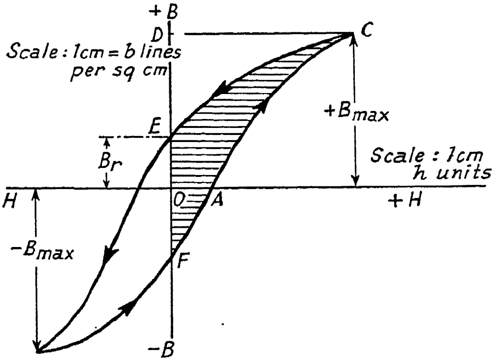


Fig. 27. Hysteresis Loss

If the current is now reversed and the above process repeated, H being finally brought back to the starting point A of the cycle of magnetization, the energy in ergs absorbed per cubic centimetre per cycle due to hysteresis will be $\frac{1}{4\pi}\int HdB$

$$= \frac{1}{4\pi} \times \text{(area of hysteresis loop)}$$

The area of the loop is, of course, measured to scale, so that the energy absorbed per cubic centimetre per cycle in ergs

$$= \frac{1}{4\pi} \times \text{area of loop in square centimetres} \times bh \quad . \quad (56)$$

where $b =$ flux density in lines per sq. cm. represented by 1 cm. on B axis

$h =$ number of c.g.s. units of H represented by 1 cm. on H axis

This expression for the energy loss in terms of the area of the loop obviously applies whether the loop is symmetrical or not.

Steinme Steinmetz has shown that the empirical law

$$W_h = k \cdot B_{max}^{1\cdot6} \quad . \quad . \quad . \quad . \quad (57)$$

gives the hysteresis loss for iron with sufficient accuracy for most practical purposes, provided the maximum flux density B_{max} lies between 1,000 and 12,000 lines per sq. cm.

W_h = energy loss in ergs per cubic centimetre per cycle

k = the hysteresis coefficient of the material, and is constant for any given material

The magnitude of k varies with the material. Its value for annealed sheet steel lies between 0·001 and 0·002 and for silicon steel is about 0·00084.

For values of B_{max} above 12,000, the energy loss increases at a higher rate than the 1·6th power of B_{max}, this rate increasing with increasing values of B_{max}. For values of B_{max} below 1,000 also the loss varies as some power of B_{max} greater than 1·6.

Steinmetz (Ref. (3)) has shown that for silicon steel the law

$$W_h = k' \cdot B_{max}^2 \quad . \quad . \quad . \quad . \quad . \quad (58)$$

is more nearly correct, k' being about 0·0000457.

With regard to the hysteresis loss in the case of an unsymmetrical cycle, Ball has shown (Ref. (7)) that this obeys the law

$$W_h = k'' B_{max}^{1\cdot6}$$

but that in this case k'' is not constant for any given material but varies with the average value of the flux density.

Thus if B_2 and B_3 are the limiting values of flux density as in Fig. 25 (*b*),

$$B_{max} = \frac{B_2 - B_3}{2}$$

Also, Ball found that if the average value of the flux density

$$B_{av} = \frac{B_2 + B_3}{2}$$

$$k'' = k + \alpha B_{av}^{1\cdot9}$$

where k is the normal hysteresis coefficient of the material.

Thus, in general, energy loss in ergs per cubic centimetre per is

$$W_h = (k + aB_{av}^{1\cdot9})B_{max}^{1\cdot6}$$

$$W_h = \left[k + a\left(\frac{B_2 + B_3}{2}\right)^{1\cdot9}\right] \left(\frac{B_2 - B_3}{2}\right)^{1\cdot6} \quad . \quad . \quad (59)$$

where B_3 is the lower value of the flux density and may, or may not, be negative.

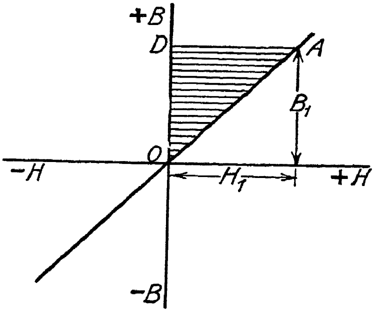


FIG. 28. ENERGY STORED IN A MAGNETIC FIELD

Values of a are: for ordinary annealed sheet steel $a = 0{\cdot}34 \times 10^{-10}$, for annealed silicon sheet steel $a = 0{\cdot}32 \times 10^{-10}$. Obviously if $B_2 = -B_3$, as in a symmetrical cycle, equation (59) reduces to $W_h = kB_2^{1\cdot6}$

Energy Stored per Unit Volume of Magnetic Field in Air. In the case of a magnetic field in air or other non-magnetic material, the hysteresis loop reduces to a straight line, i.e. the hysteresis loss is zero. If the same scales are used for both B and H, this line makes an angle of 45° with either axis, as in Fig. 28.

The energy stored in the field per cubic centimetre when the flux density is B_1 is $\frac{1}{4\pi} \int HdB$

$$= \frac{1}{4\pi} \times \text{area } OAD \text{ (to scale)}$$

$$= \frac{1}{4\pi} \times \frac{1}{2} \cdot B_1H_1$$

But $B_1 = H_1$, since the field is in air.

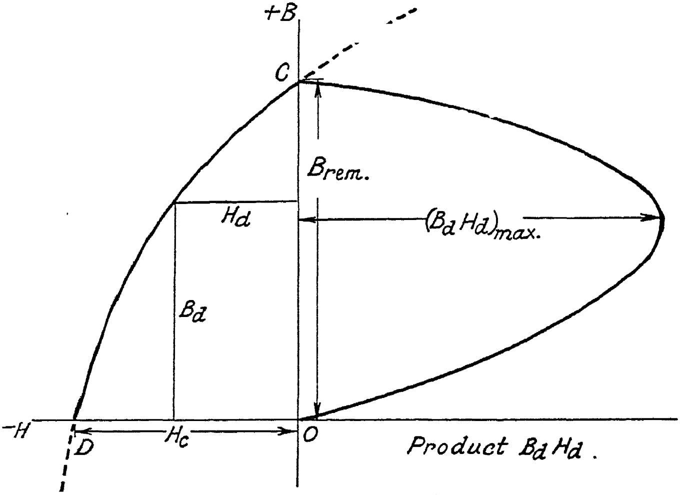
$$\therefore \text{ Energy stored in ergs per c.c.} = \frac{B_1^2}{8\pi} = \frac{H_1^2}{8\pi} \quad . \quad . \quad (60)$$

Demagnetization Curve (Permanent-magnet Design). In the design of permanent magnets, in which the flux density in the air gap is required to be as large as possible, while the magnetism must be resistant to demagnetizing forces, the remanence, coercivity, and the demagnetization curve (portion CD of Fig. 25 (a)) of the steel to be used are of great importance. For good magnet steel the product $B_r \times H_c$ should be large.

Fig. 29 shows the demagnetization curve of magnet steel, this being a portion of the hysteresis loop for the steel in which the maximum magnetization has been up to saturation point, so that OC represents the remanence and OD the coercivity.

As the demagnetizing force H_d is increased, the flux density falls as shown, and at any point, when the flux density in the steel is B_d there is, remaining in the steel, an m.m.f. per centimetre length of path in it of H_d. This m.m.f. is that which would drive the flux across an air gap in the magnetic circuit of the magnet. The products $B_d \cdot H_d$ are plotted. They exhibit a maximum value $(B_d \cdot H_d)_{max}$

which is a criterion of the value of the steel for permanent-magnet purposes. The most economical design of magnet is that for which B_d and H_d are such that their product has this maximum value. (See Refs. (8), (12), (15), (16), and (17).) The properties of some of the permanent magnet materials are given in Chap. XVII.



G. 29. Demagnetization Curve of Steel

Suppose the dimensions of the magnet are: length L_m, cross-section A_m; and of its air gap: length L_g, cross-section A_g; and let the respective flux densities be B_d and B_g. Then

$$\text{m.m.f. across the air gap} = \text{m.m.f. in the magnet}$$
$$= H_d \cdot L_m$$

Now, the energy stored in the air gap is

$$(B_g^2/8\pi)\, A_g \cdot L_g \text{ ergs}$$
$$= \frac{B_g A_g \cdot B_g L_g}{8\pi} = \frac{\text{Flux across gap} \times \text{m.m.f. across gap}}{8\pi}$$

If the flux in the gap is equal to the flux in the magnet, the energy stored in the gap

$$A_m \cdot B_d \times \underline{H_d \cdot L_m} = :$$

Since the volume of steel is $A_m \cdot L_m$, the energy in the air gap is

$$\frac{B_d \cdot H_d}{8\pi} \text{ per c.c. of steel}$$

This is obviously maximum when the product $B_d \cdot H_d$ is maximum.

TABLE I
Rationalized Form of the Numbered Formulae in Chapter I

Formula Number	Rationalized Form*
1	$F = \frac{Q_1 Q_2}{\kappa_r 4\pi r^2}$ (κ_r = the permittivity in rationalized units.)
2	$E = \frac{D}{\kappa_r}$
3	$\psi = Q$
4	$E_p = \frac{Q}{2\pi\kappa_r p}$
5	$E_r = \frac{\sigma}{\kappa_r}$
6	$\psi = Q$
7	$\psi = Q_1 + Q_2 + Q_3 \dots$
8	$E = \frac{\sigma}{\kappa_r}$
9	Same as unrationalized.
10	Same as unrationalized.
11	$V_p = \frac{Q}{4\pi\kappa_r d}$
12	$V_p = \frac{1}{4\pi}\left[\frac{Q_1}{\kappa_r d_1} + \frac{Q_2}{\kappa_r d_2} + \frac{Q_3}{\kappa_r d_3} + \dots\right]$
13	Same as unrationalized.
14	Same as unrationalized.
15	Energy per unit volume $= \frac{\kappa_r E^2}{2}$
16	$E_x = \frac{\psi}{a_x \kappa_r}$
17	$F = \frac{\sigma Q}{2\kappa_r}$
18	$H = \frac{m_1 m_2}{4\pi\mu_r r^2}$ (μ_r is the permeability in rationalized units.)

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* For ψ, D (electric flux density), κ, H and m.m.f., unrationalized unit size is $\left(\frac{1}{4\pi}\right)$ × rationalized unit size.

For μ, unrationalized unit size is (4π) × rationalized unit size.

(*Note.* In equations 43 and 45 D is a distance.)

The rationalized system of units is discussed very fully by E. Bradshaw Ref. 21) and by T. McGreevy (Ref. 22).

TABLE I—(*contd.*)

Formula Number	Rationalized Form*
19	$H = \frac{m}{4\pi\mu_r r^2}$
20	Same as unrationalized.
21	$H = \frac{B}{\mu_r}$
22	Same as unrationalized.
23	Same as unrationalized.
24	$h = \frac{J}{2}$
25	$B = H + J$
26	$B = \mu_r H$
27	Same as unrationalized.
28	Magnetic susceptibility $= \mu_r - 1$
29	Same as unrationalized.
30	Same as unrationalized.
31	$J = \frac{4\pi pH}{1 + kH}$
32	Same as unrationalized.
33	$V = \frac{m}{4\pi\mu_r r}$
34	$V_p = \frac{M \cos\phi}{4\pi\mu_r r^2}$
35	$\mathrm{F} = \frac{AB^2}{2\mu_r}$
36	$H = \frac{i}{2}$
37	$V_p = \frac{dA \,.\, i \,.\, \cos\phi}{4\pi r^2}$
38	$V_p = \frac{i\Omega}{4\pi}$
39	Same as unrationalized.
40	$f = \frac{midl \sin\theta}{4\pi r^2}$
41	Same as unrationalized.
42	$dH = \frac{idl \sin\theta}{4\pi r^2}$
43	$H = \frac{i}{2\pi D}$

* See footnote on page 47.

TABLE I—(*contd.*)

Formula Number	Rationalized Form*
44	Work done $= i$
45	$\mathrm{F} = \dfrac{i_1 i_2 l}{4\pi D}$
46	$H = \dfrac{ir^2}{2R^3}$
47	$H = \dfrac{i}{2r}$
48	Intensity at $Q = \dfrac{r^2 Ni}{2}\left[\dfrac{1}{(r^2 + x^2)^{\frac{3}{2}}} + \dfrac{1}{(r^2 + (l - x)^2)^{\frac{3}{2}}}\right]$
49	$\mathrm{F} = -\dfrac{3\pi i_A i_B \cdot a^2 b^2 x}{2(a^2 + x^2)^{\frac{3}{2}}}$
50	$H = \dfrac{Ni}{l}\cos\theta_1$
51	$H_p = \dfrac{Ni}{2l}$
52	Same as unrationalized.
53	Same as unrationalized.
54	Same as unrationalized.
55	Same as unrationalized.
56	Energy per c. c. = area of loop × bh in sq. cm.
57	Same as unrationalized.
58	Same as unrationalized.
59	Same as unrationalized.
60	Energy stored per c. c. $= \dfrac{B^2}{2\mu_r} = \dfrac{\mu_r H^2}{2}$

*See footnote on page 47

BIBLIOGRAPHY AND REFERENCES

(1) *Alternating Currents,* A. Russell, Vol. I, Ch. I.
(2) *Electrical Engineering,* T. F. Wall.
(3) *Theory and Calculation of Electric Circuits,* C. P. Steinmetz, Ch. III, IV, V.
(4) *Properties and Testing of Magnetic Materials,* T. Spooner, Ch. I to VI.
(5) "Magnetic Reluctance," A. E. Kennelly, *Trans. A.I.E.E.*, Vol. VI, p. 485.
(6) "Investigation of Magnetic Laws for Steel, and Other Materials," J. D. Ball, *G.E. Review,* Vol. XIX, p. 369.
(7) "The Unsymmetrical Hysteresis Loop," J. D. Ball, *Trans. A.I.E.E.,* 1915, p. 2693.
(8) *Applied Magnetism,* T. F. Wall.
(9) *Magnetic Induction in Iron and Other Metals,* J. A. Ewing.
(10) *Absolute Measurements in Electricity and Magnetism,*" Gray, Vol. II, Part 1, p. 268.

(11) *British Association Report*, 1889.

(12) S. Evershed, *Jour. I.E.E.*, Vol. LVIII, p. 780; and *Jour. I.E.E.*, Vol. LXIII, p. 725.

(13) "Note on the A.C. Method in Permeability Testing," L. G. A. Sims (Brit. Ass. paper), *Engineering*, September, 1937.

(14) "Standardization in Incremental Magnetism," *Engineering*, Vol. CXLIII, p. 24.

(15) *Permanent Magnets*, Ch. VI, F. G. Spreadbury.

(16) *Ferromagnetism*, Ch. IX, R. M. Bozorth.

(17) "The Economic Aspect of the Utilization of Permanent Magnets in Electrical Apparatus," E. A. Watson, *Jour. I.E.E.*, Vol. LXIII, p. 822.

(18) "Permanent Magnets and the Relation of Their Properties to the Constitution of Magnet Steels," E. A. Watson, *Jour. I.E.E.*, Vol. LXI, p. 641.

(19) "Permanent Magnets," A. Edwards, *Elec. Rev.*, June 30th, 1944.

(20) "Permanent Magnet Design," A. Edwards and K. Hoselitz, *Elec. Rev.*, Aug. 4, 1944.

(21) *Electrical Units, with special reference to the M.K.S. System*, E. Bradshaw.

(22) *The M.K.S. System of Units*, T. McGreevy.

(23) *Principles of Electricity*, A. Morley and E. Hughes.

CHAPTER II

UNITS, DIMENSIONS, AND STANDARDS

Absolute Units. An absolute system of units may be defined as a system in which the various units are all expressed in terms of a small number of fundamental units. The word "absolute" in this sense does not imply supreme accuracy: it is used as opposed to "relative." Absolute measurements do not compare the measured quantity with arbitrary units of the same kind, but are made in terms of appropriate fundamental units.

The Committee of the British Association on Electrical Units and Standards, in formulating the absolute system of units in 1863, had the idea "that the units should not be defined by a series of master standards, each defining one quantity in the way in which the units of length and mass are defined, but that each electrical unit should be defined by some natural law which expresses the relation between the quantity concerned and the fundamental quantities of length, mass and time, for which internationally accepted standards have already been established." (Ref. 66.) Electrical and magnetic units involve, in addition, the properties of the media in which the electrical or magnetic actions take place, i.e. the permittivity in the case of electrostatic forces and the permeability in the case of magnetic forces.

The British Association Committee on Practical Standards for Electrical Measurements adopted as the fundamental units of length, mass and time, the centimetre, gramme, and second respectively, and thus brought into existence the C.G.S. system of units.

systems of C.G.S. units exist; one involving only the permit-
κ of the medium as well as units of length, mass, and time; the other involving permeability as well as units of length, mass, and time. The first is known as the electrostatic C.G.S. system of units (e.s.c.g.s. or e.s.u. system), and the second as the electromagnetic C.G.S. system (e.m.c.g.s. or e.m.u. system).

The electromagnetic system is the more convenient from the point of view of most electrical measurements, and is, therefore, much more generally used than the electrostatic system. If a quantity is expressed in "C.G.S. units" without the additional designation "electromagnetic" or "electrostatic," it may be taken that the electromagnetic system is indicated.

In the electrostatic C.G.S. system the permittivity is, for purposes of definition, taken as unity, as is the permeability in the electromagnetic system.

Dimensions of Velocity, Acceleration, and Force. Since velocity $= \frac{\text{length}}{\text{time}}$, this can be expressed in the dimensional notation as

$$[v] = \frac{[L]}{[T]} = [LT^{-1}] \quad \checkmark \quad . \quad . \quad . \quad . \quad (61)$$

the square brackets indicating that the equality is dimensional only, and does not refer to numerical values.

Thus, velocity has the "dimensions" $[LT^{-1}]$.

Similarly,

$$\text{Acceleration} = \frac{\text{velocity}}{\text{time}} = \frac{\text{length}}{\text{time} \times \text{time}}$$

or, dimensionally,

$$[a] = \frac{[L]}{[T^2]} = [LT^{-2}] \;.\checkmark \quad . \quad . \quad . \quad . \quad (62)$$

$$\text{Force} = \text{mass} \times \text{acceleration}$$

thus, representing the dimension of mass by $[M]$,

$$[\mathrm{F}] = [M]\,[LT^{-2}] = [MLT^{-2}] \quad \checkmark \quad . \quad . \quad (63)$$

Dimensions in Electrostatic and Electromagnetic Systems. From Coulomb's Inverse Square Law we have

$$\mathrm{F} = \frac{Q_1 Q_2}{\kappa r^2}$$

i.e.

$$\text{Force} = \frac{(\text{quantity of electricity})^2}{\kappa \times \text{length}^2}$$

where κ is the permittivity of the medium.

Dimensionally,

$$[\mathrm{F}] = \frac{[Q]^2}{[\kappa \,.\, L^2]}$$

i.e.

$$[MLT^{-2}] = \frac{[Q]^2}{[\kappa L^2]}$$

From which

$$[Q] = [\kappa^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}] \quad \checkmark \quad . \quad . \quad . \quad . \quad (64)$$

which gives the dimensions of Q in the electrostatic system.

Also, in magnetism, the force between two magnetic poles of pole strengths m_1 and m_2, distant r apart, in a medium of permeability μ is

$$\mathrm{F} = \frac{m_1 m_2}{\mu r^2}$$

$$\text{Force} = \frac{\text{pole strength} \times \text{pole strength}}{\mu \times \text{length}^2}$$

Dimensionally,

$$[MLT^{-2}] = \frac{[m]^2}{[\mu L^2]}$$

From which

$$[m] = \mu^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} \quad . \quad . \quad . \quad . \quad . \quad (65)$$

in the electromagnetic system.

Thus, it is seen that the dimensions of these two quantities, one electrostatic and one magnetic, involve the dimensions of either κ or μ as well as those of length, mass, and time. It will be seen later that the same holds for all such quantities.

Instead of using either μ or κ as the necessary fourth fundamental dimension, any one of the electrical or magnetic magnitudes could be used. G. Giorgi (see Ref. (67)) suggested that quantity of electricity Q might be used and that this would eliminate fractional exponents in dimensional expressions. Thus, for example, the product QV (where V = potential) has the dimensions of work so that

$$[QV] = [ML^2T^{-2}]$$

from which

$$[V] = [ML^2T^{-2}Q^{-1}]$$

Dimensions of Permeability (μ) **and** (κ). If they are taken as fundamental dimensions, the dimensions of these quantities cannot be expressed in terms of length, mass, and time, but a relationship between them can be found.

As seen in the preceding paragraph, the dimensions of a quantity of electricity can be expressed in terms of κ, etc., as

$$[Q] = [\kappa^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}]$$

Now, the force exerted upon a magnetic pole, of strength m units placed at the centre of a circular wire of radius r due to a current of i flowing in an arc of the circle of length l is given by

$$F = \frac{mil}{r^2}$$

$$\therefore i = \frac{Fr^2}{ml}$$

Quantity of electricity flowing in time t, is

$$Q = it = \frac{Fr^2t}{ml}$$

Dimensionally,

$$[Q] = \frac{[MLT^{-2}]\,[L^2]\,[T]}{[\mu^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}]\,[L]}$$

substituting the expression for m from the previous paragraph.

$$\therefore [Q] = [M^{\frac{1}{2}} L^{\frac{1}{2}} \bar{\mu}^{\frac{1}{2}}] \quad . \quad . \quad . \quad . \quad . \quad (66)$$

This gives the dimensions of Q in the electromagnetic system. Since Q must have the same dimensions in either system, we have

$$[\kappa^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}] = [M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}]$$

$$\therefore\ [\kappa^{\frac{1}{2}}LT^{-1}] = [\mu^{-\frac{1}{2}}]$$

or

$$[LT^{-1}] = [\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}] \quad . \quad . \quad . \quad . \quad (67)$$

Now, the dimensions $[LT^{-1}]$ are those of a velocity.

$$\therefore\ \frac{1}{\sqrt{\mu\kappa}} = \text{a velocity}$$

In any system of units the "permeability of free space" μ_0 and the "permittivity of free space" κ_0 are related by the equation

$$\mu_0\kappa_0 = \frac{1}{c^2}$$

Where c is the velocity of light in the system of units considered. From this relationship the dimensions of any electrical quantity can be converted from those of the electrostatic system to those of the electromagnetic system, and *vice versa*.

Dimensions of Electrical and Magnetic Quantities. The dimensions of the various quantities can be derived from the known relationships between them, as shown below.

1. *Electric Current*

$$\text{Current} = \frac{\text{quantity}}{\text{time}}$$

$$[I] = \frac{[\kappa^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]}{[T]} = [\kappa^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$$

in the E.S. system.

To convert these dimensions to those of the E.M. system—involving μ instead of κ—substitute $\mu^{-\frac{1}{2}} . L^{-1}T$ for $\kappa^{\frac{1}{2}}$ from Equation (67).

Thus, in the E.M. system

$$[I] = [\mu^{-\frac{1}{2}}L^{-1}TL^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}] = [\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}]$$

2. *Electric Potential.* By definition,

$$\text{Potential} = \frac{\text{work}}{\text{quantity of electricity}}$$

Thus, representing the dimensions of potential by $[V]$,

$$[V] = \frac{[MLT^{-2}][L]}{[\kappa^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]} = [\kappa^{-\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$$

in the E.S. system.

Converting to the E.M. system, we have

$$[V] = [\mu^{\frac{1}{2}} . LT^{-1} . L^{\frac{1}{2}} . M^{\frac{1}{2}}T^{-1}] = [\mu^{\frac{1}{2}} \quad L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$$

in the E.M. system.

3. *Magnetic Flux.*

$$\begin{aligned} \text{e.m.f.} &= \text{rate of change of flux} \\ &= \frac{\text{flux}}{\text{time}} \\ \therefore \text{Flux} &= \text{e.m.f.} \times \text{time} \end{aligned}$$

Dimensionally $[\Phi] = [\kappa^{-\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]\,[T] = [\kappa^{-\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}]$
in the E.S. system, or

$$[\Phi] = [\mu^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}] \text{ in the E.M. system.}$$

The dimensions of the most important electrical and magnetic quantities in both systems, together with the relationships from which they are derived, are given in Table II. The table gives references to the pages upon which the definitions of the various units are given.

Practical and C.G.S. Units. Some of the electromagnetic C.G.S. units are too small for practical purposes, while others are too large. The British Association Committee fixed the practical unit of current and resistance as $\frac{1}{10}$ and 10^9 electromagnetic units of current and istance respectively. The magnitudes of the practical units of other quantitites, in terms of the absolute electromagnetic units, can be found from the known relationships connecting the various quantities and are given in the table.

For example,

$$\begin{aligned} \text{e.m.f.} &= \text{current} \times \text{resistance} \\ \therefore \text{The practical unit of e.m.f.} &= \text{the practical unit of current} \times \\ & \text{ the practical unit of resistance} \\ &= \frac{1}{10} \text{ of the e.m. unit of current} \\ & \times 10^9 \text{ e.m. units of resistance} \\ &= 10^8 \text{ e.m. units of e.m.f.} \end{aligned}$$

The International Conference on Electrical Units and Standards in London, 1908, agreed that the magnitudes of the fundamental electrical units should be determined on the E.M.C.G.S. system and adopted as these fundamentals (i) the Ohm = 10^9 e.m.u., (ii) the Ampere = 0·1 e.m.u., (iii) the Volt = 10^8 e.m.u. and (iv) the Watt = 10^7 e.m.u.

The *electromagnetic C.G.S. unit of resistance* is defined as the resistance of a conductor such that 1 erg of energy is expended per second when unit current passes through it.

The *electromagnetic C.G.S. unit of current* is defined as the current which, flowing in the arc of a circle 1 cm. in length and 1 cm. in radius, produces a force of 1 dyne on a unit magnetic pole placed at its centre.

The determination of the number of electrostatic C.G.S. units in one practical unit can best be illustrated by an example. In the case of e.m.f. from the corresponding dimensions given in Table II,

$$\left[\frac{1\text{ e.s.u.}}{1\text{ e.m.u.}}\right] = \left[\frac{L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\kappa^{-\frac{1}{2}}}{L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\,.\,\mu^{\frac{1}{2}}}\right]$$

Now, if the dimensions of L, M, and T are neglected, this ratio is equal to $\kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}$. But it has been shown above that

$$\begin{aligned}\kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}} &= \text{a velocity}\\ &= 2{\cdot}998 \times 10^{10}\text{ cm/sec in the c.g.s. system}\end{aligned}$$

$$\therefore \frac{1\text{ e.s.u.}}{1\text{ e.m.u.}} = 2{\cdot}998 \times 10^{10}$$

1 e.s.u. of e.m.f. or potential $= 2{\cdot}998 \times 10^{10}$ e.m.u. units of e.m.f. or potential

Note. In the following conversions the approximation $2{\cdot}998 \fallingdotseq 3$ has been used for the sake of simplification.

Thus, if the volt is equivalent to 10^8 e.m.u. it will be equivalent to $\frac{10^8}{3 \times 10^{10}} = \frac{1}{3 \times 10^2}$ of an e.s.u.

It must be noted that the above ratio of $\frac{1\text{ e.s.u.}}{1\text{ e.m.u.}}$ does not in all cases equal 3×10^{10}, but depends upon the dimensions. Thus, in the case of capacitance, the ratio $\frac{1\text{ e.s.u.}}{1\text{ e.m.u.}}$ equals, neglecting dimensions of L, M, and T, the product

$$\mu\kappa = \frac{1}{(3 \times 10^{10})^2} = \frac{1}{9 \times 10^{20}}$$

From which

1 e.s.u. of capacitance $= \frac{1}{9 \times 10^{20}}$ of 1 e.m.u. of capacitance.

The other practical units can be determined from the four fundamental units, and are as follows—

Quantity. The quantity of electricity passed through a circuit by 1 amp. in 1 sec. is 1 *Coulomb*

$$= 10^{-1}\text{ e.m.u. of quantity}$$

Work or Energy. The unit of work or energy is 1 *Joule* (or 1 watt-second), and is the energy expended by 1 watt in 1 sec.

$$1\text{ joule} = 10^7\text{ ergs (C.G.S. units of work)}$$

Capacitance. A capacitor has unit capacitance—1 *Farad*—when 1 coulomb of electricity raises the potential difference between its plates 1 volt.

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{10^{-1}}{10^{8}} = 10^{-9} \text{ e.m.u.}$$

Inductance. The practical unit of inductance is 1 *Henry*. It is the inductance of a circuit such that a rate of change of current in the circuit of 1 amp. per sec. induces an e.m.f. of 1 volt.

$$1 \text{ henry} = \frac{1 \text{ volt}}{1 \text{ amp. per sec.}} = \frac{10^{8}}{10^{-1}} = 10^{9} \text{ e.m.u.}$$

The C.G.S. unit of inductance is 1 cm.

Thus $\quad 1 \text{ henry} = 10^{9} \text{ cm.}$

Since the farad is too large a unit for many practical cases, the microfarad (represented symbolically as "μf") or the micro-micro-farad (represented by "$\mu\mu$f" symbolically) are used as more convenient units.

In the same way, the millihenry and microhenry are often used as more convenient units of inductance than the henry.

In the electrostatic system, capacitance has the dimensions $L\kappa$, which gives the electrostatic C.G.S. unit as 1 cm.

Additional names have been given to several of the C.G.S. units by the Symbols, Units, and Nomenclature (S.U.N.) Committee of the International Union of Pure and Applied Physics.* The most important of these are—

The Maxwell (the C.G.S. unit of magnetic flux),
The Gauss (the C.G.S. unit of magnetic flux density),
The Oersted (the C.G.S. unit of intensity of magnetizing field),
The Gilbert (the C.G.S. unit of magneto-motive force).

The name Weber has been given to the practical unit of flux (1 Weber = 10^8 Maxwells) with 1 V
the corresponding unit of flux density.

The M.K.S. (or Giorgi) System of Units. This system, which uses the metre, kilogramme and second as the units of length, mass and time instead of the centimetre, gramme and second, as in the C.G.S. systems, was originally suggested by Prof. G. Giorgi in 1901. After international discussions on the matter (Refs. 65, 67) the system was finally adopted by the International Electrotechnical Commission (I.E.C.) at its meeting in 1938 at Torquay, when the connecting link between the electrical and mechanical units recommended was the "permeability of free space with the value of $\mu_0 = 10^{-7}$ in the unrationalized system or $\mu_0 = 4\pi 10^{-7}$ in the rationalized system." (The question of rationalization which does not form an essential part of the M.K.S. system, but is actually a separate consideration, will be discussed a little later.)

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* Report published October, 1934.

The M.K.S. system is an absolute system of units, based on its own definitions. It is independent of the C.G.S. systems which it can be used to replace although the long-established C.G.S. systems are not likely to be entirely discarded for some time to come. Its great advantage is that its units are identical with the practical units and are the same whether built up from the electromagnetic or electrostatic theory. The rather cumbersome conversions necessary to relate the units of the electromagnetic and electrostatic C.G.S. systems to those of the practical system—given in Table II—are thus avoided.

It is important to remember that, in the M.K.S. system, the constants μ_0 (= 10^{-7}) and κ_0, the "permittivity of free space," must be used in their proper places in the expressions for the particular units being considered. They cannot be omitted—because they happen to be unity—as often occurs when the C.G.S. systems are used. From the relationship

$$\frac{1}{\sqrt{}} = c = \text{velocity of light}$$

(see also p. 54) it follows that, if $\mu_0 = 10^{-7}$ then $\kappa_0 = 1{\cdot}113 \times 10^{-10}$. L. H. A. Carr (Ref. 65) shows that μ_0 should be given dimensions $L^{-2}T^2$ and that κ_0 is a pure numeric. If this is accepted, then the dimensions of the quantities given in Table II become the same in the two systems—electromagnetic and electrostatic—and the fifth and sixth columns of the table could then be replaced by one column (substituting everywhere these dimensions for μ and taking κ as dimensionless). Thus, the first four lines of this replacing column would read

Some of the principal derived units, mechanical and electrical, in the M.K.S. system are—

Area	The square metre
Volume .	The cubic metre
Velocity .	The metre per second
Acceleration	The metre per second per second
	The Newton, which is the force which produces, in a mass of 1 kilogramme, an acceleration of 1 metre per sec. per sec. (1 Newton = 10^5 dynes).
Work or Energy	The Joule, or Newton-metre
Power . .	The Watt, or Newton-metre per second

Electric Current .	The Ampere, defined as the constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular sections, and placed 1 metre apart in a vacuum, will produce between these conductors a force equal to 2 × 10^{-7} Newton per metre of length.

The M.K.S. units of quantity of electricity, e.m.f., resistance, inductance and capacitance are respectively the Coulomb, Volt, Ohm, Henry and Farad. The unit of magnetic flux is the Weber which is equivalent to 10^8 Maxwells (or lines of force) so that a rate of change of 1 Weber per second, in the flux linking one turn, produces unit e.m.f. Flux density is measured in Webers per square metre.

The use of the constant $\mu_0 = 10^{-7}$, in defining the M.K.S. units needs some explanation.

Consider the expression for the force between parallel, current-carrying conductors. In the electromagnetic C.G.S. system, this is often written

$$F = 2\frac{i_1 i_2 l}{d}$$

where F is in dynes, i_1 and i_2 are currents in e.m. units and l and d are in centimetres.

Strictly the expression should be written

$$F = 2\frac{i_1 i_2 l}{d} \times 1$$

the figure 1 representing the permeability of air or vacuum.

Now, if F is to be in Newtons and the current i_1 and i_2 in M.K.S. units (i.e. Amperes) the expression becomes

$$F = 2\frac{i_1 i_2}{10^7}\frac{l}{d} = 2\frac{i_1 i_2 l}{d} \times 10^{-7}$$

But, if $\mu_0 (= 10^{-7})$ is included, as it must be in the M.K.S. system, we have

$$F = 2\frac{i_1 i_2 l}{d}\mu_0$$

which is in strict accordance with the M.K.S. definition of the unit of current when i_1, i_2, l and d are all made equal to unity.

It should be understood clearly that if in any calculation some medium is involved which has a permeability μ times that of air, this value must be included in addition to μ_0 so that we would then have a factor $\mu\mu_0$.

Rationalized Systems of Units. The basis of a "rationalized" system of units is the conception of *unit* flux issuing from unit magnetic pole or from unit charge instead of a flux of 4π. Clearly "rationalization" is not specially connected with the M.K.S. system: it could be used with any system founded on the same basic electric or magnetic definitions. But, if we are to adopt the new M.K.S. system, there is a good argument for adopting it in rationalized form and so performing two stages of adoption at the same time. The question was considered at the I.E.C., Torquay, meeting in 1938 but it was agreed to postpone formal endorsement until some later date.

In the rationalized system the 4π disappears in some relationships only to appear in another place in others. The advantage of rationalization must then be judged by its effect upon the relationships which one considers to be most important. In the I.E.E. papers mentioned in Ref. 65 strong arguments for rationalization are offered as well as some reasons against it. Limitations of space prevent the statement of these arguments here and the reader specially interested in the question must be referred to these papers.

In one of the papers, by H. Marriott and A. L. Cullen, a distinction is made between rationalization of theory and that of the units themselves. Table III, compiled from this paper, shows the differences between the unrationalized and rationalized forms of some of the most important formulae. In the table μ and κ are the absolute permeability and permittivity of the medium (as distinct from the values for free space).

TABLE III

Rationalized and Unrationalized Forms of Electrical and Magnetic Equations or Formulae

Equation or Formula	Rationalized	Unrationalized
Force F between isolated electric charges Q_1 and Q_2 distant x	$F = \frac{1}{\kappa}\frac{Q_1Q_2}{4\pi x^2}$	$F = \frac{1}{\kappa}\frac{Q_1Q_2}{x^2}$
Electric flux density D near a uniformly charged plane surface of charge density σ	$D = \sigma$	$D = 4\pi\sigma$
Electric field strength E near a uniformly charge plane surface of charge density σ	$E = \frac{\sigma}{\kappa}$	$E = \frac{4\pi\sigma}{\kappa}$
Electric flux density D at distance r from a point charge Q	$D = \frac{Q}{4\pi r^2}$	$D = \frac{Q}{r^2}$
Electric field E at distance r from a point charge Q	$E = \frac{1}{\kappa}\frac{Q}{4\pi r^2}$	$E = \frac{Q}{\kappa r^2}$
Electric potential V at distance r from a point charge Q	$V = -\frac{1}{\kappa}\frac{Q}{4\pi r}$	$V = -\frac{Q}{\kappa r}$
Electric energy density	$\frac{1}{2}\kappa E^2$	$\frac{\kappa E^2}{8\pi}$
Capacitance of parallel-plate capacitor, of plate area A and separation d	$C = \frac{\kappa A}{d}$	$C = \frac{\kappa A}{4\pi d}$
Capacitance of concentric capacitor of radii a, b and length l	$C = \frac{\kappa \cdot 2\pi l}{\log_e \frac{b}{a}}$	$C = \frac{\kappa l}{2 \log_e \frac{b}{a}}$

TABLE III—(*contd.*)

Equation or Formula	Rationalized	Unrationalized
Capacitance of concentric sphere capacitor of radii a and b	$C = \kappa \frac{4\pi ab}{b - a}$	$C = \frac{\kappa ab}{b - a}$
Force F between small isolated magnetic poles m_1 and m_2 distance x	$F = \frac{1}{\mu} \frac{m_1 m_2}{4\pi x^2}$	$F = \frac{1}{\mu} \frac{m_1 m_2}{x^2}$
Magnetic flux density B near a uniformly magnetized plane surface of pole strength surface density m'	$B = m'$	$B = 4\pi m'$
Magnetic field strength H near a uniformly magnetized plane surface of pole strength surface density m'	$H = \frac{m'}{\mu}$	$H = \frac{4\pi m'}{\mu}$
Magnetic flux density B at distance r from a point magnetic pole m	$B = \frac{m}{4\pi r^2}$	$B = \frac{m}{r^2}$
Magnetic energy density	$\frac{1}{2} \mu H^2$	$\frac{\mu H^2}{8\pi}$
Force F between parallel isolated current elements $I_1\delta l_1$ and $I_2\delta l_2$. The perpendicular distance between the two directions of the currents I_1 and I_2 is x	$F = \mu \frac{I_1\delta l_1 I_2 \delta l_2}{4\pi x^2}$	$F = \mu \frac{I_1\delta l_1 I_2 \delta l_2}{x^2}$
Magnetic field δH at distance x from current element $I\delta l$. θ = angle between directions $I\delta l$ and x	$\delta H = \frac{I\delta l}{4\pi x^2} \sin \theta$	$\delta H = \frac{I\delta l}{x^2} \sin \theta$
Force δF on current element $I\delta l$ situated in magnetic flux density B. ϕ = angle between the directions of B and $I\delta l$	$\delta F = BI\delta l \sin \phi$	$\delta F = BI\delta l \sin \phi$
Magnetic field strength H inside a long solenoid of n turns per unit length carrying current I	$H = nI$	$H = 4\pi nI$
Magnetic field H at a distance r from a long straight wire carrying a current I	$H = \frac{I}{2\pi r}$	$H = \frac{2I}{r}$
Ampere-turns for a closed magnetic path	$NI = \oint H \cdot dl$	$NI = \frac{1}{4\pi} \oint H \cdot dl$
E.m.f. induced in conductor δl moving transversely at velocity v in magnetic flux density B. ϕ = angle between the direction of B and the plane $(v, \delta l)$	$\delta E = Bv\delta l \sin \phi$	$\delta E = Bv\delta l \sin \phi$
E.m.f. induced by changing magnetic flux linkage	$E = -\frac{d\phi}{dt}$	$E = -\frac{d\phi}{dt}$
Inductance L of single-turn solenoid of cross-sectional area A and length d	$L = \frac{\mu A}{d}$	$L = \frac{4\pi\mu A}{d}$

In the rationalized system of units the permeability of free space is $\mu_r = 4\pi 10^{-7}$ (instead of $\mu_0 = 10^{-7}$ in the unrationalized system). Again, the permittivity of free space is $\kappa_r = \dfrac{1{\cdot}113 \times 10^{-10}}{4\pi} = 8{\cdot}854 \times 10^{-12}$ instead of $\kappa_0 = 1{\cdot}113 \times 10^{-10}$.

E. Bradshaw (Ref. 65) gives the relative sizes of the units in the C.G.S. and rationalized and unrationalized M.K.S. systems and Table IV reproduces some of the relationships for the more commonly used quantities.

Relationships Between the Mechanical, Electrical, and Thermal Practical Units. The relationships connecting the practical units for the measurement of mechanical power, energy, and heat, with those for the measurement of electrical power and energy are so important that a consideration of them here is, perhaps, not misplaced.

Power.

1 h.p. = 33,000 ft.-lb. per min. ✓
= 550 ft.-lb. per sec. ✓
= $550 \times 12 \times 2{\cdot}54 \times 453{\cdot}6 \times 981$ cm.-dynes (or ergs) per sec.
= 746×10^7 ergs per sec.

∴ Since 1 watt = 10^7 ergs per sec. ✓

1 h.p. = 746 watts ✓

$$1 \text{ ft.-lb. per sec.} = \frac{746}{550} = 1{\cdot}357 \text{ watts}$$

Energy.

1 kilowatt-hour (kWh) = 1,000 watt-hours
1 h.p.-hr. = 746 watt-hours
= 0·746 kWh.
1 ft.-lb. = $12 \times 2{\cdot}54 \times 453{\cdot}6 \times 981$ cm.-dynes (or ergs)
= $1{\cdot}357 \times 10^7$ ergs
= 1·357 Joules (or watt-sec.)

$$= \frac{1{\cdot}357}{60 \times 60 \times 1{,}000} \text{ kWh.}$$

or 1 ft.-lb. = 0·000000377 kWh.

Thermal Units.

1 gramme-calorie = $4{\cdot}18 \times 10^7$ ergs
= 4·18 joules (or watt-seconds)

1 B.Th.U. (i.e. the heat required to raise the temperature of 1 lb. water 1° F.
= 778 ft.-lb.
= $778 \times 0{\cdot}000000377$ kWh. of electrical energy
= 0·000293 kWh.

TABLE IV

K_1 = number of M.K.S. unrationalized units in 1 M.K.S. rationalized unit

K_2 = number of C.G.S. electromagnetic units in 1 M.K.S. unrationalized unit

K_3 = number of C.G.S. electrostatic units in 1 C.G.S. electromagnetic unit

K_1K_2 = number of C.G.S. electromagnetic units in 1 M.K.S. rationalized unit

$K_1K_2K_3$ = number of C.G.S. electrostatic units in 1 M.K.S. rationalized unit

K_2K_3 = number of C.G.S. electrostatic units in 1 M.K.S. unrationalized unit

c = free space velocity of propagation = $2{\cdot}998 \times 10^8$ metres/sec.

Quantity	Symbol	M.K.S. Rationalized Unit	K_1	K_2	K_3
Force	F	Newton	1	10^5	1
Energy . . .	W	Newton-metre : Joule	1	10^7	1
Power . . .	P	Joule/sec. : Watt	1	10^7	1
Current . . .	I	Ampere	1	10^{-1}	$100c$
Charge . . .	Q	Coulomb	1	10^{-1}	$100c$
Charge surface density .	σ	Coulomb/m^2	1	10^{-5}	$100c$
Potential difference .	V	Joule/coulomb : volt	1	10^8	$\frac{1}{(100c)}$
Electric field strength .	E	Volt/metre	1	10^6	$\frac{1}{(100c)}$
Resistance . . .	R	Volt/ampere : ohm	1	10^9	$\frac{1}{(100c)^2}$
Resistivity . . .	ρ	Ohm-metre	1	10^{11}	$\frac{1}{(100c)^2}$
Electric flux . .	ψ	Coulomb	4π	10^{-1}	$100c$
Electric flux density .	D	Coulomb/m^2	4π	10^{-5}	$100c$
Capacitance . .	C	Coulomb/volt : Farad	1	10^{-9}	$(100c)^2$
Permittivity . .	κ	Farad/metre	4π	10^{-11}	$(100c)^2$
Magnetic flux . .	Φ	Volt-sec : Weber	1	10^8	$\frac{1}{(100c)}$
Magnetic flux density .	B	Weber/m^2	1	10^4	$\frac{1}{(100c)}$
Magneto motive force .	F	Ampere	4π	10^{-1}	$100c$
Magnetic field strength .	H	Ampere/metre	4π	10^{-3}	$100c$
Inductance . . .	L	Weber/ampere : Henry	1	10^9	$\frac{1}{(100c)^2}$
Permeability . .	μ	Henry/metre	$1/4\pi$	10^7	$\frac{1}{(100c)^2}$
Magnetic pole strength .	m	Weber	$1/4\pi$	10^8	$\frac{1}{(100c)}$
Resistance of free space	R_0	Ohm	$1/4\pi$	10^9	$\frac{1}{(100c)^2}$

1 Centigrade heat unit (i.e. the heat required to raise the temperature of 1 lb. of water 1° C.)

$$= \frac{9}{5} \times 778 \text{ ft.-lb.}$$

$$= 0{\cdot}000528 \text{ kWh.}$$

Example 1. Calculate the number of kWh. of electrical energy obtained per hour from a generating plant whose overall efficiency is 18 per cent. Given,

Number of pounds of coal burnt per hour = 7,000 lb.
Calorific value of the coal = 12,000 B.Th.U. per lb.
Number of B.Th.U. input per hour = $7{,}000 \times 12{,}000$
Output in B.Th.U. per hour = $0{\cdot}18 \times 84 \times 10^6$
Output in kWh. per hour = $0{\cdot}18 \times 84 \times 10^6 \times 0{\cdot}000293$
= 4,420 kWh. per hour

(The power output is thus 4,420 kilowatts.)

Example 2. Calculate the number of kWh. expended in pulling a train of weight 250 tons, ½ mile up an incline of 1 in 75, at a steady speed of 20 miles an hour, by means of an electric locomotive, if 70 per cent of the energy input is usefully employed. Frictional resistance to motion may be taken as 16 lb. per ton.

Calculate also the current taken by the motors if the supply voltage is 500 volts.

$$\text{Tractive effort} = \frac{250 \times 2{,}240}{75} + 250 \times 16 \text{ lb.}$$

$$= 11{,}467 \text{ lb.}$$

$$\text{Work done} = 11{,}467 \times 2{,}640$$

$$= 30{,}250{,}000 \text{ ft.-lb.}$$

$$\therefore \text{Energy input} = \frac{30{,}250{,}000}{0{\cdot}7} \times 0{\cdot}000000377 \text{ kWh.}$$

$$= 16{\cdot}3 \text{ kWh.}$$

$$\text{Time taken to travel } \tfrac{1}{2} \text{ mile} = \frac{1}{40} \text{ hr.}$$

$$\therefore \text{Power input} = \frac{16{\cdot}3 \times 1{,}000}{\frac{1}{40}}$$

$$= 652{,}000 \text{ watts}$$

$$\therefore \text{Current} = \frac{652{,}000}{500} = 1{,}304 \text{ amp.}$$

Dimensional Equations. If a certain physical quantity y is proportional to the product of two or more other physical quantities, each of which is raised to some power which is unknown, e.g. $y \propto x^m z^n w^p$, then the unknown indices m, n, and p can be determined by substituting their dimensions for the quantities y, x, z, and w, and equating the corresponding indices of L, M, T, μ, and κ, as in the following example.

Example. The electrical power in a circuit is proportional to the voltage, and to the resistance of the circuit, each raised to some power. Determine these powers by the use of the dimensions of the quantities involved.

Let $$P \propto E^m R^n$$
or $$P = k\,.\,E^m R^n$$
where k is a number which has no dimensions.

Then, substituting the dimensions of the quantities from Table I, we have, using the electromagnetic system,

$$L^2MT^{-3} = k\,[(L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}})^m \times (LT^{-1}\mu)^n]$$

Equating corresponding indices, we have

For L—

$$2 = \frac{3}{2}m + n$$

For M—

$$1 = \frac{1}{2}m, \text{ i.e. } m = 2,\ n = -1$$

Also, *for T*—

$$-3 = -2m - n$$

which is satisfied also by $m = 2,\ n = -1$

For μ—

$$0 = \frac{1}{2}m + n$$

which is again satisfied by $m = 2,\ n = -1,$

$$\therefore P \propto \frac{E^2}{R}$$

The dimensions of the physical quantities involved can also be used to check, or detect, possible errors in equations which have been derived, perhaps, from somewhat complicated theory. This use is illustrated in the following example.

Example. It is suspected that an error has been made in the derivation of the expression

$$I = \frac{E\omega M}{\sqrt{(\omega^2M^2 + R_1R_2)^2 + \omega^2L_1R_1^2}}$$

for the current in a circuit, in terms of the voltage E, angular velocity ω, mutual inductance M, self-inductance L_1 and resistances R_1 and R_2. Ascertain if this is so and, if necessary, make a correction to ensure that the equation is dimensionally correct

ω, being an angular velocity, has the dimension T^{-1}, and M has, of course, the dimensions of inductance, i.e. $L\mu$ in the E.M. system. Then, substituting the dimensions of the various quantities in the electromagnetic system, we have—

Left-hand Side	*Right-hand Side*
$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$\dfrac{(L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}})\,(T^{-1})\,(L\mu)}{[(T^{-2}L^2\mu^2 + L^2T^{-2}\mu^2)^2 + T^{-2}\,.\,L\mu\,.\,L^2T^{-2}\mu^2]^{\frac{1}{2}}}$
	$= \dfrac{L^{\frac{5}{2}}M^{\frac{1}{2}}T^{-3}\mu^{\frac{3}{2}}}{[(T^{-2}L^2\mu^2 + L^2T^{-2}\mu^2)^2 + T^{-4}L^3\mu^3]^{\frac{1}{2}}}$

Since the sum of terms which have the same dimensions as one another must have the same dimensions as its constituent terms, the right-hand side can be written

$$\frac{L^{\frac{5}{2}}M^{\frac{1}{2}}T^{-3}\mu^{\frac{3}{2}}}{[T^{-4}L^4\mu^4 + T^{-4}L^3\mu^3]^{\frac{1}{2}}}$$

In order that the dimensions of this expression shall be the same as those of the left-hand side, the second term in the denominator should have dimensions $T^{-4}L^4\mu^4$, so that the dimensions of the denominator as a whole would be

$$[T^{-4}L^4\mu^4]^{\frac{1}{2}} = T^{-2}L^2\mu^2$$

which would give $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$ for the dimensions of the right-hand side, and would thus make the whole equation dimensionally correct.

Thus, the dimensions $L\mu$ are missing from the last term in the denominator of the right-hand side. Since these are the dimensions of inductance, the original equation, to be dimensionally correct, should have read

$$I = \frac{E\omega M}{\sqrt{(\omega^2M^2 + R_1R_2)^2 + \omega^2L_1R_1^2\mathrm{L}}}$$

L being an inductance, either self or mutual.

Determination of "c" (i.e. the ratio of the Electromagnetic to the Electrostatic Unit of Electricity). It has already been stated that

$$\mu^{-\frac{1}{2}}\kappa^{-\frac{1}{2}} = \text{a velocity} = c$$

and that this velocity can be shown, experimentally, to be that of light, i.e. $2{\cdot}998 \times 10^{10}$ or 3×10^{10} cm. per sec., very nearly.

To determine this velocity, the ratio of the electromagnetic and electrostatic values of some electrical quantity must be measured. This ratio can be measured for any of the four quantities—capacitance, resistance, e.m.f., and quantity of electricity. Of these, the first is perhaps the best, and will be described.

The method used necessitates the calculation of the capacitance of some simple form of capacitor in electrostatic C.G.S. units, and also the measurement of its capacitance in electromagnetic C.G.S. units in terms of a resistance whose value in electromagnetic C.G.S. units is known.

The value of the velocity c can be obtained from the ratio of the calculated electrostatic value to the measured electromagnetic value as below.

Let C_{ES} be the calculated value of the capacitance in e.s.u.
" C_{EM} be the measured value in e.m.u.

From Table I—

$$\frac{1\text{ e.s.u. of capacitance}}{1\text{ e.m.u. of capacitance}} = \left[\frac{\kappa \,.\, L}{L^{-1}T^2\mu^{-1}}\right] = \kappa\mu \text{ neglecting the dimensions of } L \text{ and } T$$

$$\text{But } \kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}} = c, \qquad \therefore \kappa\mu = \frac{1}{c^2}$$

$$\therefore \frac{1\text{ e.s.u. of capacitance}}{1\text{ e.m.u. of capacitance}} = \frac{1}{c^2}$$

Thus, 1 e.s.u. of capacitance $= \frac{1}{c^2}$ of 1 e.m.u. of capacitance, or the number of e.s.u. of capacitance in 1 e.m.u $= c^2$.

Now, if a certain length is expressed as L' ft. or L'' in.

$$\frac{L''}{L'} = \frac{12}{1} = \text{No. of inches in 1 ft.}$$

By analogy, $\frac{C_{ES}}{C_{EM}}$ = No. of e.s.u. of capacitance in 1 e.m.u. $= c^2$

$$\therefore \quad \sqrt{\frac{C_{ES}}{C_{EM}}} = c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Procedure. The capacitance C_{ES} having been calculated from the dimensions of the capacitor, C_{EM} must be measured. There are several ways of carrying out this measurement of capacitance in electromagnetic C.G.S. units, the best of which is Maxwell's bridge method, described by him in his *Electricity and Magnetism*, Article 775.

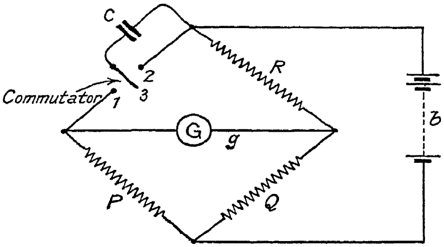

FIG. 30. CIRCUIT FOR THE DETERMINATION OF c

In this method, the capacitor C to be measured is connected in one arm of a bridge network, as shown in Fig. 30. A commutator is also connected in this arm, by means of which the capacitor is alternately charged and discharged. The commutator is driven by a small motor, supplied from a steady source, and whose speed can be varied as required.

P, Q, and R are non-inductive resistors whose values in absolute electromagnetic units are known. G is a sensitive galvanometer, of resistance g, and b is the resistance of the battery circuit.

The resistances of the leads in the capacitor branch are made negligibly small.

When the commutator is in such a position that the moving switch 3 of Fig. 30 is on contact 2, the capacitor C is discharged and the current flowing in the various arms, including the galvanometer branch, are steady currents from the battery. When switch 3 is on contact 1, the capacitor is charged to the potential of the battery and the galvanometer current is altered owing to the varying current taken by the capacitor while it is being charged. When the capacitor is fully charged, no current is taken by it, and the galvanometer current again takes up its steady value.

The galvanometer current can be made zero by suitable adjustment of the resistances P, Q, and R, and of the speed of the commutator.

The expression* for the capacitance of capacitor C is

$$C_{EM} = \frac{Q}{nPR}\left[\frac{1-\left(\dfrac{Q^2}{(P+Q+g)(Q+b+R)}\right)}{\left\{1+\dfrac{Qb}{P(Q+b+R)}\right\}\left\{1+\dfrac{Qg}{R(Q+P+g)}\right\}}\right] \quad . \quad (69)$$

n is the frequency of the commutator. To a close approximation $C_{EM} = \dfrac{Q}{nPR}$ provided $P + R$ is large compared with Q.

The resistances must be expressed in electromagnetic C.G.S. units of resistance. If expressed in ohms, the value of C will be in farads. Many investigators have measured the ratio c by the above and other methods. From their results it appears that

$$c = 2{\cdot}998 \times 10^{10} \text{ cm. per sec.}$$

while the average value obtained by many investigators for the velocity of light is $2{\cdot}9986 \times 10^{10}$ cm. per sec. These figures are taken from the *Dictionary of Applied Physics*, Vol. II, p. 960, where a record of much work carried out on the subject is given.

International and Absolute Units. Although the British Association Committee on Electrical Measurements adopted the absolute system of units in 1863, and this was confirmed at an International Conference on Electrical Units in London in 1908, this conference decided to specify material standards to be calibrated in absolute units and thereafter set up or maintained as working standards. This decision was taken because of the difficulty of making accurate absolute measurements; but the fundamental standards determined on the electromagnetic system based on the centimetre, gramme and second—the standards for which were more permanent than electrical ones—remained.

The four units established by these specifications (defined below) were known as International Units. The Ohm was chosen as the standard.

DEFINITIONS OF INTERNATIONAL UNITS. The **International Ohm** is the resistance offered to the passage of an unvarying electric current by a column of mercury at the temperature of melting ice, of mass 14·4521 g., of uniform cross-sectional area and of length 106·300 cm.

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* J. J. Thompson first gave this equation, and the theory from which it is derived is given also in Laws's *Electrical Measurements*, p. 364.

Although unnecessary for the purpose of definition, the cross-section of such a column is very nearly 1 sq. mm.

The **International Ampere** is the unvarying electric current which, when passed through a solution of silver nitrate in water, "in accordance with Specification II attached to these resolutions," deposits silver at the rate of 0·00111800 g. per sec.

The **International Volt** is the steady electric pressure which, applied to a conductor of resistance 1 International ohm, produces a current of 1 International ampere.

The **International Watt** is the electrical energy per second expended when an unvarying electric current of 1 International ampere flows under a pressure of 1 International volt.

The centimetre, gramme, and second were selected as the units of length, mass, and time, by an International Electrical Congress at Paris in 1881, and were defined by them.

One of the main reasons for adopting an absolute system of units originally was the difficulty of constructing standards which did not vary appreciably with time. Since then, however, wire resistance standards have been developed which are sufficiently permanent for their use to act as a better method of maintaining the international ohm than by occasionally setting up and measuring the "mercury" ohm. Thus it has become the practice of the national standardizing laboratories to maintain the international ohm by wire resistance standards.

As also, by 1930, it was clear that the absolute ohm and ampere could be determined as accurately as the international units, a decision was finally taken in 1946 to abandon the international units, reverting to the fundamental units defined in 1908. The date of the change-over was 1st January, 1948.

Following a number of determinations of the absolute ohm and ampere by various national standards laboratories during the period 1934–1942 the National Physical Laboratory has adopted the following conversion factors—

1 international ohm = 1·00049 absolute ohms
1 ,, ampere = 0·99985 ,, ampere
from which, 1 ,, volt = 1·00034 ,, volts
1 ,, watt = 1·00019 ,, watts
1 ,, henry = 1·00049 ,, henrys
1 ,, farad = 0·99951 ,, farad

An appendix to Reference 66 gives "Definitions of the units recommended by The International Committee on weights and Measures for Legal and Similar Purposes."

Legal Standards. For legal purposes it is necessary to lay down some simpler, if less accurate, standards than those referred to above.

It was laid down by an Order in Council (London Gazette, 1949, No. 38683, p. 3810) that the legal standards shall be—

Electrical Resistance. A standard of electrical resistance denominated one Ohm, agreeing in value within one hundredth part of 1 per cent with that of the fundamental unit, and being the resistance between the copper terminals of the instrument marked "Board of Trade Ohm Standard verified, 1894 and 1909," to the passage of an unvarying electrical current when the coil of insulated wire forming part of the aforesaid instrument is in all parts at a temperature of 14·9° C.

Electrical Current. A standard of electrical current denominated one Ampere, agreeing in value within one tenth of 1 per cent with the fundamental unit, and being the current which is passing in and through the coils of wire forming part of the instrument marked "Board of Trade Ampere Standard verified, 1894 and 1909," when on reversing the current in the fixed coils the change in the forces acting upon the suspended coil in its sighted position is exactly balanced by the force exerted by Gravity in Teddington upon the iridioplatinum weight marked *A* and forming part of the said instrument.

Electrical Pressure. A standard of electrical pressure denominated one Volt, agreeing in value within one tenth of 1 per cent with the fundamental unit and being the pressure which when applied between the terminals forming part of the instrument marked "Board of Trade Volt Standard verified, 1894 and 1909 and 1948," causes that rotation of the suspended portion of the instrument which is exactly measured by the coincidence of the sighting wire with the image of the fiducial mark *A* before and after application of the pressure and with that of the fiducial mark *B* during the application of the pressure, these images being produced by the suspended mirror and observed by means of the eyepiece.

The legal standards are maintained at the National Physical Laboratory.

Absolute Measurement of International Units. 1. MEASUREMENT OF RESISTANCE. In Table II resistance has the dimensions $LT^{-1}\mu$ in the electromagnetic system. The dimensions are those of a velocity and thus absolute measurements of resistance involve the measurement of either a velocity, or of length and time, which determine a velocity. Such measurements often involve the measurement of inductance and time, since inductance has the dimensions of length in the electromagnetic system.

At least eight different methods have been used, but only one* can be given here.

Lorenz Method. This method, originally used by Lorenz in 1873, has since been used for the absolute measurement of resistance, sometimes in a modified form, by a number of investigators.

At The National Physical Laboratory the latest determination of the Ohm, by this method, was made in the years 1933–1936 (*see* P. Vigoureux, *N.P.L. Collected Researches*, 1938, Vol. 24, p. 277). For descriptions of American work on the subject see H. L. Curtis, "A Review of the Methods for the Absolute Determination of the Ohm," *Journal of the Washington Academy of Sciences*, 1942, Vol. 32, p. 40, H. L. Curtis, C. Moon and C. M. Sparks, "A Determination of the Absolute Ohm using an Improved Self Inductor," *Journal of Research of the National Bureau of Standards*, 1938, Vol. 21, p. 375 and H. L. Curtis, "Review of Recent Absolute Determinations of the Ohm and Ampere," *Journal of Research of the National Bureau of Standards*, 1944, Vol. 33, p. 235.

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* Other methods are given in the *Dictionary of Applied Physics*, Vol. II' in the section on "Electrical Measurements."

In the original experiments a circular metal disc, mounted concentrically inside a solenoid, was driven at a uniform speed of rotation.

A steady current was passed through the solenoid, in series with which was a low resistance R, from the terminals of which leads were taken to two small brushes, one pressing on the edge of the rotating disc and another making contact with the disc near its centre. A sensitive galvanometer was included in one of these leads as shown in Fig. 31.

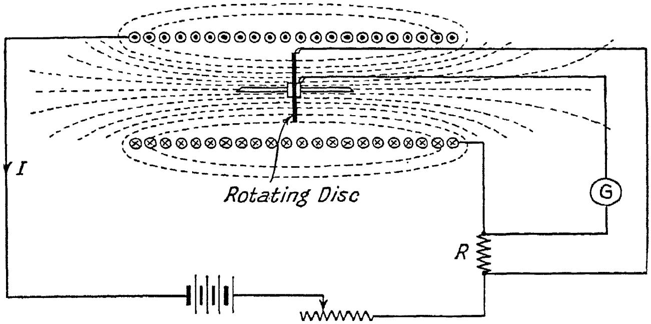

FIG. 31. LORENZ METHOD FOR THE ABSOLUTE MEASUREMENT OF RESISTANCE

As the disc rotates e.m.f.'s are induced in it, since it is placed at right angles to the field of the solenoid. The connections from the brushes on the disc to the terminals of R are so made that the induced e.m.f. in the disc is opposed by the pressure drop due to the solenoid current I in the resistor R. Thus, when the induced e.m.f. is exactly equal to the pressure drop, IR, no current passes through the galvanometer, which therefore gives no deflection.

Let M be the mutual inductance between the disc and the solenoid.

i.e. M = the magnetic flux passing perpendicularly through the disc surface when 1 e.m.u. of current flows in the solenoid.

Thus, the flux cutting the disc when I units of current flow through the solenoid = MI lines. The speed of rotation of the disc (together with the current I and resistance R, if necessary) can be adjusted until no current flows through the galvanometer.

Let N rev. per sec. be the speed of rotation for zero galvanometer deflection.

Then e.m.f. induced in the disc = MIN e.m. units of e.m.f.

Pressure drop in the resistance = IR e.m. units of e.m.f.

I and R being expressed in e.m.u.

Thus $$MIN = IR$$

or $$R = MN \text{ e.m. units of resistance} \quad . \quad (70)$$

The value of the mutual inductance M is calculated from the dimensions of the solenoid and disc, and from their relative positions, using such methods as those described in Chapter V.

As a check upon this expression from the point of view of the dimensions of the quantities involved, consider the dimensions of the product MN.

$$[M] = \frac{\text{Magnetic flux}}{\text{Current}} = \frac{[L^{\frac{3}{2}}\ M^{\frac{1}{2}}\ T^{-1}\mu^{\frac{1}{2}}]}{[L^{\frac{1}{2}}\ M^{\frac{1}{2}}\ T^{-1}\mu^{-\frac{1}{2}}]} = [L\mu]$$

$$[N] = \frac{\text{Revolutions}}{\text{Seconds}} = [T^{-1}]$$

$\therefore\ [MN] = [LT^{-1}\mu]$ which are the dimensions of resistance.

If the resistance R is that of a column of mercury of known dimensions, the resistivity of mercury can thus be obtained in absolute measure, from which the resistance of the international unit of resistance in absolute units can be calculated. Otherwise the resistance R may be some resistance whose magnitude, in terms of the international unit, is known to a high degree of precision.

The e.m.f.'s in the disc may be thought of as existing in an infinite number of radial elements, each cutting through a field of flux density equal to $\frac{MI}{\text{area of the disc}} = \frac{MI}{\pi r^2}$ where r is the radius of the disc in centimetres. The e.m.f. across the brushes is thus that induced in a radial element of length r cm., moving with a mean linear velocity of $\pi r N$ cm. per sec., through a field of flux density $\frac{MI.}{\pi r^2}$ Thus, from Equation (53), the e.m.f. induced in this element (i.e. the e.m.f. across the brushes) is

$$\frac{MI}{\pi r^2} \,.\, r \,.\, \pi r N = MIN \text{ e.m.u.}$$

which is the same as the expression given above.

Precautions Necessary to Ensure Accuracy of Measurement. To obtain an accuracy of measurement of the resistance of 1 part in 10,000, both M and N must be determined with an accuracy of a few parts in 100,000.

The speed N may be determined by stroboscopic methods (see Chapter XXII) or by a directly driven chronograph, the latter being F. E. Smith's method (see Ref. (1), (5)). He also incorporated a fly-wheel to ensure uniformity of speed.

To obtain the necessary accuracy in the value of M, both the disc and solenoid must be carefully constructed and their dimensions accurately measured. The former of the solenoid is usually a marble cylinder, very carefully machined the dimensions being obtained by the use of precision measuring apparatus. The winding is of bare copper wire, wound in grooves cut in the cylindrical surface.

Since the effective dimensions of the disc, when rotating, cannot be obtained with the same accuracy, the value of M is made as little dependent upon these dimensions as possible by suitably choosing the dimensions of the solenoid relative to the disc diameter. The disc is usually of phosphor-bronze.

The effect of the earth's magnetic field upon the e.m.f. induced in the disc is made small by arranging the plane of the latter in the magnetic meridian. Two measurements are made—one with the current I reversed—to eliminate this effect.

To reduce the effects of thermo-electric e.m.f.'s at the brush contacts, F. E. Smith used two phosphor-bronze discs of special construction, and two solenoids.

The accuracy of such absolute measurements depends upon the precision with which the component apparatus can be made and upon that of physical measurements such as those of length and time. It also depends on the variation of the dimensions with time.

Length can now be measured, under the experimental conditions applying to such work, to within 0·0001 mm. giving an accuracy of about one part in a million for the components used in these measurements. Time, or frequency, can be measured to about one part in ten million.

The overall uncertainty in the values determined for the ohm is probably not more than 20 parts in a million and the most recent comparisons of the results of determinations at various national laboratories show a total spread of only 12 parts per million for the ohm and 4 parts per million for the ampere.

2. MEASUREMENT OF CURRENT. The dimensions of current, in the electromagnetic system, being $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$, the dimensions of (current)^2 are LMT^{-2} if μ is regarded as non-dimensional. But these are the dimensions of force, so that absolute measurements of current involve the measurement of

This force may be exerted in two ways—

(*a*) By the current in a solenoid upon a suspended magnetic needle—as in a tangent or sine galvanometer.

(*b*) By the current in one part of a circuit upon another part of the circuit in series with it, and carrying the same current—as in an electrodynamometer or current balance.

Galvanometer methods suffer from the disadvantages that there is always some uncertainty about the exact position of the poles of the magnetic needle used, and also that the horizontal component of the earth's magnetic field must be separately determined with great accuracy before the results of current measurements can be interpreted.

Electrodynamometers measure current in terms of the torsion of a suspension wire or of a bifilar suspension, and this is not very satisfactory. Methods of measurement which utilize some form of current balance are therefore probably the most satisfactory, and are most commonly used.

Tangent Galvanometer Method If a current of I e.m. units flows in the coil of a tangent galvanometer, it can easily be shown that the steady deflection θ is such that—

$$\frac{Hr}{2\pi N}\tan\theta \qquad \qquad (71)$$

where r = the mean radius of the galvanometer coil in centimetres,
N = number of turns on this coil,
and H = the horizontal component of the earth's magnetic field (see Fig. 32).

Obviously the current can be obtained from this expression, in terms of the deflection, the dimensions of the coil, and H.

The following assumptions are made in deriving this expression—

(*a*) That the plane of the galvanometer coil lies exactly in the magnetic meridian, and is exactly vertical.

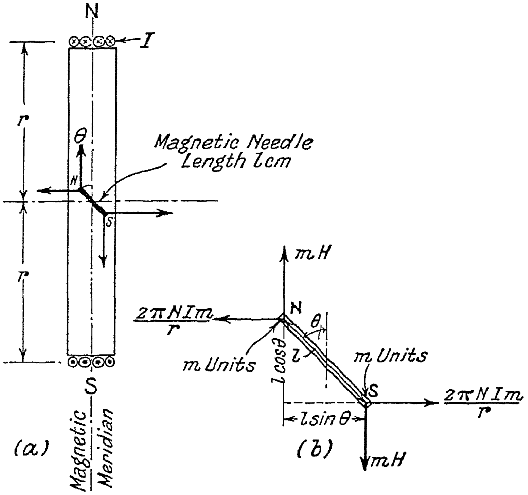

FIG. 32. TANGENT GALVANOMETER

(*b*) That the magnetic needle is infinitesimally short.

(*c*) That the needle is suspended at the *exact* centre of the coil.

(*d*) That the axis of the needle is horizontal.

These assumptions are obviously not all justifiable in practice. Again, unless the galvanometer coil has only a single layer, and is exactly circular, the value of r may be somewhat uncertain. The accuracy of the measurement depends, also, directly upon the accuracy with which H is known for the particular place at which the measurement is being made. This last is a great disadvantage of the method, since it usually necessitates a separate—and highly accurate—determination of H, and this is about as difficult a measurement as that of the current itself. Kohlrausch devised a method of measuring H and the current simultaneously (see *Philosophical Magazine*, Vol. XXXIX), but the method does not appear to have been adopted generally.

Corrections can be applied to allow for some of the divergencies between practice and theory. Two of these—given by F. E. Smith in the *Dictionary of Applied Physics*, Vol. II, p. 231—are as follows—

(*a*) To allow for the fact that all the turns on the galvanometer coil cannot be coincident in space, the field intensity at the centre of the coil is taken as

$$\frac{\pi NI}{d} \log_e \frac{r + d + \sqrt{(r + d)^2 + b^2}}{r - d + \sqrt{(r - d)^2 + b^2}}$$

instead of $\frac{2\pi NI}{r}$ as assumed in the elementary theory of the galvanometer. In this expression $2b$ = axial length of the coil, $2d$ = radial depth of the coil, both in centimetres. This expression is due to A. Gray, and is given in his *Absolute Measurements*, Vol. II, Part I.

(*b*) To allow for the fact that the centre of the needle is not exactly at the centre of the coil, the correction factor to be applied to the field intensity due to the current is

$$1 + \frac{3}{2} \cdot \frac{\delta y^2 + \delta z^2 - 2\delta x^2}{r^2}$$

where δx, δy, and δz are the displacements of the centre of the needle relative to the centre of the coil. These displacements are measured, of course, in three, mutually perpendicular, directions, δx being measured along the axis of the coil.

If, however, corrections are to be applied to allow for all departures from the theoretical assumptions, the method becomes very cumbersome.

Helmholtz modified the tangent galvanometer by adding a second coil and placing the needle midway between the two coils in the uniform field produced by this arrangement (see Chapter I). The correction for axial displacement of the centre of the needle from the centre of a coil is thus rendered unnecessary.*

Rayleigh Current Balance. The principle of this instrument will first of all be discussed. If a current-carrying coil is placed with its plane parallel to that of another current-carrying coil and in such a position that their axes are coincident, a force—either of attraction or repulsion—will exist between the coils, depending upon the current directions. This force is proportional to the product of the two currents in the coils. If the coils are connected in series, so that the same current flows through both, the force between them is proportional to the square of the current passing. This force can be measured if one of the coils is movable, and is suspended from one arm of a balance, the force thus being "weighed"; hence the name "current weigher" given to such instruments. Lord Rayleigh and Mrs. Sidgwick, in their experiments for the determination of the electro-chemical equivalent of silver, used two parallel coaxial fixed coils with a moving coil suspended between them, the three coils being so arranged relative to one another that the force upon the moving coil was maximum. This arrangement is shown in Fig. 33.

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* The theory of this galvanometer is given in Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II, Part 1, p. 254.

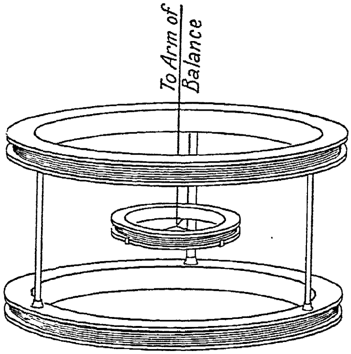

FIG. 33. ARRANGEMENT OF COILS CURRENT BALANCE

The force acting on the moving coil, and measured by the balance, is given by

$$F = I^2 \frac{dM}{dx} \text{ dynes} \quad . \quad (72)$$

where I is in electromagnetic C.G.S. units and is the current in the three coils in series. M is the mutual inductance of the coils, and depends upon their numbers of turns, and upon their dimensions and relative positions. dx is an element of length along the axis of the three coils. The value of M can be calculated from the dimensions of the coils by means of formulae given by Gray (Ref. (7)) or by J. V. Jones (Ref. (8)).

In the above apparatus, if the three coils are so placed that the moving coil is at a distance of half their radius from each of the fixed coils, the value of $\frac{dM}{dx}$ becomes dependent only on the ratio $\frac{\text{radius of fixed coil}}{\text{radius of moving coil}}$. Under these circumstances, also, very little error is introduced by a slight inaccuracy in the axial position of the moving coil (see Chapter I).

Bosscha (Ref. (9)) introduced an electrical method of measuring the ratio of the coil radii which does away with the necessity for measuring the mean radii of the coils themselves—somewhat uncertain measurements in the case of multi-layer coils.

The measurement of current in absolute units, by means of the Rayleigh balance, thus becomes little more than a careful weighing, very accurate measurements of dimensions being avoided. This is perhaps the greatest advantage of this form of current balance. Other advantages, common to all forms of current balance, are that neither measurement of the horizontal component of the earth's magnetic field, nor a determination of the torsion constants of a suspension are required.

The weighing is usually carried out by observing the change in the weights necessary to balance the moving coil when the current in it is reversed, this having the effect, of course, of reversing the force upon the moving coil. It should be noted that the expression for the force, given above, is in dynes; and, since the weights used in the weighing will be grammes, the value of g—the acceleration due to gravity—must be known. In some forms of current balance the accuracy with which g is known determines the accuracy of the current measurement.

Many forms of current balance have been constructed on this principle, and have been used for the determination of the International ampere in absolute units.* At The National Physical Laboratory the most recent determination of the Ampere was made during the years 1930–35 (see P. Vigourleux, *N.P.L. Collected Researches*, 1938, Vol. 24, p. 173).

In all cases a precision balance of special form is used. Great care is necessary in the construction to ensure that the flexible leads for the purpose of leading current into the moving coil or coils exert no appreciable torque upon the moving system. Other important points are the selection of truly non-magnetic material for the bobbins of the coils. Marble is perhaps the best material from this point of view. Brass is suitable if selected with care. The cooling of the coils, also, is very important, water jackets being used for the fixed coils and a water-cooled chamber being provided for the moving coil.

3. DETERMINATION OF THE INTERNATIONAL VOLT IN ABSOLUTE UNITS. The value of the international volt in terms of the absolute unit of e.m.f. is, from its definition, obtained by Ohm's Law, using the values of the ohm and ampere in absolute measure which have been determined by many investigators as stated above. Thus, from the adopted value 1 International ampere $= 0{\cdot}099985$ e.m.u., and 1 International ohm $= 1{\cdot}00049 \times 10^{9}$ e.m.u., it follows that 1 International volt $= 0{\cdot}099985 \times 1{\cdot}00049 \times 10^{9} = 1{\cdot}00034 \times 10^{8}$ e.m.u.

Standard Resistors. The standard resistor known as the "legal ohm," as representing, for general commercial purposes, the unit of resistance, has already been referred to. Since it is considerably easier to compare resistances than to determine their value in absolute measure, it is convenient to have available standard resistors which can be used as reference standards. For general purposes, measurements of resistance can be made with sufficient accuracy by comparison with such standards. The values of sub-standard resistors can be determined by comparison with these, such sub-standards being used in the calibration of laboratory standards of resistance.

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* Detailed descriptions of these pieces of apparatus are given in the *Dictionary of Applied Physics*, Vol. II. pp. 235, etc., and in Laws's *Electrical Measurements*, p. 90.

One form of standard resistor consists of a coil of platinum-silver wire non-inductively wound on a metal bobbin. The wire is wound as shown in Fig. 34. In this bifilar method of winding, the wire is doubled back on itself before winding. This gives the effect of two wires, side by side, carrying currents in opposite directions. The magnetic fields due to the two currents neutralize one another, giving a very small inductance.

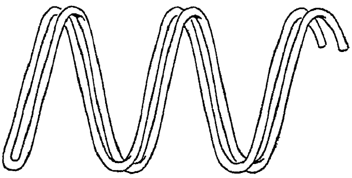

FIG. 34. BIFILAR WINDING

The coil is insulated from the metal bobbin by a layer of shellaced silk which is baked before the wire is wound on. The wire is laid in one layer in order that the cooling shall be as efficient as possible, it being essential that the coil shall not be appreciably heated during use. After winding, the coil is usually shellaced and baked at a temperature of about 140° C. This serves the double purpose of drying out the coil and of anneali the wire, the latter l sary in order to remove conditions of strain, due to bending the wire, and so ensure greater permanence of the resistance of the coil. The coil is fixed inside an outer cylindrical metal case, which has an ebonite top to which the coil and bobbin are attached, and the space between the coil and the outer cylinder is filled with paraffin wax. The terminals consist of long copper rods, hard-soldered to the resistance coil, the ends of these terminals being amalgamated. In use, the coil is maintained at a constant temperature for some hours before measurements are made. This is done by immersing the major portion of it in water.

The Board of Trade ohm is of this form.

More recent designs for standard resistance coils differ mainly in the new materials used for the coil former; e.g. Barber, Gridley and Hall (Ref. 73) have described the construction at N.P.L. of coils wound on Perspex discs. The coils are hermetically sealed in nitrogen.

Many other forms of standard resistor have been constructed, the most important being those designed and constructed by the Standards Laboratories of different countries, such as the German Physikalisch-Technische Reichsanstalt, the American Bureau of Standards, and the National Physical Laboratory of this country.*

After experiments with platinum wire and a gold-chromium alloy it has been concluded that forms of manganin are the best materials for standard resistance coils. By suitable heat treatments and mountings, freeing the coils from strain, and by enclosing them in hermetically sealed containers, it is possible to obtain a constancy over a year of less than one part in ten million.

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* Several forms are described in the *Dictionary of Applied Physics*, Vol. II, p. 700, etc., and in the publications mentioned in Refs. (2), (10), (11), (12), (66), (69), (70), (71), (72), at the end of this chapter.

The reference standard of the N.P.L. consists of a group of five or more coils of nominal value 1 ohm which have shown the greatest relative constancy over preceding years. The average value of the group is assumed to have remained constant and values with an accuracy of one part in a million are assigned to the reference standards for the purpose of international comparisons.

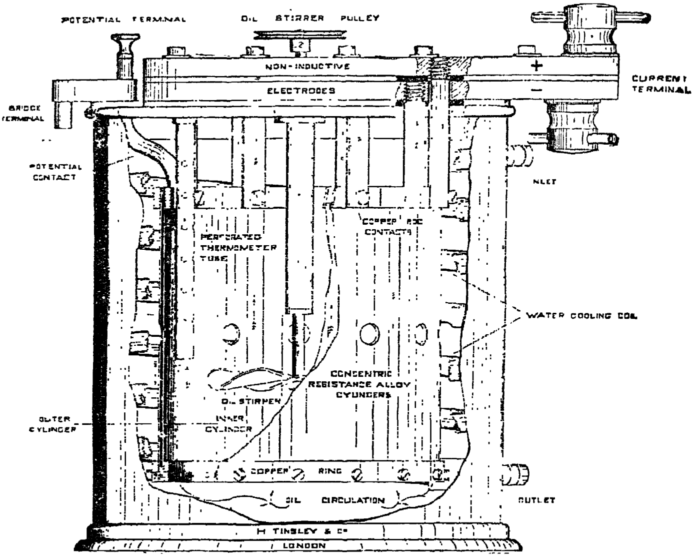
Requirements of Standard Resistors. The most important properties of resistors which are to be used as standards of reference are—

1. Permanence. The necessity for this property is obvious. In order to avoid variation, with time, of the resistance value of the finished standard, annealing during manufacture is essential. Thorough drying out by baking after covering the wire insulation with shellac is also necessary and, if the coil or strip is immersed in oil for cooling purposes, care must be taken to ensure that the oil is free from acid and water, in order to avoid corrosion of the resistance alloy.
2. Robust construction.
3. A small temperature coefficient of resistance, in order that the correction for temperature variations shall be small.
4. Small thermo-electric effects, when a current is passed through it.

5. Such resistors should also have as low an inductance as possible and should be capable of carrying an appreciable current without overheating.

Low Resistance Laboratory Standards. In the case of low resistance standards such as those used for potentiometer work, the currents to be carried are often very large and adequate cooling must be provided, this being done by immersing the resistor in oil (first grade paraffin oil being often used for this purpose), the oil being stirred by a motor-driven stirrer and water cooling being provided. A distinction should be made, however, between resistors of this latter class and those which are used for reference purposes only, and are not required to carry large currents. These low resistance standards are fitted with potential terminals as well as current terminals. The potential terminals fix the points on the resistor between which the nominal resistance of the standard is measured. The current terminals, by means of which the resistor is connected to the supply circuit, should be at an appreciable distance from the tapping points of the potential leads in order that the current distribution shall be uniform, throughout the cross-section of the resistance material, before the tapping points are reached (*see* Ref. **(13)**).

Fig. 35 shows a low resistance standard of the Drysdale-Tinsley non-inductive type, designed to carry heavy currents such as may be required in potentiometer and other work. In addition to the ordinary current and potential terminals, it has, fitted to the potential terminals, mercury contacts for use in a standardizing bridge (see Chap. VII). This type of resistor is manufactured by Messrs. H. Tinsley & Co., and is designed for use with either direct or alternating currents (up to 1,000 ∼). The resistance material used is manganin, silver-soldered to copper rings, which are screwed to heavy copper lugs to which they are also soldered with tin-lead solder. The manganin resistance strips are in the form of concentric cylinders, through which the current passes axially in opposite directions, thus giving a very low inductance. A range of resistors of the type shown in the figure is manufactured, having resistance values from 0·02 ohm down to 0·0001 ohm. The watts dissipated are 200 for resistors from 0·02 ohm down to 0·005 ohm and 500 from 0·001 ohm down to 0·0001 ohm.



(*H. Tinsley & Co., Ltd.*)

FIG. 35. DRYSDALE-TINSLEY NON-INDUCTIVE LOW RESISTANCE STANDARD (500 Watt Type)

Fig. 36 shows the construction of air-cooled low resistance standards manufactured by W. G. Pye & Co.

Resistance Materials. It is desirable that a material to be used in the construction of standard resistances should possess the following properties—

(*a*) High specific resistance, in order that the standard resistor, when constructed, may be reasonably compact.

(*b*) Permanence. There should be as little variation in resistance with time as possible.

(*c*) It should have a low thermo-electric force with copper.

(*d*) Low temperature coefficient, in order that the correction for temperature variation may be small.

(*e*) It should not easily oxidize, and should be unaffected by moisture, acids, etc.

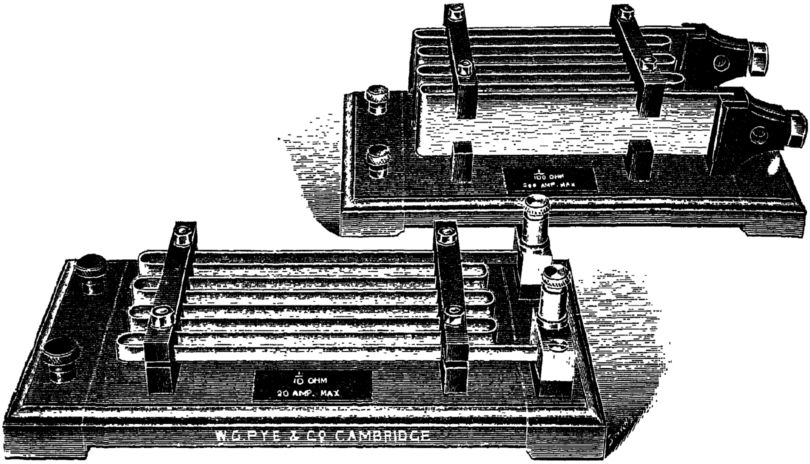
In addition, it should, if possible, be easily worked and jointed.

From inter-comparison, over a long period of years, by various investigators, of a number of standard coils made up in 1864 by Mathiessen and Hockin, on behalf of the British Association, it appeared that platinum was the best material from the point of view of permanence though later work has shown the superiority of manganin. Platinum has the disadvantage of a high temperature coefficient—about 0·4 per cent per 1° C. Many alloys, such as

TABLE V

PROPERTIES OF OTHER RESISTANCE MATERIALS

Material	Composition (approx.)	Specific Resistance (microhms per cm.-cube)	Temperature Coefficient (% per ° C.)	Thermo-electric e.m.f. against Copper (Micro-volts)	Remarks
Therlo	Copper 71 % Aluminium 16·5% Manganese 10·5% Iron 2%	47 (at 20° C.)	0·0005	Very low	Comparatively new material. Properties similar to manganin.
Platinum-silver	1 part platinum, 2 parts silver	31·6	0·03	Small	High temperature coefficient.
Constantan	Copper and nickel	50 (at 20° C. approx.)	− 0·001	40	Cheap. Easy to work. High thermo-electric e.m.f. is a disadvantage.
Eureka	Copper 60 % Nickel 40%	As for Constantan.			
German silver	Copper 63 % Zinc 22% Nickel 15%	30 (at 20° C.)	0·03	35	The presence of zinc in alloys produces unstable properties.
Platinoid.	German silver with addition of about 1% tungsten.	34 to 40	0·02 to 0·03	20	Tungsten improves the permanence.
Nichrome		95 (at 20° C.) approx.	0·04		Used for resistances of rougher class, especially at high temperatures. Is non-corrosive.
Platinum		11	0·36		Used in resistance thermometry.
Iron		12	0·4		Used for resistances when its magnetic properties and high temperature coefficient are unimportant.



(*W. G. Pye & Co., Ltd.*)

FIG. 36. AIR-COOLED LOW RESISTANCE STANDARDS

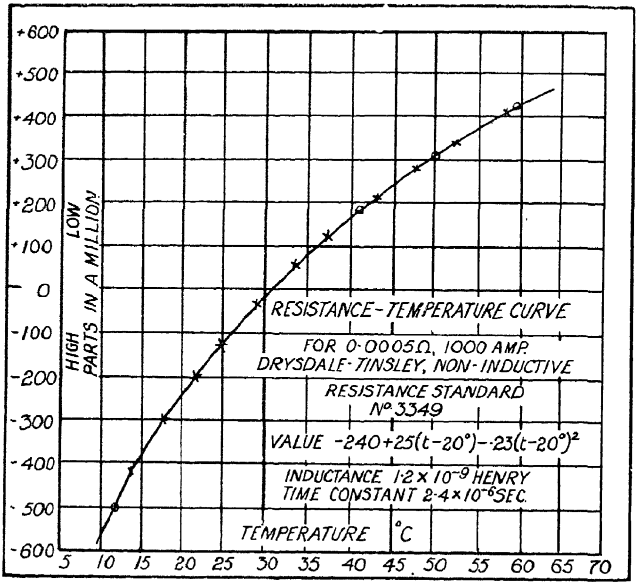


FIG. 37. RESISTANCE-TEMPERATURE CURVE FOR DRYSDALE-TINSLEY NON-INDUCTIVE RESISTANCE STANDARD

platinum-silver, platinum-iridium, German silver, manganin, etc., have been used as resistance materials, and much research, beginning with the work of Mathiessen (Ref. (14)) has been carried out upon the subject.

Manganin. Weston, in 1889, discovered that alloys of copper, manganese, and nickel, have a very small temperature coefficient. He gave the name "manganin" to such alloys. Lindeck (Ref. (12)), Bash, and others, have since further investigated the properties of the material, and it has been found that the composition—84 per cent copper, 12 per cent manganese, 3·5 per cent nickel, and 0·5 per cent iron—has an extremely low temperature coefficient and is most suitable for resistance purposes.

The temperature coefficient of manganin changes sign as the temperature increases, the point at which the change occurs depending upon the iron content.

Fig. 37 shows a temperature-resistance curve for a standard of low resistance of the type shown in Fig. 35, such a curve being supplied by the makers with each standard supplied.

A representative value for the specific resistance of manganin is 50 microhms per cm. cube at 20° C., and of the temperature coefficient + 0·0004 per cent per 1° C. at 20° C. The thermo-electric e.m.f. against copper is from 3 to 8 microvolts per 1° C.

The annealing of manganin to remove bending strains, as mentioned above, must be carried out with the material out of contact with air, as oxidation will otherwise take place, resulting in a high temperature coefficient. This is best done by heating the material to a temperature of 600–700° C. in an atmosphere of some chemically inert gas, although the method often adopted is to subject the material to a gradual annealing over a period of several hours at a temperature of about 140° C., the material being coated with a protecting coat of shellac varnish.

The disadvantage of this method is that, as discovered by Rosa and Babcock, of the Bureau of Standards in 1907, the shellac absorbs moisture from the atmosphere, which causes it to swell and this stresses the wire, causing appreciable variation of resistance with time.

Current Standards. It is obviously impossible to set up a standard of current in the same sense that a standard of resistance can be set up. Current is defined either in terms of the electrochemical equivalent of silver, or in terms of readings obtained when the current is passed through some standard instrument such as the current balance already mentioned. Arguments have been advanced in favour of concretely defining the volt instead of the ampere for the purpose of International units. The ease with which any voltage can be measured by comparison with the voltage of a standard cell, using a potentiometer, and the accuracy with which a standard cell can be set up, being two of the advantages of using the volt as one of the primary International units.

Kelvin Current Balance. The current balance as used for the determination of the absolute value of the ampere has already been described. Lord Kelvin designed an instrument the action of which depends upon the same principle which has since been largely used for the accurate measurement of current. Secondary current balances are of this type, the instrument mentioned in the definition of the legal ampere being a special form of Kelvin balance.

An instrument of this type has been used for many years in the measurement of currents of from 0·1 to 10 amperes. The instrument consists of six coils, four fixed and two moving, the latter being carried on a beam which can rotate in a vertical plane, like the beam of a chemical balance. Instead of a knife edge, as the means of pivoting this beam, it is suspended at its centre by two flexible copper ribbons, each consisting of a large number of fine wires.

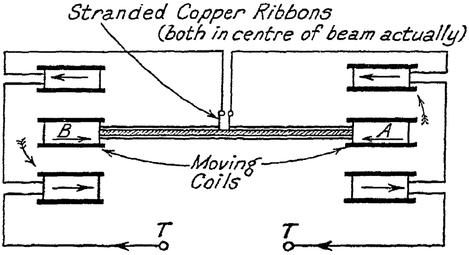


FIG 38. ARRANGEMENT OF COILS KELVIN CURRENT BALANCE

These ribbons also act as leads to the moving coils. The latter are situated between the two pairs of fixed coils as shown in Fig. 38, all six coils being connected in series, the connections being such that the currents flow as shown. Under these conditions the top fixed coil on the right attracts the adjacent moving coil A, while the bottom fixed coil repels A. On the left the top fixed coil repels the adjacent moving coil B, while the bottom fixed coil attracts B. The total effect is thus to cause an anti-clockwise movement of the beam carrying the moving coils. This anti-clockwise torque is balanced by means of weights carried by a small carriage which runs on a graduated bar attached to the moving beam. This carriage is moved by means of cords which pass through holes in the case of the instrument. To ensure that the weights shall always be placed in the same position on the carriage, the latter is fitted with two small conical pins, which fit into holes in the weights.

To use the instrument, a known weight, whose value is suitable for use with the current to be measured, is placed on the carriage, and a counterpoise of the same value is placed in the aluminium V-shaped trough attached to the right-hand moving coil. The carriage is then moved to zero at the left-hand end of the graduated scale, and the clamping device, for removing the weight of the moving system from the copper ligaments when the instrument is not in use, is freed. The moving system should then be balanced as indicated by the pointers—one at each end of the beam—which move over small vertical scales attached to the base. A means of adjustment is provided to obtain complete balance with zero current if this condition should not be obtained without.

When current flows through the instrument, the anti-clockwise torque produced by it is balanced by moving the carriage, with its weights, along the scale to the right. If $2l$ cm. is the length of the scale, and balance is obtained with a movement of the weight of x cm. from zero, then, the moving system being suspended at its centre, the total turning moment due to the weights, each of weight W grammes (say), is $Wl - W(l-x) = Wx$ cm.-gm. At balance this turning moment is equal to that due to the current, which latter is proportional to the square of the current. The current I is calculated from the equation

$$I = \qquad . \quad (73)$$

where D is the displacement of the moving weight in scale divisions for balance, and K is a constant for the instrument which depends also upon the weight used. A fixed inspectional scale for approximate readings is fitted behind the moving scale, the former being graduated in terms of $2\sqrt{D}$. Four sliding weights and four counterpoise weights are supplied with the instrument, in order to obtain different ranges, the carriage constituting the smallest of the sliding weights. These four weights are in the ratio 1, 4, 16, 64. The first being the weight of the carriage; the last three are 3, 15, and 63 times the weight of the carriage respectively.

Kelvin balances can be used with alternating current as well as direct, since, the currents in all the coils being the same (because they are in series), all the magnetic fields of the coils reverse direction together, thus producing a turning moment which is always in the same direction.

Voltage Standards—Standard Cells. The Conference on Electrical Units and Standards of 1908 suggested the use of the Weston normal cell as a concrete standard of e.m.f. The advantages of the Weston cell over the Clark cell, as given by Wolff (Ref. (4)) may be summarized briefly as follows—

1. A temperature coefficient of less than $\frac{1}{20}$th that of the Clark cell.

2. Small hysteresis effects attending temperature variations, while in the Clark cell such effects are large, particularly in old cells.

3. Much longer life than the Clark cell, which has a tendency to crack at the point of introduction of the negative terminal wire.

4. In the Clark cell a layer of gas is sometimes formed in the cell which interrupts the circuit. This does not occur in Weston cells.

Both types of cells are essentially voltaic cells of a special form, the metals used in them—mercury, zinc, and cadmium—being such as can be obtained with a high degree of purity. This is essential in a cell whose e.m.f. is to be as permanent as possible.

Weston Standard Cell. This cell, patented by Weston in 1892, consists essentially of mercury, as the positive element, and cadmium amalgam—a solution of 1 part of cadmium in 7 parts of mercury—as the negative element. The electrolyte is a saturated solution of cadmium sulphate, and the depolarizer mercurous sulphate. To ensure saturation of the electrolyte, cadmium sulphate crystals are added to it. Lord Rayleigh suggested the *H* form shown in Fig. 39, the two limbs of which are hermetically sealed. The connections to an external circuit are made by platinum wires sealed into the glass. In both Clark and Weston cells it is most essential that the mercurous sulphate shall be especially pure, as impurities in it have a much greater effect upon the permanence of the e.m.f. of the cell than have slight impurities in most of the other chemicals employed.

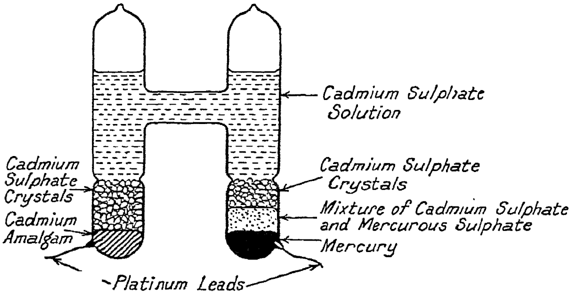

Cadmium Sulphate Solution
Cadmium Sulphate Crystals
Cadmium Amalgam
Cadmium Sulphate Crystals
Mixture of Cadmium Sulphate and Mercurous Sulphate
Mercury
Platinum Leads

FIG. 39. WESTON STANDARD CELL

The voltage of the cell when constructed in accordance with the standard specification is 1·01824 International volts or 1·01859 absolute volts at 20° C. This falls 40 microvolts per degree Centigrade rise in temperature.

Clark Cell. This cell, described by its inventor, Latimer Clark (Ref. (19)) was recommended by him as a standard of e.m.f. It has been largely used as such, the reasons for its not being adopted as an international standard being given in the previous paragraphs. It consists of a mercury positive electrode and a zinc amalgam negative electrode (10 per cent zinc in mercury). The electrolyte is a saturated solution of zinc sulphate, crystals of which are added to ensure saturation of the solution. The mercury positive electrode is covered with a paste of zinc and mercurous sulphates, together with finely divided mercury, in a saturated zinc sulphate solution. The mercurous sulphate is for the purposes of depolarization and, as in the Weston cell, must be very carefully prepared if instability of e.m.f. is to be avoided. Two forms of Clark cell are shown in

Fig. 40. The *H* type shown (*a*) was designed by Lord Rayleigh, while the cell shown in (*b*) was designed by Kahle, of the German Reichsanstalt.

No specification for this cell was drawn up by the International Committee, but a full specification is given in the *Dictionary of Applied Physics*, Vol. II. The e.m.f. of the Clark cell at 15° C. is 1·4326 International volts, equivalent to 1·4331 absolute volts, the law of variation of e.m.f. with temperature being given, in the *Dictionary of Applied Physics*, as

$$E_t = E_{15} - 0{\cdot}00119\ (t-15) - 0{\cdot}000007\ (t-15)^2$$

between the limits 10° C. to 25° C.

Lord Rayleigh's formula for the e.m.f. variation with temperature, given in 1886, was

$$E_t = E_{15} \,.\, \{1 - 0{\cdot}00077\ (t-15)\}$$

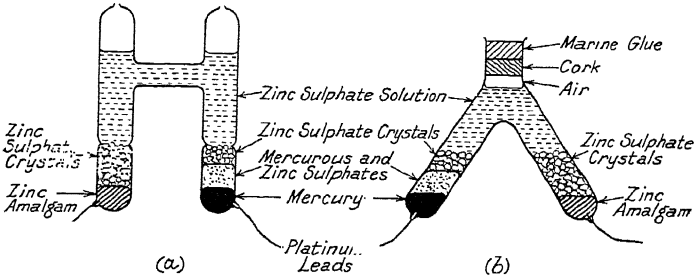


FIG. 40. CLARK STANDARD CELLS

Precautions when Using Standard Cells. Great care must be taken to ensure that when in use no appreciable current is taken from a standard cell, as the e.m.f. is only strictly constant on open circuit. The voltage falls when a current is taken from such cells, and although they recover after a time, such disturbances are undesirable and may lead to considerable errors in measurement. Standard cells are thus only used in null methods of measurement such as in measurements by the potentiometer. A high resistance should be connected in series with the standard cell, which protects it during the initial manipulations of the apparatus and which can be cut out when approximately balanced conditions are obtained.

Care should be taken also in moving a standard cell, as any appreciable shaking up of the chemicals in the cell tends to produce variations of e.m.f.

For storage purposes a dry position having a fairly uniform temperature of about 15° to 20° C. should be selected, in order to avoid troubles from hysteresis effects due to temperature variations, and to avoid any possibility of leakage currents owing to moisture on the insulating material between the terminals of the cell.

The use of a group of standard cells as a working standard in international comparisons of voltage is described in Refs. (66) and (69).

Standards of Mutual and Self-Inductance. It has been seen previously (Table II) that the dimensions of inductance in the electromagnetic system are those of length. Thus standards of inductance, both self and mutual, depend for their value upon their dimensions, together with the number of turns of wire in them, this latter being a mere number which has no dimensions.

The self-inductance of a coil, or the mutual inductance of a system of coils, can be calculated from the dimensions of the coils by the use of formulae which have been given by many workers on this subject.*

In the construction of a primary standard of inductance, whether mutual or self, some form must be adopted for which a rigidly accurate formula exists for calculation purposes. The design should be such as to facilitate the accurate measurement of the dimensions of the standard, for, if the formula used is rigidly correct, the errors in the calculated value, as compared with the actual value of the inductance, will depend very largely upon the accuracy of such measurements. There should, also, be no doubt about what lengths should be taken as the effective dimensions of the standard. For this reason the coils are usually single layer, and are often wound with bare wire laid in a screw thread cut in a marble cylinder. Other factors influencing the design are that the dimensions should be subject to as little variation as possible with time in order to ensure permanence of the inductance of the standard, and also that the bobbins used for the coils should be absolutely non-magnetic.

It has been found that marble is the best material for the purpose, its advantages being: (*a*) it does not warp and is unaffected by moisture and atmospheric conditions; (*b*) its electrical resistance is very high, so that it serves as an insulator when bare wire is wound on it; (*c*) its magnetic susceptibility (which would be zero for a completely non-magnetic material) is about $-0{\cdot}97 \times 10^{-6}$, corresponding to a permeability of 0·999988, as given by Coffin (Ref. (25)); (*d*) its coefficient of expansion is only about 0·000004 per degree Centigrade; (*e*) it is comparatively cheap and easy to work, so that any desired shape can be obtained.

It is essential that metal shall be avoided as far as possible in the construction of such coils, as eddy currents set up in metal parts may appreciably affect the value of the inductance of the standard. For the same reason, standards constructed for use with heavy currents, when the conductors must be of large section, employ stranded wire to reduce the eddy current effect. Capacitance effects should also be avoided as far as possible, and the resistance of the windings should be low compared with the inductance.

Measurements of the dimensions of coils to be used as primary standards are carried out by means of a precision measuring apparatus, one form of which is described by Coffin (Ref. (25)).

Primary Standards of Mutual Inductance. Such standards are always fixed standards—i.e. they are of single value. Variable standards of inductance will be described in a later chapter. The general form of such primary standards is a single layer coil, uniformly wound and of circular cross-section, its axial length being large compared with its cross-sectional diameter, which forms the primary circuit, with a coil of small axial length placed at its centre, the latter forming the secondary circuit. The secondary coil may be wound on top of the primary coil, so that their cross-sections are as nearly coincident as possible, or it may be wound on a separate bobbin and placed inside the primary coil, the first form being the better from the point of view of ease of construction and measurement.

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* References to publications giving such formulae are given at the end of the chapter.

The flux density (lines per cm²) at the centre of the primary coil is

$$\cos \theta_1 = B \text{ in lines per square centimetre}$$

where N is the number of turns on the coil, l its axial length in centimetres, and i the current, in e.m.u., flowing in it. θ_1 is the angle between the axis of the coil and a line drawn from the centre point of the axis to a point on the circumference of an end turn of the coil (see Fig. 22).

If the secondary coil, placed at the centre of the primary, has n turns, and is of cross-section a sq. cm., the flux linkages with this secondary coil per unit current in the primary (which is the mutual inductance) is

$$\frac{nBa}{i} = \frac{4\pi\, Nna}{l} \cos \theta_1$$

or

$$M = \frac{4\pi Nna \cos \theta_1}{l} \times 10^{-9} \text{ henries} \tag{74}$$

It should be noted, however, that in the derivation of Equation (74) assumptions are involved which are not quite justifiable for the purpose of calculation of mutual inductance for standards purposes, and that more exact formulae are applied in practice. The above equation gives a fairly close approximation.

Campbell Primary Standard of Mutual Inductance. The Campbell type of primary standard (Ref. (26)) consists of a primary coil of bare copper wire wound under tension in a screw thread cut in a marble cylinder. It is a single layer coil and is divided into two equal parts connected in series and displaced from one another by a distance equal to three times the axial length of one of them. The secondary coil, consisting of a number of layers of wire wound in a channel cut in the circumference of a marble ring, is placed so that it is concentric and coaxial with the primary coil cylinder. This coil is situated midway between the two portions of the primary coil, and a means of adjustment is provided to enable the coil to be brought into the correct position relative to the primary coil.

With this construction the magnitude of the mutual inductance obtainable is much greater than is possible if both primary and secondary coils are single-layered, whilst the difficulty of accurately measuring the effective radius of the multi-layered secondary is overcome by arranging its dimensions so that small variations of radius or of axial position have a negligible effect upon the mutual inductance. With the relative positions of the secondary and the two portions of the primary coil as stated above, maximum mutual inductance is obtained by making the mean radius of the secondary coil about 1·46 times that of the primary coil. This means that the circumference of the secondary coil is situated in the position of zero magnetic field when current flows in the primary coil (see Fig. 41). Thus the mutual inductance will not be appreciably affected by small errors in measurement of the secondary coil radius, or by a small departure from the true midway position between the two halves of the primary winding.

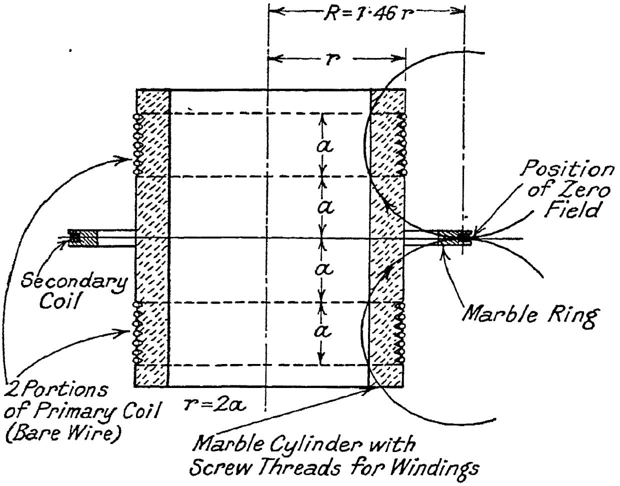
R = 1·46 r
r
a
a
a
a
Position of Zero Field
Secondary Coil
Marble Ring
2 Portions of Primary Coil (Bare Wire)
r = 2a
Marble Cylinder with Screw Threads for Windings

FIG. 41. CONSTRUCTION OF CAMPBELL PRIMARY STANDARD OF MUTUAL INDUCTANCE

The mutual inductance is calculated by J. V. Jones's formula, mentioned previously.

The data for the National Physical Laboratory primary standard constructed on this principle are, as given by Campbell—

Primary Coil—

Number of turns	75 in each half
Diameter	30 cm.
Axial length of each half . . .	15 cm.
Distance between inner ends of the two halves	15 cm.

Secondary Coil—

Number of turns	485
Mean diameter	$43{\cdot}73_7$ cm.
Axial depth	1·00 cm.
Radial depth	0·86 cm.

The mutual inductance of this standard is given as 10·0178 millihenries.

Secondary Standards of Mutual Inductance. Such secondary standards are used as standards of mutual inductance for general laboratory purposes. Since they are not absolute standards it is not essential that their dimensions shall be determined with great accuracy, it being merely essential that they shall have a mutual inductance which is as near as possible to the nominal value for which they are designed. When constructed they are compared with a primary standard, and their mutual inductance is adjusted, if necessary, until it is within, say, 1 part in 10,000 of their nominal value. Such standards are constructed to have nominal values which are either multiples or fractions of the inductance of the primary standard.

Requirements. Since the most important requirement of such pieces of apparatus is that their mutual inductance shall remain constant under all conditions of use, they should have the following characteristics—

(*a*) Their inductance should not vary with time to any appreciable extent. For this reason the materials used must be carefully chosen to avoid warping, and the coils must be firmly fixed in position to avoid relative displacement.

(*b*) Their construction should be such that the mutual inductance varies as little as possible with changes of temperature.

(*c*) Their inductance should be independent of the supply frequency as far as possible. To ensure this, the wire used should be stranded, each strand being insulated from the neighbouring ones, in order to reduce eddy current effects in the wire. The intercapacitance of the windings should be small, also, and the insulation should be as perfect as possible.

Secondary standards usually consist of two coils wound on a bobbin of marble or hard, paraffined wood, the coils being separated by a flange. The wire is stranded copper, with double silk coverings. After winding, the coils and bobbin are immersed in hot paraffin wax. When withdrawn and allowed to cool, the wax firmly fixes the wires in the coils in position.

Adjustment to the value of mutual inductance required is carried out by carrying one end of one of the coils through a further arc of a circle in order to give the effect of a fraction of a turn. Campbell (Ref. (27)) gives a method of adjustment utilizing a third coil, of small diameter, concentric and coaxial with the other two, and connected in series with the secondary (Fig. 42). Adjustment is by alteration of the number of turns on the small coil, a variation of one in the number of turns on the small coil having the effect of a variation of a fraction of a turn on the larger coil.

Primary Standards of Self-inductance. Although mutual inductances are more generally regarded as the primary standards of inductance, owing to the greater accuracy with which their value can be calculated from their dimensions, standards of self-inductance have been constructed at several of the national laboratories already mentioned. Their magnitudes are calculated from formulae previously referred to, and permanence is ensured by winding the coil of bare hard-drawn copper wire under tension in a screw thread cut in a marble cylinder which has been very carefully ground so as to be as nearly truly cylindrical as possible. Descriptions of such standards at the Bureau of Standards and at the Physikalisch Technische Reichsanstalt have been given by J. G. Coffin (Ref. (25), and by Grüneisen and Giebe (Ref. (28)), respectively. These two coils are of inductances 216·24 mh and about 10 mh respectively.

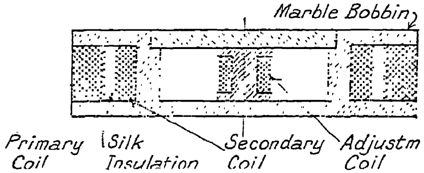

FIG. 42. CAMPBELL SECONDARY STANDARD OF MUTUAL INDUCTANCE

Secondary Standards of Self-inductance. As in the case of secondary standards of mutual inductance, self-inductance secondary standards are constructed to have a nominal value which is usually a simple fraction of 1 henry. Such standards are compared with a primary standard of inductance and are used as reference standards for general laboratory work.

For the purpose of obtaining the largest possible time constant (i.e. ratio $\left(\frac{\text{inductance}}{\text{resistance}}\right)$ when winding an inductance coil, Maxwell recommended the use of the relative dimensions given in Fig. 43. An approximate formula for the inductance of a coil having these relative dimensions is

$$L \doteqdot 6\pi N^2 r \times 10^{-9} \text{ henries}$$

where N = number of turns on coil

r = mean radius of the coil in centimetres.

Since $r = 1{\cdot}85a$

$$L \doteqdot 11{\cdot}1\pi N^2 a \times 10^{-9} \text{ henries} \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Later work by Shawcross and Wells (Ref. 51) on this subject showed that Maxwell did not consider enough terms in the formula which he used in this calculation, and that a coil of shape somewhat similar to that of Fig. 43 but having a mean diameter 3a (instead of 3·7a) gives a slightly greater time constant (0·5 per cent greater).

The formula for the inductance of such a coil (dimensions in centimetres)

$$L : 16{\cdot}83\, N^2 r \times 10^{-9} \text{ henries}$$
$$L : 25{\cdot}24\, N^2 a \times 10^{-9} \text{ henries} \quad . \quad . \quad . \quad . \quad . \quad . \quad (75a)$$

Actually the maximum time constant is obtained by making the mean diameter $2{\cdot}95a$, but $3a$ is more convenient and is a sufficiently close approximation.

Coils for use as secondary standards are wound of silk-covered stranded copper wire on bobbins of marble, or of mahogany impregnated with paraffin. After winding, the coils are immersed in melted paraffin wax for some time. Upon being removed and cooled, the paraffin wax solidifies and rigidly fixes the coil wires in position.

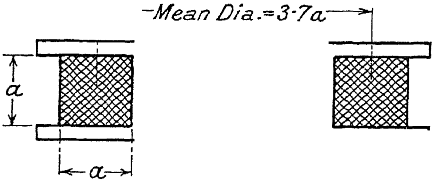

FIG. 43 MAXWELL'S DIMENSIONS FOR SELF-INDUCTANCE STANDARD

The chief cause of non-cyclic temperature coefficient and general instability of inductance is always the non-equality of expansion of the conductor and the radial dimension (W) of the former. A design for stable inductances with sensibly zero temperature coefficient, due to W. H. F. Griffiths, is illustrated in Fig. 44. Members (B) of Keramot are supported rigidly by the end cheeks (D) of special laminated Bakelite. The conductor (A), of thin strip copper or Litzendraht wire, is wound in grooves. The pins (P) form the *chief* locating means for the relative positioning of the B and D members. The temperature coefficient of linear expansion of the D members varies with the direction of the radius owing to its laminated manufacture.

In order that the *resultant* temperature coefficient of the dimension W shall exactly equal that of the linear expansion coefficient of the conductor A, the following condition must be satisfied—

$$\begin{aligned} \delta'(W + X) - \beta X &= \delta''(W + Y) - \beta Y \\ &= \delta'''(W + Z) - \beta Z = \alpha W \end{aligned}$$

where δ', δ'', and δ''' are the measured temperature coefficients of expansion of the D members in the radial directions p, q and r respectively and the expansion coefficients of the conductor and B members are α and β respectively, the latter being uniform in all directions.

From the above expression it is seen that the radial dimensions of the pins P_x, P_y, and P_z are given respectively by—

$$X = \frac{\alpha - \delta'}{\delta' - \beta} \cdot W$$

$$Y = \frac{\alpha - \delta''}{\delta'' - \beta} \cdot W$$

$$Z = \frac{\alpha - \delta'''}{\delta''' - \beta} \cdot W$$

Having thus, by the above determination of the correct dimensions $X + W$, $Y + W$, and $Z + W$, ensured that the temperature coefficient δ of the former may be safely replaced by that, α, of the conductor, it can be shown that the resultant temperature coefficient of inductance is

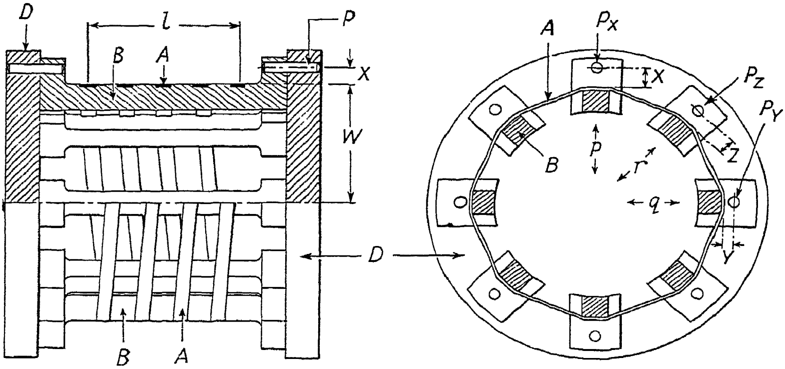

FIG. 44. CONSTRUCTION OF SULLIVAN-GRIFFITHS TEMPERATURE-COMPENSATED SELF-INDUCTANCE STANDARD

where γ is a variable, depending upon the ratio of length to diameter, which has been enumerated for all possible shapes of coil by Griffiths.

The principle can be applied to multi-layered coils up to 1 henry with equal success. The temperature coefficients of *true* inductance are always less than 5×10^{-6} per ° C. and can by careful design be reduced to 10^{-6}.

Griffiths* shows how the temperature coefficient of these inductances may be affected by frequency within certain bands and discusses the effects of humidity, current and self-capacitance upon their ultimate stability.

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* "Recent Improvements in Air Cored Inductances," *The Wireless Engineer*, Vol. XIX, No. 220, pp. 8–19 and No. 220, pp. 56–63. (See also H. W. Sullivan, 1954 catalogue.)

Primary Standards of Capacitance. Such standards are capacitors whose capacitance can be accurately calculated, by means of an exact formula, from their dimensions. Capacitance in the electrostatic system of units has the dimensions of length (Table II). The capacitance of absolute capacitors can thus be expressed in terms of lengths—i.e. of their dimensions—and it is therefore of prime importance that such dimensions shall be very accurately known and also that these dimensions shall not vary once the capacitor has been constructed. Owing to the fact that air is the only dielectric whose permittivity is definitely known and which is free from absorption and dielectric loss (see Chap. IV), it is always used as the dielectric in primary standard capacitors. Three types have been used as primary standards, viz. the concentric spheres type, the concentric cylinders type with "guard rings" (see Chap. IV), and the parallel plate type with guard plates. Of these, the last is perhaps the least satisfactory, as it requires very careful adjustment if the calculated value of capacitance is to be accurately realized. The necessity for the guard rings and the formulae for the calculation of the capacitance of these types will be considered in Chapter IV.

The disadvantages of air as a dielectric in such capacitors are as follows—

(a) Its "dielectric strength" (see Chap. IV) is low, which necessitates a comparatively long gap between plates in order to withstand breakdown of the air when a voltage is applied.

(b) Its permittivity is low compared with solid dielectrics, which fact, combined with the long gap referred to above, means that an air capacitor is very bulky if the capacitance is to be other than very small.

(c) Dust particles, settling in the gap between the plates, cause leakage troubles unless precautions, such as thorough drying of the air in the capacitor, are taken to avoid this. The minimum distance between plates to ensure freedom from dust troubles should be 2 to 3 millimetres.

(d) Since there is no solid dielectric between the plates to act as a spacer, the plates must be rigidly fixed in position by supports of some solid dielectric. Very few of such insulating materials are satisfactory for this purpose, owing to their tendency to warp and cause displacement of the plates from their original position. Fused quartz and amberite are used for such purposes.

Absolute standards of capacitance were originally developed in connection with the measurement of *c*—the ratio of the electromagnetic to the electrostatic C.G.S. unit of quantity—as described earlier in the chapter. Rosa and Dorsey (Ref. (30)) have described fully several types of absolute standards of capacitance constructed by them for this purpose.

Standard Air Capacitors for High-voltage Testing. In recent years the development of methods of measuring the dielectric loss and power factor of capacitors at high voltages (see Chap. IV) has led to the construction of several types of standard air capacitors for use in making such measurements.

These are either of the parallel plate or concentric cylinder type, ? guard rings being employed in each case in order to 1 shield the capacitor from external electrostatic influences and to 2 render more definite the effective area of the electrodes, so that the area to be used in calculating the capacitance from the dimensions shall be subject to no uncertainty. The air gap between the plates must be large in order to withstand the applied voltage, and the edges of the plates must be rounded in order to avoid brush discharges (which would produce a loss of power due to ionization of the air at such edges. For the same reason the surfaces of the plates must be free from irregularities, which necessitates a very careful grinding of these surfaces during the construction. High-voltage capacitors of the parallel plate type have been used by various investigators, including Shanklin (Ref. (31)), who used a high-tension plate suspended from the ceiling by insulating cord, with two low-tension plates, one on either side, the latter being provided with earthed guard rings. This was used up to 60,000 volts. Rayner, Standring, Davis, and Bowdler (Ref. (32)), at the National Physical Laboratory, employed a somewhat similar construction, the high-tension plate in this case having a rounded edge of 3 in. radius. A full description of the capacitor is given by them in the paper referred to. Dunsheath (Ref. (33)) has described a parallel plate capacitor used by him for the same purpose.

The concentric cylinder type of capacitor, developed by Petersen (Ref. (34)) has been more generally adopted, and is more satisfactory than the parallel plate type owing to the difficulty of efficiently screening the latter. Petersen's form consists of a cylindrical low tension electrode with a guard cylinder of the same diameter at each end. This is surrounded by the high-tension cylinder, which is concentric with the inner one and which projects beyond the ends of the low-tension cylinder by a considerable length at each end. The ends of this high-tension cylinder are bell-shaped. Freedom from brush discharge is thus obtained, whilst the screening is efficient and the capacitance of the arrangement is easily calculable, within fairly narrow limits, from the formula

$$C = \frac{l}{2\log_e \frac{D}{d}} \text{ (e.s. units)}$$

or

$$C = \frac{l}{2\log_e \frac{D}{d}} \times \frac{10^{12}}{9 \times 10^{11}} \text{ micro-microfarads}$$

i.e.

$$C = \frac{l}{1{\cdot}8 \log_e \frac{D}{d}} \text{ micro-microfarads} \quad . \quad . \quad . \quad (76)$$

where l is the active length in centimetres of the low-tension electrode, d being its diameter and D the internal diameter of the outer electrode.

Rayner (Ref. (35)), Semm (Ref. (36)), Churcher and Dannatt (Ref. (37) and (50)), and others have used capacitors of this type. Fig. 45 shows the construction of a standard capacitor designed by Churcher

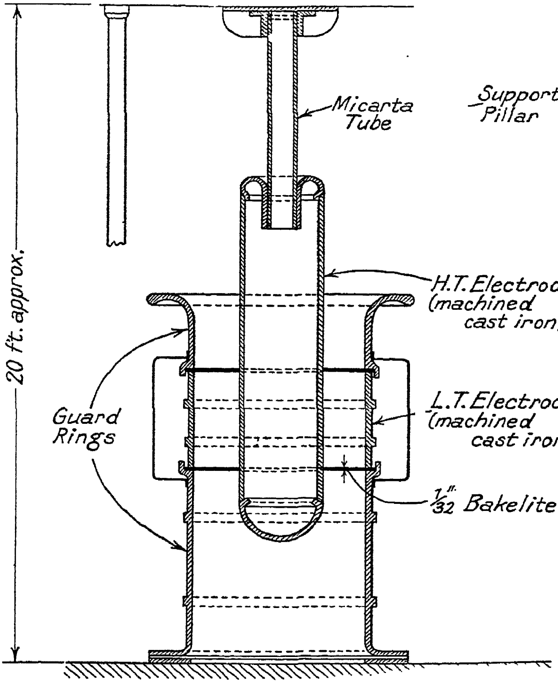
20 ft. approx.
Micarta Tube
Support Pillar
H.T. Electrode (machined cast iron)
Guard Rings
L.T. Electrode (machined cast iron)
1/32" Bakelite

FIG. 45. CHURCHER AND DANNATT STANDARD AIR CAPACITOR FOR 300 kV.

and Dannatt for use at 300 kV (r.m.s.). The electrodes are of machined cast iron having a specially smooth finish to avoid surface irregularities which cause premature breakdown of the capacitor when the voltage is applied. The high-tension cylinder is suspended inside the low-tension cylinder from separate supports, and is insulated from the latter by a Micarta tube.

An accuracy of 0·2 per cent in the calculated capacitance of the capacitor was aimed at in the design. The average breakdown voltage is 310 kV (r.m.s.).

The original paper (Ref. (50)) should be referred to for details. Fig. 45 is drawn to scale, but much detail is omitted in order to show the main features more clearly.

Compressed-gas Capacitors. Fig. 46 shows the construction of the compressed-gas capacitor for a maximum working voltage of 250 kV (r.m.s.) as manufactured by Metropolitan-Vickers. The main features are the high-tension electrode *A*, consisting of a steel tube fitting tightly inside a Micarta tube, the latter being long enough to give the necessary insulation to ground for the working voltage. Connection from the electrode to the top plate *B* is made through a spring contact. The high-tension terminal, with domed head, is fitted to the centre of a stress distributor *C* (used for voltages above 150 kV), which is secured to the top plate by small screws.

The low-tension electrode is supported on a central post terminating in the cap D, which acts as a guard ring, this being insulated from the effective part of the electrode by an insulating collar. The lead from the electrode, which is screened throughout its length, is brought down to a screened terminal box.

The capacitance of the capacitor is $50\mu\mu$ F., and its loss angle is less than 0·00001 degree. The gas used may be either air or nitrogen and it must be clean and dry. The working gas pressure is 150 lb. per sq. in. gauge reading.

Secondary Standard Capacitor. Capacitors for use as secondary standards are calibrated capacitors whose dimensions need not be accurately known since the magnitude of their capacitance is not calculated from their dimensions. It is essential that they shall have a capacitance which does not vary with time, and therefore care must be exercised in selecting materials which will not warp and so alter the dimensions. The plates, also, must be rigidly fixed in position, and, if possible, there should be no appreciable expansion of them with moderate increases of temperature. The insulation should be very efficient and the construction such that leakage is avoided. Sharp edges must also be avoided in order to eliminate troubles from brush discharges within the capacitor. Air is used as the dielectric in such capacitors, in order that they shall be free from dielectric losses, and the leads to the terminals are made as short as possible, to reduce the I^2R loss to a minimum. A cover, to prevent the accumulation of dust between the plates, must be provided, and it is desirable also that the air should be dried before entering the interior as moisture is conducive to leakage.

By the use of a number of plates instead of merely two, as in the primary standards, much greater capacitances can be obtained without excessive bulk. Capacitances up to about 0·02 μF are obtainable compared with those of the order of 100 to $200\mu\mu$F in the case of primary capacitors.

Glazebrook and Muirhead (Ref. (38)) designed a secondary standard air capacitor for the committee of the British Association in

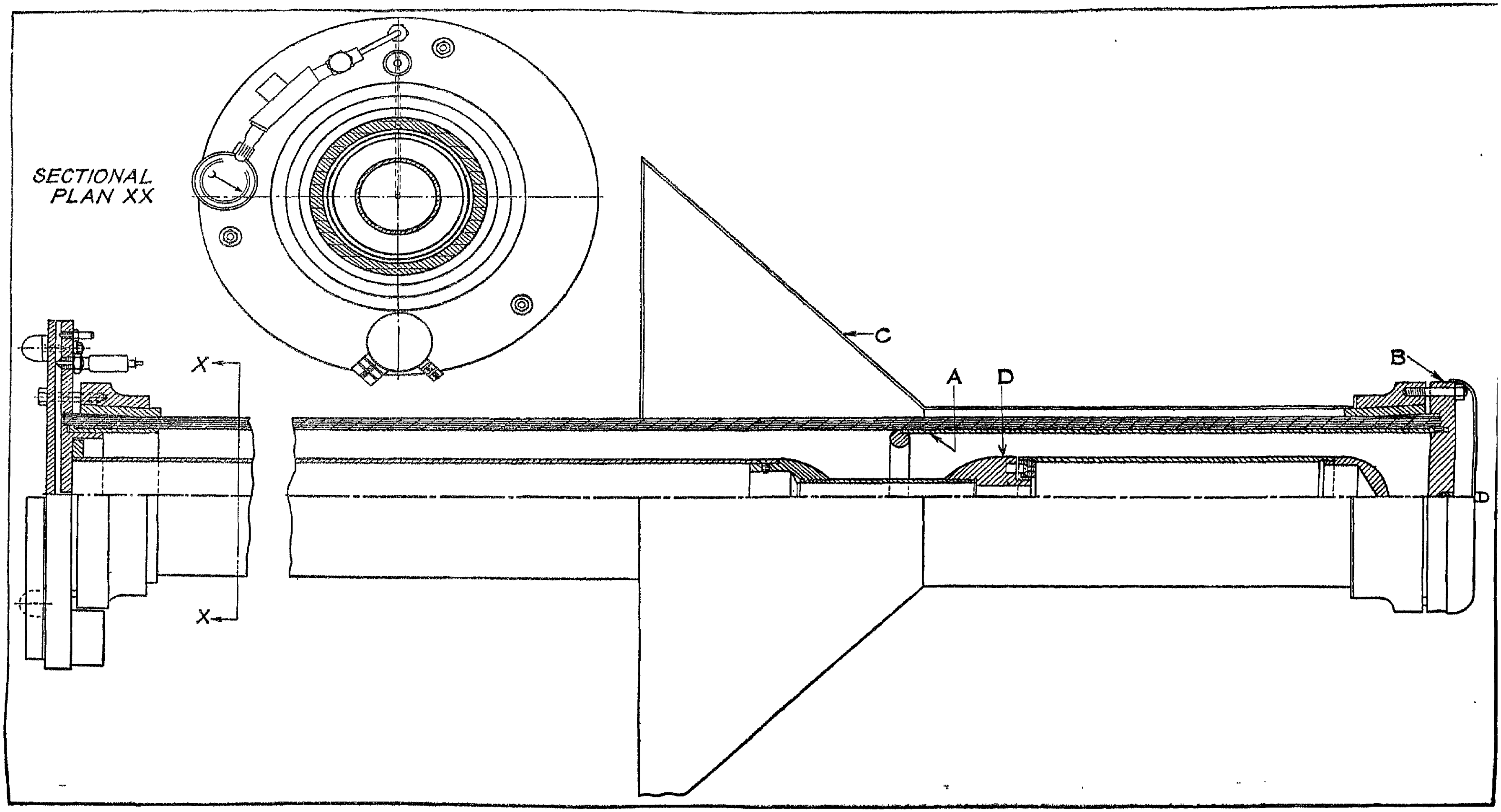

FIG. 46. A 250KV COMPRESSED-GAS CAPACITOR

(Metropolitan-Vickers Elec. Co., Ltd.)

1890. It consisted of twenty-four concentric brass tubes, the thickness of whose walls was about $\frac{1}{32}$ in. Twelve of these tubes were supported in a vertical position by a conical brass casting, the outside surface of which formed a series of twelve steps over which the tubes fitted and to which they were screwed. This casting, with its tubes attached, was carried by three ebonite pillars about 3 in. high. The other twelve tubes were fitted to a similar stepped brass casting, which was carried by the outside case so that these tubes hung downwards in the air spaces between the other twelve cylinders. The terminal of the insulated cylinders was in the form of a brass rod passing through a central hole in the upper brass casting, and insulated from it by an ebonite plug, this rod being screwed into the bottom brass casting. The internal air was dried by a small dish of sulphuric acid placed inside the case. The capacity of this capacitor was about 0·021 microfarad.

Giebe (Ref. (47)) in 1909 described a modified form of the above capacitor constructed by him, and also a plate type which he found to be superior to the former and which is shown in Fig. 47A. It consists of a large number of thin, circular plates of magnalium—a magnesium-aluminium alloy—with a space of about 2 mm. between successive plates. In one form there are 71 plates in all—35 connected to one terminal and 36 to the other. Hague* gives a full description of this capacitor.

Messrs. H. W. Sullivan, Ltd., manufacture a range of standard air capacitors in which the insulation between the two conducting systems consists of small pieces of silica-quartz, a material having very low dielectric loss. Their long-period permanence is 1 part in 20,000 and the temperature coefficient less than 1 part per 100,000 per degree centigrade. This extraordinarily low temperature coefficient is brought about by a method of compensation described fully by W. H. F. Griffiths (Ref. (63)). These fixed standards of capacitance have a special terminal system to eliminate errors of stray lead capacitance. The two main terminals of the condenser remain connected to the testing apparatus and the exactly standardized capacitance is inserted by means of a strap (see Fig. 47B).

The Capacitance Increment Standard (H. W. Sullivan, Ltd.) introduces into a calibrating circuit, by a rotary motion of a variable air capacitor, four very accurately known capacitances of, say, 100, 200, 300, and 500$\mu\mu$ F. or 10, 20, 30, and 50$\mu\mu$ F. These capacitances are inserted strictly as circuit increments and may thus be used as a very accurate means of calibrating variable capacitors by either direct or substitution bridge methods.

Recent improvements in precision variable air capacitors (silica insulated) have been responsible for stabilities which are now measured in parts in a million. In the Sullivan and Griffiths instrument

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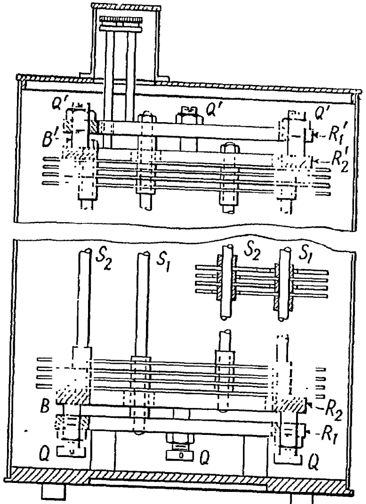
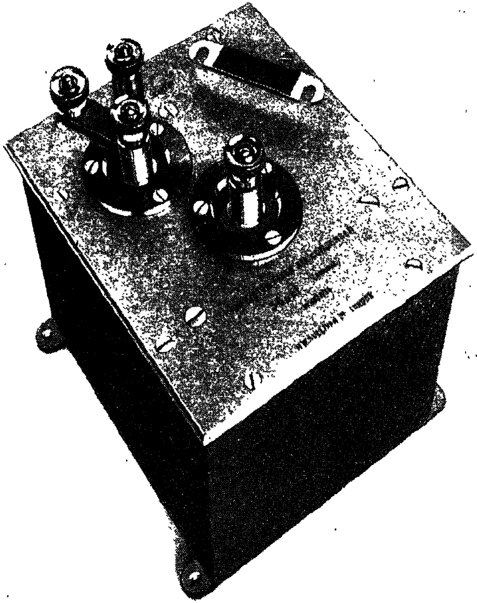
* *A.C. Bridge Methods*, 2nd Edition, p. 122.



(From *Alternating Current Bridge Methods*. Hague)

FIG. 47A. GIEBE'S PLATE AIR CAPACITOR



(*H. W. Sullivan, Ltd.*)

FIG. 47B. SULLIVAN-GRIFFITHS FIXED AIR CAPACITOR

the capacitance accuracy is 1 part in 10^4 and it is made to be unaffected by variable stray electrostatic fields in the vicinity of the terminals and connecting leads by a special screened lead device. Power factors of 0·00001 or less are maintained up to frequencies of 10^5 or 10^6 c/s, and the temperature coefficient is compensated by means of a bi-metallic method also due to W. H. F. Griffiths (Ref. (63)).

Laboratory Standards of Capacitance. The secondary standards described above are unsuitable for general laboratory purposes. As laboratory standards, capacitors having a solid dielectric instead of air are used. The dielectrics used for these purposes are mica and paraffined paper, the former being the better. Both of these materials have a permittivity greater than that of air (mica 3 to 8, paper 2 (about)), and therefore give a greater capacitance for a given size than when air is the dielectric. They have also high resistivity and dielectric strength, both of which characteristics are necessary for the purpose for which they are used. Fixed value standards from 0·01 to 1·0 μF. and single and multi-decade standards up to 5 μF. continuously subdivisible down to $1\mu\mu$ F. have a *direct reading* accuracy of 0·01 per cent. Extraordinarily low power factors of 0·00005 are obtained on the higher capacitances. Temperature coefficients as low as 0·001 per cent per °C. are obtained and the internal inductance and series resistance are reduced to such low values that frequency corrections of both capacitance and power factor are unnecessary up to quite high frequencies even for an accuracy of 0·01 per cent (15 kc/s for 1 μF., 200 kc/s for 0·01 μF.).

Paraffined paper capacitors are not so reliable as the mica type, and are not suitable as standards for precision work. They have a greater dielectric loss (and therefore power factor) than mica capacitors, the power factor as stated by Grover (Ref. (40)) for a range of them varying from 0·0017 to 0·017. Grover also found the frequency variation to be of the order of 4 parts in 1,000 for a frequency range of 50∼ to 1,000∼, the capacitance decreasing with increase of frequency. The manufacture of paper and other capacitors is described by Mansbridge (Ref. (41)).

BIBLIOGRAPHY AND REFERENCES

(1) *Dictionary of Applied Physics*, Vol. II.

(2) *British Association Reports on Electrical Measurements, A Record of the History of "Absolute Units" and of Lord Kelvin's Work in Connection with These* (Camb. Univ. Press, 1912).

(3) *Report to the International Committee on Electrical Units and Standards of a Special Technical Committee* (Bureau of Standards).

(4) "The So-called International Electric Units," F. A. Wolff, *Bull. Bur. Stands.*, Vol. I, No. 1 (1904).

(5) "Absolute Measurements of a Resistance by a Method based on that of

Lorenz," F.E. Smith, *Collected Researches of the National Physical Laboratory*, Vol. XI, No. 13.

(6) *Absolute Measurements in Electricity and Magnetism*, Gray, Vol. II, Pt. 1.

(7) *Absolute Measurements in Electricity and Magnetism*, Gray, Vol. II, Pt. 2, p. 400.

(8) *British Association Report*, 1889.

(9) *Pogg. Ann.*, 1854.

(10) *Phil. Mag.*, 1889.

(11) *Phys. Soc. Proc.*, 1892.

(12) *B.A. Report*, 1892.

(13) Searle, *Electrician*, 1911.

(14) *B.A. Report*, 1862.

(15) "A Tubular Electrodynamometer for Heavy Currents," P. G. Agnew, *Bull. Bur. Stands.*, Vol. VIII (1912), p. 651.

(16) "Clark and Weston Cells," F. A. Wolff and C. E. Waters, *Bull, Bur. Stands.*, Vol. IV, No. 1 (1907).

(17) "A New Determination of the Electromotive Force of Weston and Clark Cells, by an Absolute Electrodynamometer," K. E. Guthe, *Bull. Bur. Stands.*, Vol. II, No. 1 (1906).

(18) Hulett, *Phys. Review*, 22 and 23 (1906).

(19) Latimer Clark, *Trans. Roy. Soc.*, Vol. CLXIV (1874).

(20) "The Electrode Equilibrium of the Standard Cell," Wolff and Waters, *Bull. Bur. Stands.*, Vol. IV, No. 1 (1907).

(21) *Bull. Bur. Stands, Circular*, Nos. 169, 320, 455, 468.

(22) J. V. Jones, *Roy. Soc. Proc.* (1898), LXII.

(23) Maxwell, *Electricity and Magnetism.*

(24) Nagaoka, *Tokyo Math. Phys. Soc.* (1903) and (1911).

(25) "Construction and Calculation of Absolute Standards of Inductance," J. G. Coffin, *Bull. Bur. Stands.*, Vol. II, No. 1 (1906).

(26) A. Campbell, *Roy. Soc. Proc.* (1907), LXXIX, and (1912) LXXXVII.

(27) A. Campbell, N.P.L. *Collected Researches*, Vol. IV, Art. viii (1908).

(28) Grüneisen and Giebe, *Zeitschrift fur Instrumentenkunde*, XXXI (1911) and XXXII (1912).

(29) *A.C. Bridge Methods*, Hague.

(30) "A New Determination of the Ratio of the Electromagnetic to the Electrostatic Unit of Electricity," Rosa and Dorsey, *Bull. Bur. Stands.*, Vol. III (1907), p. 433.

(31) G. B. Shanklin, *Gen. Elec. Rev.*, Vol. XIX, p. 842 (1916).

(32) "Low Power Factor Measurements at High Voltages," Rayner, Standring, Davis, and Bowdler, *Jour. I.E.E.*, Vol. LXVIII, p. 1132 (1930).

(33) *High Voltage Cables*, Dunsheath, p. 56.

(34) W. Petersen, *Hochspannungstechnik*, 104, p. 92 (1911).

(35) "The Design and Use of an Air Condenser for High Voltages," E. H. Rayner, *Jour. Sci. Insts.*, Vol. III, pp. 33, 70, and 104 (1926).

(36) A. Semm, *Archiv. fur Elektrotechnik*, Vol. IX, p. 30.

(37) "The Use of the Schering Bridge at 150 kV," B. G. Churcher and C. Dannatt, *World Power*, Vol. V, p. 238 (1926).

(38) "On the Air Condensers of the British Association," R. T. Glazebrook and A. Muirhead, *B.A. Report*, 1890, Appendix II.

(39) H. L. Curtis, *Bull. Bur. Stands.*, VI, p. 431 (1911).

(40) F. W. Grover, *Bull. Bur. Stands.*, VII, p. 495 (1911).

(41) "The Manufacture of Electrical Condensers," Mansbridge, *Jour. I.E.E.*, Vol. XLI, p. 535 (1908).

(42) "Dielectric Properties of Mica," E. Bouty, *Proc. Inst. Civil Engs.*, 107, p. 535.

(43) "Mica Condensers," E. Bouty, *Elect. World*, 22, p. 417.

(44) *Electrical Condensers*, Coursey.

(45) *Electrical Measuring Instruments*, Drysdale and Jolley

(46) *Electrical Measurements*, Laws.

(47) "Normal Luftkondensatoren und ihre absolute Messung," Giebe, *Zeits. fur Inst.*, Vol. XXIX (1909).

(48) Hunter and Bacon, *Trans. Amer. Electro-chemical Soc.*, XXXVI, 323.

(49) "The Permittivity and Power Factor of Micas," C. Dannatt and S. E. Goodall, *Jour. I.E.E.*, Vol. LXIX, p. 490 (1931).

(50) "The Use of Air Condensers as High-Voltage Standards," B. G. Churcher and C. Dannatt, *Jour. I.E.E.*, Vol. LXIX, p. 1019 (1931).

(51) Shawcross and Wells, *Electrician*, Vol. LXXV, p. 64.

(52) "*Electrical Measurements*." H. L. Curtis.

(53) "The M.K.S. System of Units." A. E. Kennelly, *Jour. I.E.E.*, Vol. LXXVIII, p. 235.

(54) "The Electrical Stability of Condensers," H. A. Thomas, *Jour. I.E.E.*, Vol. LXXIX, p. 297 and Vol. LXXXI, p. 277.

(55) *Memorandum on the M.K.S. System of Practical Units*, by G. Giorgi.

(56) *The Theory of Dimensions and Its Application for Engineers*, by F. W. Lanchester.

(57) "La métrologie électrique classique et les systèmes d'unités qui en dérivent: Examen critique," G. Giorgi, *Revue générale de l'électricité*, 10th October, 1936.

(58) "La métrologie électrique nouvelle et la construction du système électrotechnique absolu M.K.S.," G. Giorgi, *Revue générale de l'électricité*, 24th July, 1937.

(59) "Mémorandum du Prof.-Ing. G. Giorgi concernant le choix de la quatrième unité fondamentale." *Procès-verbaux des séances du Comité International des Poids et Mesures*, 2e série, tome XVII, 1935.

(60) "Recent Developments in Electrical Units," A. E. Kennelly, *Electrical Engineering*, February, 1939.

(61) "Fundamental Electrical Measurements," Sir F. E. Smith, *Jour. I.E.E.*, Vol. LXXXI, p. 701.

(62) "Standards and Standardization," W. H. F. Griffiths, *The Wireless Engineer*, Vol. XX, p. 109.

(63) "The Temperature Compensation of Condensers," W. H. F. Griffiths, *The Wireless Engineer*, Vol. XIX, No. 222, pp. 101–11, and No. 223, pp. 148–57.

(64) "Law Linearity of Semi-circular Plate Variable Condensers," *The Wireless Engineer*, Vol. XXII, p. 107.

(65) Symposium of papers on the M.K.S. system of units, *Jour. I.E.E.* Vol. xcvii, Pt. I, p. 235, 1950.

(66) "Units and Standards of Measurement employed at the National Physical Laboratory," Vol. III (Dept. of Sc. and Ind. Res., 1952).

(67) *Memorandum on the M.K.S. System of Electrical and Magnetic Units*, B.S.I. No. 1637, 1950.

(68) "Transcription of Electrodynamic Relations into Different Systems," H. Pelzer, *E.R.A. Report, L/T* 261.

(69) "The Accuracy of Measurement of Electrical Standards," A. Felton, *Journ. I.E.E.*, Vol. XCVIII, Pt. II, p. 694.

(70) "Absolute Electrical Measurements," L. Hartshorn, *Reports on Progress in Physics* (Physical Society) 1938, Vol. V, p. 302.

(71) "Standards of Electrical Measurement," L. Hartshorn, *Jour. I.E.E.*, 1942, Vol. 89, p. 526.

(72) "Establishment and Maintenance of the Electrical Units," F. B. Silsbee, *National Bureau of Standards Circular No.* 475, 1949.

(73) "A Design for Standard Resistance Coils," C. R. Barker, A. Grindley and J. A. Hall, *Journal of Scientific Instruments*, Vol. 29, No. 3, p. 65, March, 1952.

(74) "The Power Factor and Capacitance of Mica Capacitors at Low Frequencies," P. R. Bray, *Jour. Sci. Insts.*, Vol. 30. No. 2, p. 49.

(75) "Alternating Current Resistance Standards," A. H. M. Arnold, *Proc. I.E.E.*, Vol. 100, Pt. II, p. 319.

(76) *Electrical Units (with Special Reference to the M.K.S. System)*, E. Bradshaw.

(77) "Recent Improvements in Air Cored Inductances," *The Wireless Engineer*, Vol. XIX, No. 220, pp. 8–19, and No. 221, pp. 56–63.

(78) "The Direct Reading of the Frequency of Resonant Circuits," *The Wireless Engineer*, Vol. XX, No. 242, pp. 524–38.

(79) "Wide Range Variable Condenser for Special Laws," *The Wireless Engineer*, Vol. XI, No. 131, pp. 415–8.

(80) "Electrical Standards for Research and Industry," a catalogue published by Messrs. H. W. Sullivan, Ltd. in 1954.

(81) *The M.K.S. System of Units*, T. McGreevy, 1953.

(82) *The Metre-Kilogram-Second System of Electrical Units*, R. K. Sas and F. B. Pidduck.

(83) *Principles of Electricity*, A. Morley and E. Hughes, 1953.

(84) *M.K.S. Units and Dimensions*, G. E. M. Jauncey and A. S. Langsdorf, 1940.

(85) *Dimensional Methods and their Application*, C. M. Focken, 1953.

(86) *Dimensions in Engineering Theory*, G. W. Stubbings.

(87) *Dimensional Analysis*, H. E. Huntley.

CHAPTER III

CIRCUIT ANALYSIS

In alternating current circuits generally, and especially in networks, the symbolic (j) notation is of great use in simplifying the calculation of the various quantities involved. For this reason it will be considered here before proceeding to work in which such calculations are necessary.

Fig. 48 shows a vector representing (say) a voltage, which, expressed in the usual trigonometrical notation, is given by

$$v = V_{max} \sin (\omega t + a)$$

This vector could otherwise be defined by stating its resolved components in the horizontal and vertical directions—i.e. along axes OX and OY. Thus

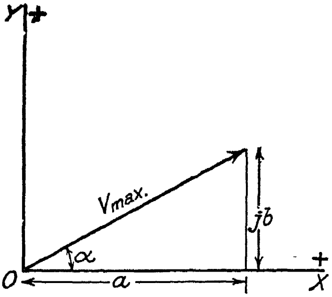


FIG. 48. RECTANGULAR CO-ORDINATES OF A VECTOR

$$v = V_{max} \cos a \text{ (horizontally)} + V_{max} \sin a \text{ (vertically)}$$

The commonest of the symbolic methods employs this means of expression, the horizontal component being written simply as $V_{max} \cos a$ and the vertical component being distinguished by placing a letter j in front of it. Thus, symbolically, the vector is expressed as

$$[V] = V_{max} \cos a + jV_{max} \sin a = V_{max} [\cos a + j \sin a]$$

or $$[V] = a + jb$$

where a and b are its horizontal and vertical components, the brackets [ ] indicating that the notation is symbolic.*

---

* The fact that a quantity is expressed symbolically may be indicated also by a dot placed under the symbol, thus— $\underset{.}{E}$ $\underset{.}{I}$ etc.

In the same way, the vectors shown in Fig. 49 can be represented symbolically as

$$[V_1] = a + jb$$
$$[V_2] = -c + jd$$
$$[V_3] = -e - jf$$
$$[V_4] = k - jh$$

respectively, the directions OX and OY being positive and directions OX' and OY' negative.

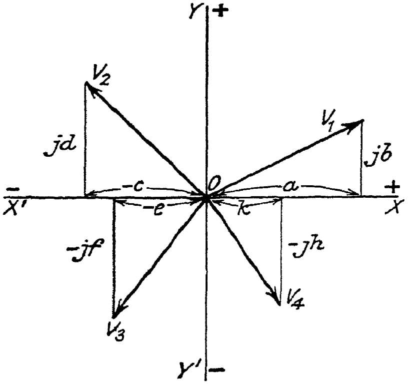


FIG. 49. SYMBOLIC REPRESENTATION

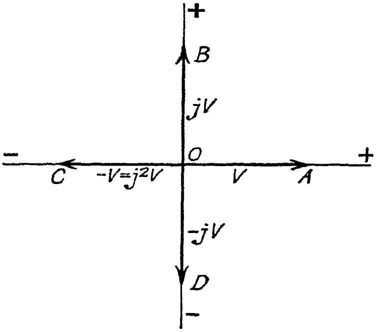


FIG. 50

Actual Value of the Operator "j". In Fig. 50, the vectors OA, OB, OC, and OD are all of the same magnitude V. Expressing them symbolically, we have

$$OA = V$$
$$OB = jV$$
$$OC = -V$$
$$OD = -jV$$

From this it appears that the multiplication of a vector V, such as OA, by j means that it is rotated through 90° in an anti-clockwise direction. Then, multiplying OB by j, we rotate it through another 90° to OC. Thus,

$$OA \times j \times j = OC$$

or
$$j^2 V = -V$$

i.e.,
$$j^2 = -1$$
$$j = \sqrt{-1}$$

The operator "j" is thus an imaginary quantity, and can be treated as having the value $\sqrt{-1}$ in all calculations in which it occurs.

Addition of Vectors. Vectors are added by adding together their horizontal and vertical components separately. Thus the sum of vectors $[V_1] = a + jb$ and $[V_2] = c + jd$ is $[V] = (a + c) + i(b + d)$ and its phase angle relative to the horizontal axis OX is

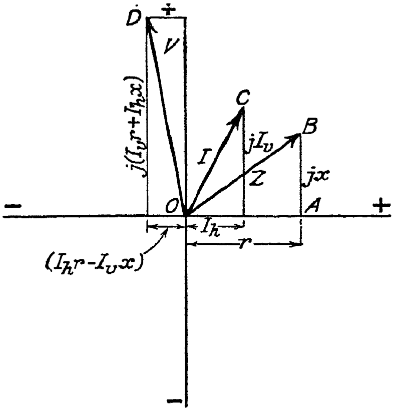


FIG. 51

$\tan^{-1}\left(\frac{b + d}{a + c}\right)$. The actual magnitude of V is $\sqrt{(a + c)^2 + (b + d)^2}$.

Subtraction of Vectors. If a vector $[V_1] = a + jb$ is to be subtracted from a vector $[V_2] = c + jd$, then the resultant is

$$\begin{aligned}[V] = [V_2] - [V_1] &= c + jd - (a + jb) \\ &= c - a + j\,(d - b)\end{aligned}$$

its actual value being

$$\sqrt{(c - a)^2 + (d - b)^2}$$

and its phase angle

$$\tan^{-1}\left(\frac{d - b}{c - a}\right)$$

Multiplication of Complex Quantities. The product of two complex quantities $[V_1] = a + jb$ and $[V_2] = c + jd$ is

$$\begin{aligned}[V] = (a + jb)\,(c + jd) &= ac + j\,(bc + ad) + j^2bd \\ &= ac - bd + j\,(bc + ad) \\ &= A + jB\end{aligned}$$

where A and B are the horizontal and vertical resolved components of the product $[V]$. The numerical value of V is obviously $\sqrt{A^2 + B^2}$.

As an example, an impedance expressed as $[Z] = r + jx$, when multiplied by a current $[I] = I_h + jI_v$ gives a voltage

$$\begin{aligned}[V] &= (I_h + jI_v)(r + jx) \\ &= (I_h r - I_v x) + j(I_v r + I_h x)\end{aligned}$$

To find the numerical value of V,

$$\begin{aligned}V^2 &= (I_h r - I_v x)^2 + (I_v r + I_h x)^2 \\ &= (I_h^2 + I_v^2)(r^2 + x^2)\end{aligned}$$

Thus $V = (\sqrt{I_h^2 + I_v^2})(\sqrt{r^2 + x^2})$

and since $\sqrt{I_h^2 + I_v^2} = I$, we have

$$V = I\sqrt{r^2 + x^2}$$

which is, of course, the result which would be obtained by trigonometrical methods. Fig. 51 illustrates this example. The triangle OAB is the "impedance triangle," giving the symbolic expression $r + jx$ for the impedance, while OC and OD are the current and voltage vectors respectively.

Division of a Vector Quantity by a Complex Quantity. If a vector quantity $a + jb$ is to be divided by a complex quantity $c + jd$ the quotient is $\frac{a + jb}{c + jd}$.

Rationalizing the denominator, we have

$$\begin{aligned}[V] = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} &= \frac{ac + j(bc - ad) - j^2bd}{c^2 - j^2d^2} \\ &= \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + j\frac{(bc - ad)}{c^2 + d^2} \\ &= C + j \, . \, D\end{aligned}$$

where C and D are the resolved parts of the resultant vector.

The numerical value of V is, as before, given by $V = \sqrt{C^2 + D^2}$.

Other Forms of Representation. EXPONENTIAL FORM. This is really an extension of the trigonometrical form of expression. It was seen that a vector quantity could be expressed in the form

$$[V] = V(\cos a + j \sin a)$$

If the angle a is in radians, $\sin a$ and $\cos a$ can be expanded as below—

$$\sin a = a - \frac{a^3}{\underline{|3}} + \frac{a^5}{\underline{|5}} - \frac{a^7}{\underline{|7}} + \dots$$

$$\cos a = 1 - \frac{a^2}{\underline{|2}} + \frac{a^4}{\underline{|4}} - \frac{a^6}{\underline{|6}} + \dots$$

$$\therefore [V] = V\left[\left(1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \frac{\alpha^6}{\underline{|6}} + \ldots\right) + j\left(\alpha - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} - \frac{\alpha^7}{\underline{|7}} + \ldots\right)\right]$$

$$= V\left[1 + j\alpha - \frac{\alpha^2}{\underline{|2}} - \frac{j\alpha^3}{\underline{|3}} + \frac{\alpha^4}{\underline{|4}} + \frac{j\alpha^5}{\underline{|5}} - \frac{\alpha^6}{\underline{|6}} - \frac{j\alpha^7}{\underline{|7}} + \frac{\alpha^8}{\underline{|8}} + \ldots\right]$$

Substituting j^2 for -1 in the above we have

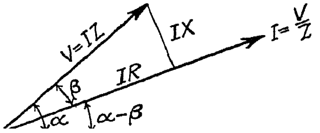
$$[V] = V\left[1 + j\alpha + \frac{j^2\alpha^2}{\underline{|2}} + \frac{j^3\alpha^3}{\underline{|3}} + \frac{j^4\alpha^4}{\underline{|4}} + \frac{j^5\alpha^5}{\underline{|5}} + \frac{j^6\alpha^6}{\underline{|6}} + \frac{j^7\alpha^7}{\underline{|7}} + \frac{j^8\alpha^8}{\underline{|8}} + \ldots\right]$$

or $[V] = V\varepsilon^{j\alpha}$,

since the series is the expansion of $\varepsilon^{j\alpha}$, where ε is the base of natural logarithms. Thus, if a current I is given by $\frac{V}{Z}$ where the voltage $[V] = V\varepsilon^{j\alpha}$ and the impedance $[Z] = Z\varepsilon^{j\beta}$, then

$$[I] = \frac{[V]}{[Z]} = \frac{V\varepsilon^{j\alpha}}{Z\varepsilon^{j\beta}} = \frac{V\varepsilon^{j(\alpha-\beta)}}{Z}$$

This is illustrated in Fig. 52.



. 52. EXPONENTIAL FORM OF REPRESENTATION

POLAR FORM (1). This form of representation, suggested by Prof. Diamant (*Trans. Am. I.E.E.*, Vol. XXXV, p. 957), has not been very generally applied, but is nevertheless useful in some types of problems.

In this method, the vector quantity $[V] = V(\cos\alpha + j\sin\alpha)$ is expressed as VJ^m where J represents an operator which, when applied to a vector, rotates it through an angle of 90°. In this respect it is similar to j. The index m is the ratio of the angle which the vector makes with the horizontal axis to one right angle. Thus, in the vector V mentioned above, $m = \frac{\alpha}{\pi/2}$, expressing the angles in circular measure. If m is positive, the rotation is anti-clockwise, and if negative the rotation is clockwise. The 3-phase voltage vectors shown in Fig. 58 could be expressed in this form as

$$[E_1] = EJ^0$$
$$[E_2] = EJ^{+\frac{4}{3}}$$
$$[E_3] = EJ^{-\frac{4}{3}}$$

Since $m = \dfrac{a}{\pi/2}$ $\therefore a = m\dfrac{\pi}{2}$ and the vector V can be written

$$
\begin{aligned}
[V] &= V\left(\cos m\frac{\pi}{2} + j\sin m\frac{\pi}{2}\right)\\
&= V\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)^m \text{ from De Moivre's Theorem}\\
&= V\,(0 + j\,.\,1)^m\\
&= V\,(j^m) = VJ^m
\end{aligned}
$$

Thus j and J have the same meaning.

Polar Form (2). Another form of representation which is fairly frequently used is $V\underline{/\alpha}$ meaning that the vector is of length V and is rotated in an anti-clockwise direction, so that it makes an angle of a with the horizontal. This form is merely conventional. Using this mode of expression, the three-phase vectors referred to in the previous paragraph can be expressed as

$$
\begin{aligned}
[E_1] &= E\underline{/0}\\
[E_2] &= E\,\underline{/120^\circ} \quad \text{or } E\,\underline{\Big/\frac{2\pi}{3}}\\
[E_3] &= E\,\underline{/-120^\circ} \text{ or } E\,\underline{\Big/-\frac{2\pi}{3}}
\end{aligned}
$$

The product of two vectors $[E_1] = E_1\underline{/\alpha}$ and $[E_2] = E_2\underline{/\beta}$ may be expressed as

$$E = E_1E_2\underline{/\alpha + \beta}$$

and the quotient of two such vectors as

$$E = \frac{E_1}{E_2}\underline{/\alpha - \beta}$$

Application of the Symbolic Method to Alternating Current Problems. The application of the rectangular form of representation to problems in a.c. circuits can be illustrated by means of examples. Several different types of circuits and problems are given below.

Example 1 (Simple Series Circuit). A sinusoidal voltage of r.m.s. value 100 volts and frequency 50 cycles per second is applied to the circuit shown in Fig. 53. Calculate the current in the circuit and find its phase relative to that of the applied voltage.

The impedances in the circuit can be expressed symbolically

$$\begin{aligned}
\text{Impedance of } R_1 &= 3 \\
\text{,,} \quad L_1 &= j\omega L_1 = j \times 314 \times 0{\cdot}0159 = 5j \\
\text{,,} \quad R_2 &= 4 \\
\text{,,} \quad L_2 &= j\omega L_2 = j \times 314 \times 0{\cdot}0477 = 15j \\
\text{,,} \quad C &= -j \times \frac{1}{\omega C} = \frac{-j10^6}{314 \times 318} = -10j
\end{aligned}$$

[NOTE. The negative sign in the capacitor impedance is explained by considering the current as a horizontal vector, when the voltage drop across the capacitor will be vertically downwards (since it lags 90° in phase behind

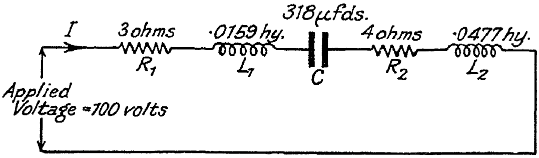


FIG. 53

the current). Thus, if the current is expressed symbolically as I, the voltage drop (given by current × impedance of capacitor) is $-j\dfrac{I}{\omega C}$, i.e. the impedance is $\dfrac{-j}{\omega C}$.]

The total impedance of the circuit is the sum of these symbolic expressions. Thus

$$\begin{aligned}
[Z] &= 3 + 4 + 5j + 15j - 10j \\
&= 7 + 10j
\end{aligned}$$

Then

$$\begin{aligned}
Z^2 &= 7^2 + 10^2 = 149 \\
Z &= \sqrt{149} = 12{\cdot}2
\end{aligned}$$

and the current is $\dfrac{100}{12{\cdot}2} = 8{\cdot}22$ amp.

If the current vector is horizontal, so that

$$[I] = 8{\cdot}22 + j\,.\,0$$

the voltage is

$$V] = [I][Z] = [8{\cdot}22 + j\,.\,0][7 + 10j] = 57{\cdot}54 + 82{\cdot}2j$$

The voltage thus leads the current by an angle ϕ such that $\tan\phi = \dfrac{82{\cdot}2}{57{\cdot}54} = \dfrac{10}{7}$. It is, of course, the angle of the impedance triangle.

Example 2 (Series-Parallel Circuit). A sinusoidal voltage of r.m.s. value 100 volts and frequency 50 cycles per second, is applied to the circuit shown in Fig. 54. Calculate the current in the main circuit and the currents in the two branch circuits. Take the voltage vector as horizontal, so that $[V] = 100 + j\,.\,0$.

$$\left.\begin{aligned}
\text{Impedance of } R_1 &= 8 \\
\text{,,} \quad L_1 &= j\omega L_1 = 314 \times 0{\cdot}0477j = 15j \\
\text{,,} \quad C_1 &= \frac{-j}{\omega C_1} = \frac{-10^6 j}{314 \times 159} = -20j
\end{aligned}\right\} \begin{aligned}&\text{Total impedance}\\ &= 8 - 5j\end{aligned}$$

$$\left.\begin{array}{l}\text{Impedance of } R_2 = 10 \\ \quad ,, \quad L_1 = j\omega L_2 = 314 \times 0{\cdot}0636j = 20j\end{array}\right\} \begin{array}{l}\text{Total impedance} \\ = 10 + 20j\end{array}$$

$$\left.\begin{array}{l}\text{Impedance of } R_3 = 7 \\ \quad ,, \quad C_2 = \dfrac{-j}{\omega C_2} = \dfrac{-j10^6}{314 \times 318} = -10j\end{array}\right\} \begin{array}{l}\text{Total impedance} \\ = 7 - 10j\end{array}$$

$$\text{Admittance of branch I} \quad = \frac{1}{10 + 20j} = \frac{10 - 20j}{10^2 + 20^2}$$
$$= 0{\cdot}02 - 0{\cdot}04j = [Y_1]$$

$$\text{Admittance of branch II} \quad = \frac{1}{7 - 10j} = \frac{7 + 10j}{7^2 + 10^2}$$
$$= 0{\cdot}047 + 0{\cdot}067j = [Y_2]$$

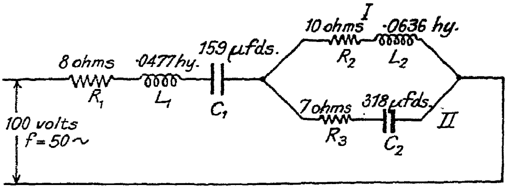


FIG. 54. SERIES-PARALLEL CIRCUIT

$$\begin{array}{l}\text{Total admittance of the two} \\ \quad \text{branches in parallel}\end{array} \quad = [Y_1] + [Y_2] = 0{\cdot}067 + 0{\cdot}027j$$

$$\begin{array}{l}\text{Total impedance of the two} \\ \quad \text{branches in parallel}\end{array} \quad = \frac{1}{0{\cdot}067 + 0{\cdot}027j} = \frac{0{\cdot}067 - 0{\cdot}027j}{0{\cdot}067^2 + 0{\cdot}027^2}$$
$$= 12{\cdot}8 - 5{\cdot}16j$$

Thus the total impedance of the complete circuit is

$$8 - 5j + 12{\cdot}8 - 5{\cdot}16j = 20{\cdot}8 - 10{\cdot}16j$$

and the current I in the main circuit is given by

$$I = \frac{100}{20{\cdot}8 - 10{\cdot}16j} = \frac{100\,(20{\cdot}8 + 10{\cdot}16j)}{20{\cdot}8^2 + 10{\cdot}16^2}$$
$$= 3{\cdot}88 + 1{\cdot}9j$$

Its numerical value is therefore $\sqrt{3{\cdot}88^2 + 1{\cdot}9^2}$

$$= 4{\cdot}32 \text{ amp.}$$

and its phase relative to the applied voltage is $\tan^{-1} \dfrac{1{\cdot}9}{3{\cdot}88}$ leading (since the imaginary term in the expression for the current is positive).

The voltage drop across the two parallel branches is

$$(3{\cdot}88 + 1{\cdot}9j)(12{\cdot}8 - 5{\cdot}16j) = 59{\cdot}46 + 4{\cdot}3j$$

$$\text{Current in branch I} = \frac{59{\cdot}46 + 4{\cdot}3j}{10 + 20j}$$

$$\frac{(59{\cdot}46 + 4{\cdot} \text{[illegible]} - 20j)}{10^2 + 20^2} \qquad 1{\cdot}37 - 2{\cdot}29j$$

Its numerical value is $\sqrt{1{\cdot}37^2 + 2{\cdot}29^2} = 2{\cdot}67$ amp. and it lags behind the applied voltage V by an angle $\tan^{-1} \frac{2{\cdot}29}{1{\cdot}36}$.

$$\text{Current in branch II} = \frac{59{\cdot}46 + 4{\cdot}3j}{7 - 10j}$$

$$= \frac{(59{\cdot}46 + 4{\cdot}3j)(7 + 10j)}{7^2 + 10^2} = 2{\cdot}51 + 4{\cdot}2j$$

Its numerical value is $\sqrt{2{\cdot}51^2 + 4{\cdot}2^2} = 4{\cdot}88$ amp. and it leads the applied voltage by a phase angle $\tan^{-1} \frac{4{\cdot}2}{2{\cdot}5}$.

Adding the two branch currents gives $3{\cdot}88 + 1{\cdot}9j$ which is the original expression for the current in the main circuit.

Application of Symbolic Method to a Network Problem. Fig. 55 shows Wien's arrangement of Maxwell's method of comparing a self inductance with a capacitance by means of a bridge network. V.G. is a vibration galvanometer used as a detector for frequencies within the commercial range.

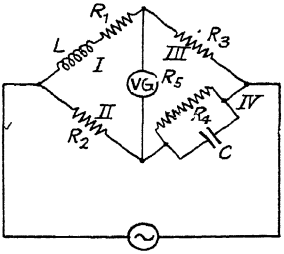


FIG. 55. WIEN BRIDGE NETWORK

The conditions for balance with this network are that

$$L$$

as can be seen from the following.

At balance, when no current flows through the galvanometer circuit,

$$\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4}$$

where Z_1, Z_2, Z_3, and Z_4 are the impedances of branches I, II, III, and IV respectively.

$$[Z_1] = R_1 + j\omega L$$
$$[Z_2] = R_2$$
$$[Z_3] = R_3$$

Total admittance of branch IV

$$[Y_4] = \frac{1}{R_4} + \frac{1}{\frac{-j}{\omega C}}$$

$$= \frac{1}{R_4} + \frac{j\omega C}{-j^2} = \frac{1}{R_4} + j\omega C$$

$\therefore$ Total impedance of branch IV

$$= \frac{1}{[Y_4]} = \frac{R_4}{1 + j\omega C R_4} = [Z_4]$$

$$\therefore \frac{R_1 + j\omega L}{R_3} = \frac{R_2}{\left(\dfrac{R_4}{1 + j\omega CR_4}\right)} = \frac{R_2}{R_4}(1 + j\omega CR_4)$$

Cross-multiplying,

$$R_1R_4 + j\omega LR_4 = R_2R_3 + jR_2R_3\omega CR_4$$

Equating real and imaginary terms, we have

$$R_1R_4 = R_2R_3$$

and $$j\omega LR_4 = jR_2R_3\omega CR_4$$

from which $$L = R_2R_3C$$

Thus $$R_1R_4 = R_2R_3 = \frac{L}{C} \quad . \quad . \quad . \quad (77)$$

The symbolic method can be used, also, to calculate the current in the galvanometer circuit when the bridge network is out of balance. In Fig. 56 the network is represented simply by impedances, and the cyclic currents (used by Maxwell to simplify network calculations) X, $X + Y$, and A, are assumed to flow in the three meshes as shown.

Z_5 is the impedance of the galvanometer circuit, Z_6 that of the alternator branch, and V the alternator voltage.

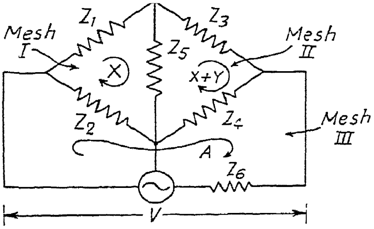


FIG. 56

Then the current in the galvanometer circuit is $(X + Y - X) = Y$. Using Kirchhoff's second law—that the algebraic sum of the potential differences in any closed circuit is zero— we have

Mesh I.

$$Z_1X + Z_5(-Y) + Z_2(X - A) = 0$$
$$= 0$$

Mesh II.

$$Z_3(X + Y) + Z_4(X + Y - A) + Z_5Y = 0$$

r $$X(Z_3 + Z_4) + Y(Z_3 + Z_4 + Z_5) - AZ_4 = 0$$

Mesh III.

$$Z_2(A - X) + Z_4(A - X - Y) + Z_6A = V$$

r $$-X(Z_2 + Z_4) - YZ_4 + A(Z_2 + Z_4 + Z_6) = V$$

Thus the three equations, from which Y is to be obtained, are—

$$X(Z_1 + Z_2) - YZ_5 - AZ_2 - 0 = 0 \quad . \quad \text{(i)}$$
$$X(Z_3 + Z_4) + Y(Z_3 + Z_4 + Z_5) - AZ_4 - 0 = 0 \quad . \quad \text{(ii)}$$
$$X(Z_2 + Z_4) - YZ_4 + A(Z_2 + Z_4 + Z_6) - V = 0 \quad . \quad \text{(iii)}$$

expressing the equations in the form most suited to the solution by the method of determinants. Then, from the algebraic theory of determinants, we have

$$\frac{Y}{\begin{vmatrix} Z_1+Z_2 & -Z_2 & 0 \\ Z_3+Z_4 & -Z_4 & 0 \\ -(Z_2+Z_4) & Z_2+Z_4+Z_6 & -V \end{vmatrix}} = \frac{1}{\begin{vmatrix} Z_1+Z_2 & -Z_5 & -Z_2 \\ Z_3+Z_4 & Z_3+Z_4+Z_5 & -Z_4 \\ -(Z_2+Z_4) & -Z_4 & Z_2+Z_4+Z_6 \end{vmatrix}}$$

or, expressing the determinants by Δ_y and Δ respectively, we have $\frac{Y}{\Delta_y} = \frac{1}{\Delta}$ so that $Y = \frac{\Delta_y}{\Delta}$

$$= \frac{V(Z_1Z_4 - Z_2Z)}{\begin{vmatrix} Z_1+Z_2 & -Z_2 & -Z_2 \\ Z_3+Z_4 & Z_3+Z_4+Z_5 & -Z_4 \\ -(Z_2+Z_4) & -Z_4 & Z_2+Z_4+Z_6 \end{vmatrix}}$$

which is the expression for the galvanometer current. The numerical value of this current can be obtained, in any particular case, by substituting in the above equation the symbolic expressions for the impedances of the various branches.

Network Containing a Mutual Inductance. Fig. 57 shows the connections of Heaviside's mutual inductance bridge for the determination of a self-inductance in terms of a mutual inductance. R_1, R_2, R_3, and R_4 are non-inductive resistances. while L_1 and L_2 are self inductances, there being, in addition, mutual inductance M between the alternator circuit and branch IV. The inductance L_2 forms the secondary of this mutual inductance.

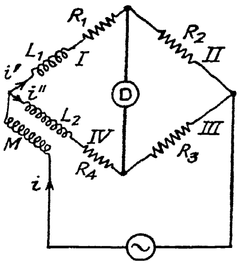


FIG. 57. HEAVISIDE BRIDGE

The treatment of a problem when mutual inductance is present is somewhat different from that used in the previous network.

Since, at balance, the voltage across the detector branch is zero, the volt drops across branches I and IV are equal. Thus,

$$i'(R_1 + j\omega L_1) = (R_4 + j\omega L_2)i'' + j\omega Mi \quad . \quad . \quad . \quad (i)$$

The mutual inductance term $j\omega Mi$ represents the voltage induced in arm IV by a current of i in the alternator branch. The convention is employed that this voltage is in the opposite direction to the current i, i.e. in the same direction as the current i''.

We have also that

$$i = i' + i''$$

and, since no current flows in the detector branch, the current in arm II is also i' and in arm III is i''. Thus

$$R_2 i' = R_3 i''$$

Substituting for i in equation (i)

$$i'(R_1 + j\omega L_1) = (R_4 + j\omega L_2)i'' + j\omega M(i' + i'')$$

or

$$i'(R_1 + j\omega L_1 - j\omega M) = (R_4 + j\omega L_2 + j\omega M)i''$$

Substituting $\frac{R_2}{R_3} i'$ for i''

$$i'(R_1 + j\omega L_1 - j\omega M) = (R_4 + j\omega L_2 + j\omega M)\frac{R_2}{R_3} i'$$

$$R_1 + j\omega L_1 - j\omega M = \frac{R_4 R_2}{R_3} + j\omega L_2 \frac{R_2}{R_3} + j\omega M \frac{R_2}{R_3}$$

Equating real and imaginary terms we have

$$R_1 = \frac{R_4 R_2}{R_3} \text{ or } R_1 R_3 = R_2 R_4 \quad (78)$$

Also,

$$j\omega L_1 - j\omega M = j\omega L_2 \frac{R_2}{R_3} + j\omega M \frac{R_2}{R_3}$$

from which

$$R_3(L_1 - M) = R_2(L_2 + M) \quad (79)$$

Application of the Symbolic Method to Polyphase Circuit Calculations. The full consideration of such problems is both outside the scope of this work, and is too lengthy for inclusion here, but the application of the symbolic notation to one three-phase circuit problem will be considered.

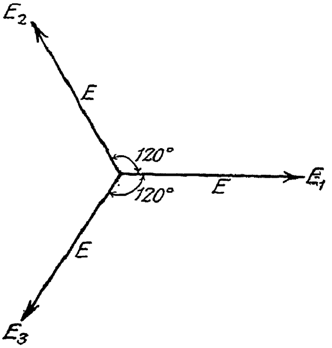


FIG. 58. THREE-PHASE VOLTAGE VECTORS

Fig. 58 shows a balanced system of three-phase voltages E_1, E_2, and E_3. When in the phase positions shown, these can be expressed symbolically as

$$[E_1] = [E + j0]$$
$$[E_2] = [-E\cos 60° + jE\sin 60°]$$

$$= E\,[-0{\cdot}5 + 0{\cdot}866j]$$
$$[E_3] = [-E\cos 60^\circ - jE\sin 60^\circ]$$
$$= E\,[-0{\cdot}5 - 0{\cdot}866j]$$

The symbolic sum of the voltages is, of course, zero.

Fig. 59 shows a three-phase network with alternator phase voltages E_1, E_2, and E_3, having positive directions as shown. I_1, I_2, and I_3 represent the phase (and line) currents, and z_1, z_2, and z_3 the line impedances. P, Q, and S are the mesh currents, and Z_1, Z_2, Z_3 are the phase impedances, including the impedances of both alternator and load phases and the line impedances.

Then, whether the system is balanced or not, the following equations hold, all the quantities being expressed in symbolic notation.

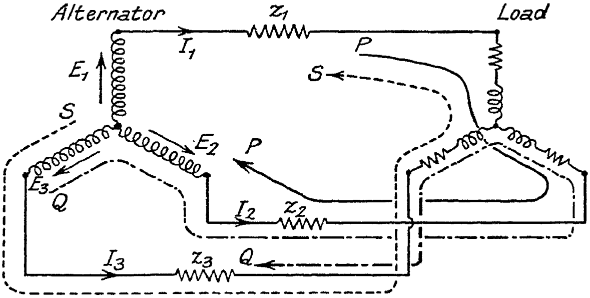


FIG. 59. THREE-PHASE CIRCUIT

Mesh P.

$$E_1 - Z_1(P - S) - Z_2(P - Q) - E_2 = 0$$

Mesh Q.

$$E_2 - Z_2(Q - P) - Z_3(Q - S) - E_3 = 0$$

Mesh S.

$$E_3 - Z_3(S - Q) - Z_1(S - P) - E_1 = 0$$

Substituting line currents for mesh currents, we have

$$E_1 - Z_1I_1 + Z_2I_2 - E_2 = 0$$
$$E_2 - Z_2I_2 + Z_3I_3 - E_3 = 0$$
$$E_3 - Z_3I_3 + Z_1I_1 - E_1 = 0$$

or

$$E_1 - E_2 = Z_1I_1 - Z_2I_2$$
$$E_2 - E_3 = Z_2I_2 - Z_3I_3$$
$$E_3 - E_1 = Z_3I_3 - Z_1I_1$$

Eliminating I_2 and I_3 by the use of the relationship $I_1 + I_2 + I_3 = 0$, we have

$$\frac{E_1 - E_2}{Z_2} + \frac{E_1 - E_3}{Z_3} = I_1 \left(1 + \frac{Z_1}{Z_2} + \frac{Z_1}{Z_3}\right)$$

$$\therefore I_1 = \frac{E_1 - E_2}{Z_2 Z_1\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)} + \frac{E_1 - E_3}{Z_3 Z_1\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)}$$

$$= \frac{E_1 - E_2}{Z_1 + Z_2 + \frac{Z_1 Z_2}{Z_3}} + \frac{E_1 - E_3}{Z_1 + Z_3 + \frac{Z_1 Z_3}{Z_2}} \quad . \quad . \quad (80)$$

Expressions can be found for I_2 and I_3 in the same way.

Example. The three-phase star-connected load shown in Fig. 60 (*a*) is connected to a three-phase system having a line voltage of 440 volts. Assuming the line voltages to be unaffected by the unbalanced load, calculate the current flowing in the branch containing the capacitor. The supply frequency = 50 cycles per sec.

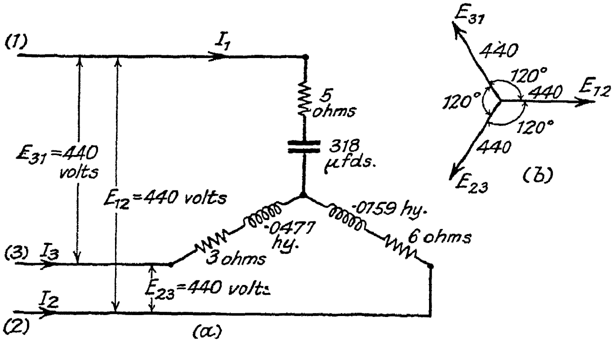


FIG. 60. THREE-PHASE STAR-CONNECTED LOAD CIRCUIT

Fig. 60 (*b*) shows the phase relationships of the three line voltages.

Then $E_{12} = E_1 - E_2$, $E_{23} = E_2 - E_3$, $E_{31} = E_3 - E_1$

where E_1, E_2, and E_3 are the phase voltages of the supply as used in the expression for I_1 in the above paragraph.

From the figure

$$[E_{23}] = -440 \cos 60 - j \,.\, 440 \sin 60$$
$$= -440\,(0{\cdot}5 + 0{\cdot}866j)$$
$$[E_{31}] = -440 \cos 60 + j \,.\, 440 \sin 60$$
$$= -440\,(0{\cdot}5 - 0{\cdot}866j)$$

The impedances of the load branches are—

$$[Z_1] = 5 - \frac{j10^6}{314 \times 318} = 5 - 10j.$$

$$[Z_2] = 6 + 314 \times 0{\cdot}0159j = 6 + 5j$$

$$[Z_3] = 3 + 314 \times 0{\cdot}0477j = 3 + 15j$$

Then $$[I_1] = \frac{E_1 - E_2}{Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}} + \frac{E_1 - E_3}{Z_1 + Z_3 + \frac{Z_1Z_3}{Z_2}}$$

$$[I_1] = \frac{440}{5 - 10j + 6 + 5j + \frac{(5 - 10j)(6 + 5j)}{3 + 15j}}$$

$$+ \frac{440\,(0{\cdot}5 - 0{\cdot}866j)}{5 - 10j + 3 + 15j + \frac{(5 - 10j)(3 + 15j)}{6 + 5j}}$$

$$= \frac{440}{9{\cdot}78 - 10{\cdot}57j} + \frac{440\,(0{\cdot}5 - 0{\cdot}866j)}{28 - 4{\cdot}1j}$$

$$= 440\,[0{\cdot}069 + 0{\cdot}023j]$$

$$= 30{\cdot}4 + 10{\cdot}1j$$

Thus the numerical value of I_1 is $\sqrt{30{\cdot}4^2 + 10{\cdot}1^2} = 32{\cdot}1$ amp. and it leads E_{12} by a phase angle $\tan^{-1}\frac{10{\cdot}1}{30{\cdot}4}$.

The application of the symbolic method to alternating current bridge networks is fully dealt with by Hague (Ref. (1)), and the application to general three-phase networks is given by Dover (Ref. (2)). As the matter in this chapter is necessarily brief, these works should be consulted by readers desiring fuller information on the subject.

Symmetrical Components. The method of calculation referred to as that of "Symmetrical Components" involves, and is an extension of, the symbolic methods already described. It is especially applicable to the solution of problems in connection with unbalanced polyphase networks, and simplifies the calculation when it would be very difficult, if not impossible, by other methods. The most usual polyphase system is, of course, the three-phase, and the symmetrical components method will be discussed here with reference to such a system.

The method, which was largely developed by C. L. Fortescue (Ref. 8), involves the analysis of an unbalanced system of three-phase vectors into three systems which are each balanced but which have different phase sequences. These are referred to as the *positive-sequence*, *negative-sequence*, and *zero-sequence* systems respectively and constitute the symmetrical components of the three original unbalanced vectors. In other words, each of these vectors is split up into three components, each of which forms part of a balanced system.

Fig. 61 shows positive-, negative-, and zero-sequence systems of vectors in diagrams (i), (ii), and (iii). It must be clearly understood that all vectors are assumed to rotate in an anti-clockwise direction in accordance with the standard convention. The sequence is determined, not by any differences in direction of rotation, but by the order in which the vectors pass any fixed position. Thus, in the positive sequence this order is *a*, *b*, *c*, whereas in the negative sequence it is *a*, *c*, *b*. The three zero-sequence vectors are in phase with one another so that they pass any fixed position together. All three systems of vectors are balanced, so that E_{a1}, E_{b1} and E_{c1} are all equal in magnitude as are E_{a2}, E_{b2} and E_{c2}, and E_{a0}, E_{b0}, E_{c0}.*

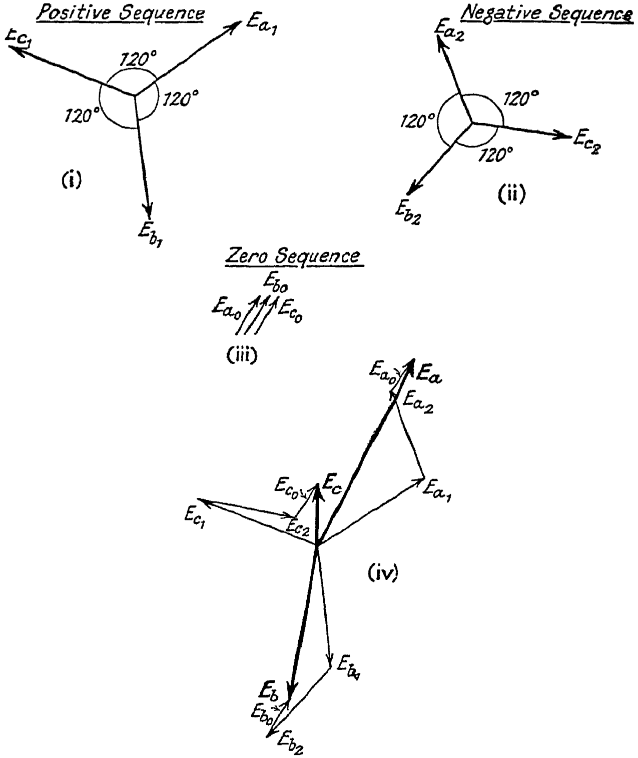


FIG. 61. SYMMETRICAL COMPONENTS

In diagram (iv) of Fig. 61 the three vectors E_{a1}, E_{a2}, and E_{a0} of diagrams (i), (ii), and (iii) are added vectorially to give vector E_a, as are E_{b1}, E_{b2}, E_{b0} and E_{c1}, E_{c2}, E_{c0} to give E_b and E_c respectively. It will be noticed that the result is an unbalanced system of vectors E_a, E_b, and E_c. Conversely, the unbalanced system E_a, E_b, and E_c can be split up into, or replaced by, the three balanced systems shown in diagrams (i), (ii), and (iii), these three diagrams being supposed to correspond to the same instant of time, so that they may be superposed in one diagram to show the correct phase relationships between the nine vectors.

---

* The notation used here is that commonly adopted in published work on the subject of symmetrical components.

We must now consider the mathematical treatment of symmetrical components. For this purpose we introduce a vector or operator a which is comparable with the operator j already used, except that the multiplication of a vector by a rotates it through an angle of 120° in an anti-clockwise direction instead of by 90° as does multiplication by j.

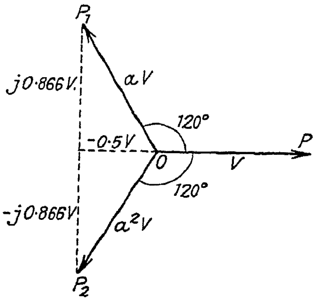


FIG. 62

Referring to Fig. 62, OP represents a vector V (or $V + j\,.\,0$), while OP_1 is the vector aV and OP_2 the vector a^2V.

Obviously

$$OP_1 = aV = -0{\cdot}5\ V + j\,.\,0{\cdot}866V$$
$$= (-0{\cdot}5 + j\,.\,0{\cdot}866)V$$

and

$$OP_2 = a^2V = -0{\cdot}5V - j\,0{\cdot}866V$$
$$= (-0{\cdot}5 - j\,0{\cdot}866)V$$

so that $a = -0{\cdot}5 + j\,0{\cdot}866$

and $a^2 = -0{\cdot}5 - j\,0{\cdot}866$

Again, if we multiply vector OP_2 by a we rotate it a further 120° to OP, so that

$$= OP = V \text{ or } a^3 = 1^*$$

Summarizing, we have therefore,

$$a = -0{\cdot}5 + j\,0{\cdot}866$$
$$a^2 = -0{\cdot}5 - j\,0{\cdot}866$$
$$a^3 = 1$$
$$a^4 = a^3\,.\,a = -0{\cdot}5 + j\,0{\cdot}866$$

and so on.

It is important to note, also, that

$$1 + a + a^2 = 1 - 0{\cdot}5 + j\,0{\cdot}866 - 0{\cdot}5 - j\,0{\cdot}866 = 0$$

Adopting the exponential notation of page 108, a is the unit vector $1.\varepsilon^{j\frac{2\pi}{3}}$ or $1.\varepsilon^{j120°}$ so that any vector $M.\varepsilon^{j\alpha}$, when multiplied by a, becomes $M\varepsilon^{j(\alpha+120°)}$. The vector OP may be written in this notation as $V\varepsilon^{j\cdot 0}$, OP_1 as $V\varepsilon^{j120°}$, and OP_2 as $V\varepsilon^{j240°}$.

---

* 1, a and a^2 are the cube roots of unity.

Using the operator a in relation to the symmetrical components of Fig. 61 we have, for example, $E_{c1} = aE_{a1}$ and $E_{b1} = a^2 E_{a1}$, so that we obtain the following equations—

$$\left.\begin{aligned} E_{a1} &= E_{a1} \\ E_{b1} &= a^2 E_a \\ E_{c1} &= a\,E_a \end{aligned}\right\} \text{ Positive-sequence system}$$

$$\left.\begin{aligned} E_{b2} &= a\,E_{a2} \\ E_{c2} &= a^2\,E_{a2} \end{aligned}\right\} \text{ Negative-sequence system}$$

$$E_{a0} = E_{b0} = \qquad \text{Zero-sequence system}$$

Hence the three unbalanced vectors E_a, E_b, and E_c may be expressed by the symbolic expressions below—

$$E_a = E_{a0} + E_{a1} + E_{a2} = E_{a0} + E_{a1} + E_{a2} \quad . \quad . \quad . \quad \text{(i)}$$

$$E_b = E_{b0} + E_{b1} + E_{b2} = E_{a0} + a^2E_{a1} + aE_{a2} \quad . \quad . \quad \text{(ii)}$$

$$E_c = E_{c0} + E_{c1} + E_{c2} = E_{a0} + aE_{a1} + a^2E_{a2} \quad . \quad . \quad \text{(iii)}$$

The symmetrical components E_{a0}, E_{a1}, and E_{a2} may be derived, in terms of the three unbalanced vectors E_a, E_b, and E_c, from these equations as shown below.

Positive-Sequence Components. Utilizing the values obtained in equations (i), (ii), and (iii) we have for the sum of E_a, aE_b, and a^2E_c

$$\begin{aligned} E_a + aE_b + a^2E_c &= E_{a0}\,(1 + a + a^2) \\ &\quad + E_{a1}\,(1 + a^3 + a^3) + E_{a2}\,(1 + a^2 + a^4) \\ &= 3E_{a1} \qquad \text{since } a^4 = a,\ a^3 = 1 \\ &\qquad\qquad \text{and } 1 + a + a^2 = 0 \end{aligned}$$

Hence
$$E_{a1} = \frac{E_a + aE_b +}{3}$$

Negative-sequence Components. Again, from (i), (ii) and (iii)

$$\begin{aligned} E_a + a^2E_b + aE_c &= E_{a0}\,(1 + a + a^2) \\ &\quad + E_{a1}\,(1 + a^4 + a^2) \\ &\quad + E_{a2}\,(1 + a^3 + a^3) \\ &= 3E_{a2} \end{aligned}$$

Hence
$$E_{a2} = \frac{E_a + a^2E_b + aE_c}{3}$$

Zero-sequence Components. Adding (i), (ii) and (iii) we have

$$\begin{aligned} E_a + E_b + E_c &= 3E_{a0} + E_{a1}\,(1 + a^2 + a) \\ &\quad + E_{a2}\,(1 + a + a^2) \\ &= 3E_{a0} \end{aligned}$$

Hence

$$E_{a0} = E_{b0} = E_{c0} = \frac{E_a + E_b + E_c}{3}$$

The zero-sequence components are thus each equal to one-third the vector sum of the three unbalanced vectors.

These statements and equations may, from the use of the symbol E throughout, be taken to apply to voltage vectors, but they apply in exactly the same way to current or impedance vectors.

From the above, two important facts are immediately apparent. First, if a system is balanced, the zero-sequence and negative-sequence components are both zero, since we may write $E_b = a^2E_a$ and $E_c = aE_a$ from which

$$E_{a0} = \frac{E_a + a^2E_a + aE_a}{3} = \frac{E_a(1 + a + a^2)}{3} = 0$$

$$E_{a2} = \frac{E_a + a^4E_a + a^2E_a}{3} = \frac{E_a(1 + a + a^2)}{3} = 0$$

The positive-sequence components are equal to the balanced vectors themselves since

$$E_{a1} = \frac{E_a + a^3E_a + a^3E_a}{3} = \frac{3E_a}{3} = E$$

Again, although three vectors may not constitute a balanced system, yet if their resultant is zero (i.e. if the vector sum $E_a + E_b + E_c$ is zero) the zero-sequence components must be zero since $E_{a0} = (E_a + E_b + E_c)/3 = 0$. It follows, therefore, that in a mesh-connected system there are no zero-sequence components of voltage and in a star-connected three-wire system with an insulated neutral point there are no zero-sequence components of current.

It must be realized that in the few pages of available space here no more than a brief introduction to the method of symmetrical components can be given. For fuller treatment, including the application of the method to the calculation of networks upon which there are faults, for which purpose it is particularly suited, the reader should refer to the works mentioned at the end of the chapter.

Before concluding, however, the application of the method will be illustrated by an alternative solution of the problem given on page 118.

Example. Let E_a, E_b, and E_c be the voltages, line to neutral, applied to the impedances 1, 2, and 3 in Fig. 60.

Then

$$\begin{aligned} E_a &= (5 - 10j)I_1 &&= (5 - 10j)[I_{a0} + I_{a1} + I_{a2}] \\ E_b &= (6 + 5j)I_2 &&= (6 + 5j)[I_{b0} + I_{b1} + I_{b2}] \\ E_c &= (3 + 15j)I_3 &&= (3 + 15j)[I_{c0} + I_{c1} + I_{c2}] \end{aligned}$$

where I_1, I_2, and I_3 are the three line currents, expressed in symbolic notation, and I_{a0}, I_{a1}, I_{a2}, etc., their symmetrical components. Since there is no fourth wire the zero-sequence components I_{a0}, I_{b0}, and I_{c0} are zero, so that we have

$$E_a = (5 - 10j)\ [I_a + I_{a2}] \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (x)$$

$$E_b = (6 + 5j)\ [I_{b1} + I_{b2}] = (6 + 5j)\ [a^2I_{a1} + aI_{a2}] \quad . \quad . \quad . \quad (y)$$

$$E_c = (3 + 15j)\,[I_{c1} + I_{c2}] = (3 + 15j)\,[aI_{a1} + a^2I_{a2}] \quad . \quad . \quad . \quad (z)$$

Now, from the vector diagram of Fig. 60

$$E_a - E_b = 440\,;\ E_b - E_c = a^2 \,.\, 440\,;\ E_c - E_a = a \,.\, 440$$

From (x) and (y), by subtraction, giving a and a^2 their known values,

$$E_a - E_b = I_{a1}\,[3{\cdot}66 - 2{\cdot}3j] + I_{a2}\,[12{\cdot}34 - 12{\cdot}7j] = 440$$

Again, from (y) and (z)

$$E_b - E_c = I_{a1}\,[15{\cdot}84 - 2{\cdot}8j] + I_{a2}\,[-\ 18{\cdot}84 + 12{\cdot}8j]$$
$$= a^2 \,.\, 440 = -\ 220 - 381j$$

And, from (z) and (x)

$$E_c - E_a = I_{a1}\,[-\ 19{\cdot}5 + 5{\cdot}1j] + I_{a2}\,[6{\cdot}5 - 0{\cdot}1j]$$
$$= a \,.\, 440 = -\ 220 + 381j$$

Only the first two of these equations are needed to evaluate I_{a1} and I_{a2}. First eliminating I_{a1} by multiplying the first equation by $[15{\cdot}84 - 2{\cdot}8j]$ and the second by $[3{\cdot}66 - 2{\cdot}3j]$ and subtracting, we have for I_{a2}

$$I_{a2} = 12{\cdot}6 + 18{\cdot}8j$$

$$\text{and}\quad I_{a1} = \frac{440 - [12{\cdot}34 - 12{\cdot}7j]I_{a2}}{3{\cdot}66 - 2{\cdot}3j} = 17{\cdot}8 - 8{\cdot}5$$

Then

$$I_1 = I_{a1} + I_{a2} = 30{\cdot}4 + 10{\cdot}1j$$

which is the same result as previously obtained on page 119.

We may proceed to find I_2 and I_3 as follows—

$$I_2 = a^2Ia_1 + aI_{a2}$$
$$= -\ 38{\cdot}86 - 9{\cdot}55j$$
$$I_3 = aI_{a1} + a^2I_{a2}$$
$$= 8{\cdot}46 - 0{\cdot}55j$$

Graphical methods of determining the symmetrical components of three unbalanced voltages or currents are discussed fully in Chapter XIII of the book by Wagner and Evans mentioned in Ref. (7). The measurement of such quantities is also dealt with in Chapter XIV of the same book. Specially constructed meters for the analysis of unbalanced voltages and currents into their symmetrical components are described in a paper by T. A. Rich (Ref. (11)).

BIBLIOGRAPHY AND REFERENCES

(1) *Alternating Current Bridge Methods*, B. Hague.
(2) *Theory and Practice of Alternating Currents*, A. T. Dover.
(3) *Electrical Power Transmission and Interconnection*, C. Dannatt and J. W. Dalgleish.
(4) *Electric Circuits*, O. G. C. Dahl.
(5) *Alternating Current Phenomena*, C. P. Steinmetz.

(6) *Theory and Calculation of Electric Circuits*, J. L. La Cour and O. S. Bragstad.

(7) *Symmetrical Components*, C. F. Wagner and R. D. Evans.

(8) "Method of Symmetrical Co-ordinates Applied to the Solution of Polyphase Networks," C. L. Fortescue, *Trans. A.I.E.E*, Vol. XXXVII, Pt. II, p. 1027.

(9) "Symmetrical Components and their Application to the Phase Convertor," F. M. Denton, *Journ. I.E.E.*, Vol. LXXI, p. 663.

(10) "Experimental Analysis of Double Unbalances," E. W. Kimbark, *Trans. A.I.E.E.*, Vol. LIV, p. 159.

(11) "The Symmetrical-component Meter," T. A. Rich, *Gen. Elect. Review*, May, 1935.

(12) "Complex Vectors in 3-phase circuits," A. Pen-Tung Sah, *Trans. A.I.E.E.* Vol. LV, p. 1356.

(13) *Symmetrical Component Analysis of Unsymmetrical Polyphase Systems*, R. Neumann.

(14) *Elements of Symmetrical Component Theory*, G. W. Stubbings.

(15) *Application of the Method of Symmetrical Components*, W. V. Lyon.

(16) "The Determination of Symmetrical Components by Multiple Magnetic Deflection of a Cathode-ray Beam," F. de la C. Chard, *Jour. I.E.E.*, Vol. LXXXIII, p. 684.

(17) "Symmetrical Components," E. W. Golding, *Electrician*, 27th Dec., 1935.

(18) "The Theory of Symmetrical Components and its Application to Protective Devices," E. W. Golding, *Mining Electrical Engineer*, June, 1938.

CHAPTER IV

CAPACITORS, CAPACITANCE, AND DIELECTRICS

General Considerations. In Chapter I capacitance was defined with reference to a number of conductors having different charges and being at different potentials. Self- and earth-capacitances were also discussed. Before proceeding to develop formulae for the capacitances of various common arrangements of conductors encountered in practice it may be well to give these matters a little further consideration.

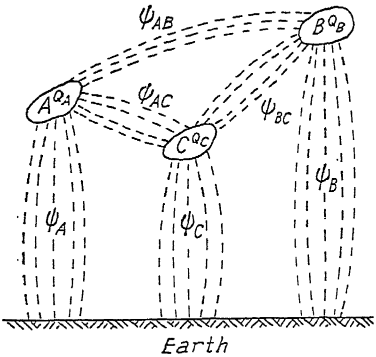


FIG. 63. SYSTEM OF CHARGED CONDUCTORS NEAR TO EARTH

Fig. 63 shows a general system of conductors in air, situated at various distances from earth and from one another. If all these conductors are at the same potential above earth, varying quantities of electric flux will pass from them to earth, these fluxes depending, in each case, upon the size and shape of the conductor, and upon its position relative to earth—i.e. upon the "earth capacitance" of each conductor. No flux will pass from one conductor to another, since they are all at the same potential above earth. The quantities of positive electricity existing upon the various conductors will be different, since their earth capacitances are different and their potentials the same.

Suppose the capacitances of the various conductors to earth are given by C_A, C_B, C_C, etc. Suppose now that the conductors are charged to different potentials V_A, V_B, V_C, etc., above earth. In this case, not only will some flux pass from each conductor to earth, but, in addition, flux will pass between any one conductor and each of the others in the system. Each of these inter-conductor fluxes will be proportional to the difference of potential of the conductors between which it exists, and its direction will, of course, depend upon which of the two conductors concerned is at the higher potential. If conductor A is at a higher potential than any of the other conductors, fluxes will flow from it, which may be represented by ψ_{AB}, ψ_{AC} ψ_{AD}, and so on, the second suffix letter indicating, in each case, the conductor to which the particular flux radiated from A flows. If B is at the second highest potential, the fluxes radiating from it are $-\psi_{BA}$, ψ_{BC}, ψ_{BD}, etc., and for conductor C, $-\psi_{CA}$, $-\psi_{CB}$, ψ_{CD}, etc., assuming it to be the third highest in potential. As stated above there will be, in each case, an earth flux which may be represented by ψ_A, ψ_B, ψ_C, etc.

It may be supposed that a portion of the total charge of each conductor is associated with each of the fluxes radiating from that conductor. These portions of charge will, of course, be proportional to the corresponding fluxes, and therefore will be proportional to the differences in potential between the pairs of conductors. Representing these portions of charge, in the case of A by Q_{AB}, Q_{AC}, Q_{AD}, etc., and in the case of B by Q_{BA}, Q_{BC}, Q_{BD}, etc., and so on, we have for the total charges on the various conductors

$$\begin{aligned}
Q_A &= C_A V_A + C_{AB}(V_A - V_B) + C_{AC}(V_A - V_C) + C_{AD}(V_A - V_D) + \ldots \\
Q_B &= C_B V_B + C_{AB}(V_B - V_A) + C_{BC}(V_B - V_C) + C_{BD}(V_B - V_D) + \ldots \\
Q_C &= C_C V_C + C_{AC}(V_C - V_A) + C_{BC}(V_C - V_B) + C_{CD}(V_C - V_D) + \ldots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (81)
\end{aligned}$$

Thus, if there are n capacitors, each one has n component capacitances, including its earth capacitance.

In most cases in practice we are concerned with two (or it may be three or four) conductors, which are so near together, compared with their distances from other conductors and from earth, that the capacitances due to the latter can be neglected. Thus, in the case of a capacitor having two plates, A and B, near together, it is only the capacitance C_{AB} which is considered, and this is spoken of as the capacitance of the capacitor. In the cases considered in the following pages earth capacitances and inter-capacitances with conductors other than those forming the arrangement under consideration, will be neglected unless otherwise stated. The earth capacitance, and intercapacitance with other conductors, may, however, be of considerable importance if the capacitor is of small capacitance and large dimensions. In the case of capacitors of capacitance $\frac{1}{10}$ microfarad and over, earth capacitances are usually negligible.

Capacitance of Various Systems of Conductors. 1. CAPACITANCE OF AN ISOLATED SPHERICAL CONDUCTOR. Suppose the spherical conductor to be perfectly insulated and at an infinite distance from all other conductors. Let its radius be R cm. and let the medium surrounding it have permittivity κ.

If a charge of Q units of electricity be given to the sphere, the intensity of the electric field at any point outside it is the same as it would be if the charge were concentrated at the centre of the sphere. Thus, the intensity at any point P, distant x cm. from the centre of the sphere, is, from Equation (1),

$$E = -^{Q}$$

and the potential of the sphere is given by

$$V = \int_{R} \frac{Q}{\kappa x^2}\,dx = \frac{Q}{\kappa R}$$

$$\therefore \text{ The capacitance of the isolated sphere} = \frac{Q}{Q/\kappa R} \tag{82}$$

If the sphere is in air, its capacitance in electrostatic c.g.s. units is equal to its radius R, expresed in centimetres; or, in air,

$$C \qquad R$$

2. CAPACITANCE OF A SPHERICAL CONDUCTOR INSIDE A CONCENTRIC HOLLOW CONDUCTING SPHERE. Let the radii of the inner and outer spheres be R_1 and R_2 cm. respectively, the latter being the radius of the inner spherical surface of the outer sphere. Let κ be the permittivity of the medium between them.

If a charge of $+ Q$ units be given to the inner sphere a charge of $- Q$ units will be induced on the inner surface of the outer sphere. Since, as shown in Chapter I, the intensity at any point inside a hollow charged conductor is zero, the intensity at any point between the two spheres will be that due to the inner sphere only. Taking any point P, distant x cm. from the centre of the inner sphere, and, as before, considering the charge on this sphere to be concentrated at its centre, we have, for the intensity at P,

$$E_P = \frac{Q}{\kappa x^2}$$

The potential difference between the spheres is given by

$$V = \int_{R_1}^{R_2} \frac{Q}{\kappa x^2} \,.\, dx \,. = \left[-\frac{Q}{\kappa x}\right]_{R_1}^{R_2}$$

---

* The potentials and capacitances here and in the rest of the chapter are expressed in e.s.c.g.s. units but can be expressed in m.k.s. units if κ_0, the permittivity of free space, is inserted (*see* Ch. II).

† This conversion to farads—by dividing by 9 × 10^{11}—will be omitted in the rest of the chapter.

$$\therefore V = \frac{Q}{\kappa}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Hence, the capacitance of the arrangement is

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{\kappa}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} = \frac{\kappa R_1 R_2}{R_2 - R_1} \quad . \quad . \quad . \quad (83)$$

3. CAPACITANCE BETWEEN TWO SPHERES AT A RELATIVELY GREAT DISTANCE APART. In this case each sphere will have its own "self-capacitance," and also a mutual capacitance with the other sphere. Suppose that the two spheres have equal and opposite charges, and are at a relatively great distance apart, and infinitely distant from all other bodies.*

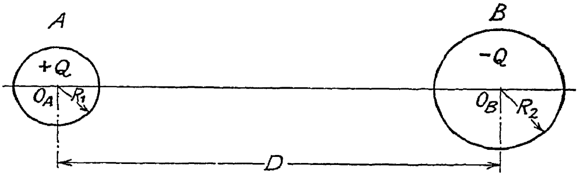


FIG. 64. TWO CHARGED SPHERES

Under these conditions, if the charges upon spheres A and B are $+Q$ and $-Q$ units, and their potentials V_1 and V_2, then the capacitance between the spheres is

$$C = \frac{Q}{V_1 - V_2}$$

Let the spheres have radii R_1 and R_2 cm. respectively, and let their distance apart be D cm. in air (see Fig. 64). Then the potential at the centre O_A of sphere A due to its own charge is $\frac{Q}{R_1}$. If the second sphere is distant from sphere A, the potential at O_A due to the charge on B is $-\frac{Q}{D}$

$$\therefore V_1 = \frac{Q}{R_1} - \frac{Q}{D}$$

By similar reasoning

$$V_2 = -\frac{Q}{R_2} + \frac{Q}{D}$$

---

* The capacitance of a system of two charged spheres in the general case has been fully investigated by Russell (Ref. (12)).

Thus the capacitance between the spheres

$$C = \frac{Q}{V_1 - V_2} = \frac{Q}{\dfrac{Q}{R_1} - \dfrac{Q}{D} - \left(\dfrac{-Q}{R_2} + \dfrac{Q}{D}\right)} = \frac{1}{\dfrac{1}{R_1} - \dfrac{2}{D} + \dfrac{1}{R_2}}$$

or

$$C = \frac{R_1 R_2 D}{D(R_1 + R_2) - 2R_1R_2}$$

If the medium is not air but has a permittivity κ, then

$$C = \frac{\kappa . R_1R_2D}{D(R_1 + R_2) - 2R_1R_2} \quad . \quad . \quad . \quad . \quad . \quad (84)$$

If the spheres are equal

$$C = \frac{\kappa\, RD}{2\ (D - R)}$$

where R is the common radius.

Russell (*loc. cit.*) gives the *capacitance of the two spheres in parallel,* i.e. when connected by a thin wire so that they are at the same potential as

$$C_P = \left(R_1 + R_2 - \frac{2R_1R_2}{D}\right) \frac{D^2}{D^2 - R_1R_2} \quad . \quad . \quad . \quad . \quad (85)$$

(in air), using the symbols as above. If the spheres have equal radii R, then

$$C_P = \frac{2RD}{D + R} \text{ or } C_P = \frac{2\kappa RD}{D + R}$$

when in a medium of permittivity κ.

For *two equal spheres close together,* the capacitance between the spheres is given approximately by

$$C = \frac{R}{2}\left(1 + \frac{x}{6R}\right)\left(1{\cdot}2704 + \frac{1}{2}\log_e \frac{R}{x} + \frac{x}{18R}\right) \quad . \quad . \quad (86)$$

electrostatic c.g.s. units in air, where R is the common radius and x the nearest distance between them ($= D - 2R$).

4. CAPACITANCE BETWEEN TWO CONDUCTING PLATES. Consider two equal conducting plates, placed parallel to one another, and at a distance D cm. apart, this distance being small compared with the dimensions of the plates, so that the fringing effect at the edges of the plates can be neglected. Let the area of each plate (one side only) be A sq. cm., and let the charges on the plates be $+ Q$ and $- Q$ electrostatic units.

From Chapter I the intensity at a point between the plates is $\frac{4\pi\sigma}{\kappa}$ where σ is the density of the charge and equals $\frac{Q}{A}$. Then the potential difference between the plates is

$$V = \int_0^D \frac{4\pi Q}{\kappa A} \, . dx = \frac{4\pi QD}{\kappa . A}$$

Thus, $$C = \frac{Q}{V} = \frac{Q}{4\pi QD/\kappa A} = \frac{\kappa A}{4\pi D} \quad . \quad . \quad . \quad . \quad (87)$$

Suppose that instead of there being only one dielectric in between the plates, there are several parallel layers of dielectrics of thicknesses D_1, D_2, D_3, etc., and having permittivities κ_1, κ_2, κ_3, etc., respectively, as in Fig. 65.

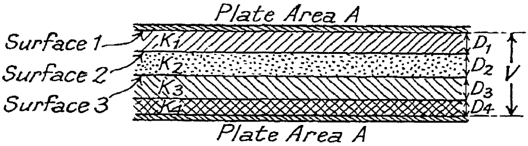


FIG. 65. DIELECTRICS IN SERIES IN A PLATE CAPACITOR

Then potential difference between surfaces (1) and (2) is

$$V_{12} = \int_0^{D_1} \frac{4\pi Q}{\kappa_1 A}\, dx = \frac{4\pi Q}{\kappa_1 A} D_1$$

while that between surfaces (2) and (3) is

$$V_{23} = \frac{4\pi Q}{\kappa_2 A} \cdot D_2$$

and so on. Thus, the total potential difference V between the parallel conducting plates is

$$\begin{aligned} V &= V_{12} + V_{23} + V_{34} + \ldots \\ &= \frac{4\pi Q}{A}\left(\frac{D_1}{\kappa_1} + \frac{D_2}{\kappa_2} + \frac{D_3}{\kappa_3} + \ldots\right) \end{aligned}$$

and the capacitance between the plates is therefore

$$C = \frac{Q}{V} = \frac{A}{4\pi\left(\frac{D_1}{\kappa_1} + \frac{D_2}{\kappa_2} + \frac{D_3}{\kappa_3} + \ldots\right)} \quad . \quad . \quad (88)$$

Effect of Additional Plates. If two more similar plates are added, one of which is connected to each of the existing plates (Fig. 66), and the same dielectric placed between them, then the effective area for the whole capacitor thus formed is $3A$, and the capacitance is thus increased to $\frac{3\kappa A}{4\pi D}$.

In general, since the use of N plates creates $N - 1$ spaces (each of width D cm.) the capacitance of such a capacitor with N plates is

$$C = \frac{\kappa \, . \, (N - 1) \, A}{4\pi D} \quad . \quad . \quad . \quad . \quad (89)$$

By this means the capacitance of a plate capacitor can be made large whilst using plates with only a comparatively small surface area.

Although these formulae must be considered as approximations, if the plates are close together they are sufficiently accurate for most practical purposes, even though the capacitor may be in the vicinity of other conductors.

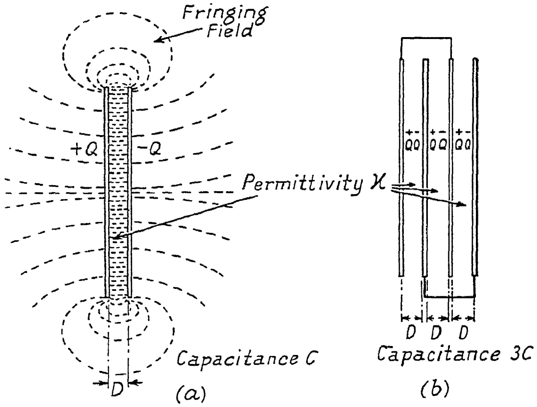


Fig. 66. Plate Capacitor

5. Capacitance Between Two Long, Parallel Conducting Cylinders. This problem can be resolved into two separate cases, namely: (*a*) when the cylinders are at a distance apart which is great compared with their diameters; (*b*) when they are comparatively close together.

In the former case it is considerably easier to calculate the capacitance between them than in the latter. This case will be considered first.

Case (*a*).

Fig. 67 represents two long parallel conducting cylinders, perpendicular to the plane of the paper, each of diameter d cm. placed at a distance D cm. apart in air, D being great compared with d and the cylinders being at a great distance from all other conductors.

Let $+ Q$ and $- Q$ units be the charges per centimetre axial length on A and

B respectively. In this case it may be assumed that the charges are concentrated at the axes of the cylinders.

From Equation (4), the intensity at P, distant x from cylinder A, due to this cylinder is $\frac{2Q}{x}$ which is the force (in dynes, if Q is in electrostatic units and x in centimetres) upon unit charge placed at P. This force is in the direction AB. Similarly, cylinder B would exert a force (of attraction) upon unit charge at P of $\frac{2Q}{D-x}$ dynes, also in the direction AB. Thus the total force upon unit charge at P is $2Q\left(\frac{1}{x}+\frac{1}{D-x}\right)$ dynes in direction AB. The potential difference between the cylinders—which is the work done in moving unit charge from the surface of one cylinder to the surface of the other—is

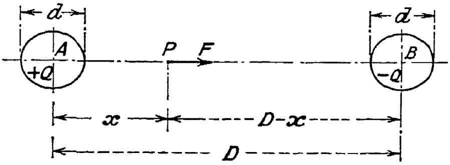


FIG. 67. PARALLEL CYLINDERS

$$\int_{x=\frac{d}{2}}^{x=D-\frac{d}{2}}\left[2Q\left(\frac{1}{x}+\frac{1}{D-x}\right)\right]dx = 2Q\left[\log_e x - \log_e(D-x)\right]$$
$$= 4Q\log_e\frac{2D-d}{d}$$

i.e. potential difference between the cylinders

$$V = 4Q\log_e\frac{2D-d}{d} \qquad (90)$$

∴ The capacitance between the cylinders *per centimetre axial length*

$$= \frac{Q}{V} = \frac{1}{4\log_e\frac{2D-d}{d}}$$

If the permittivity of the medium between the cylinders is κ, then, of course,

$$C = \frac{1{\cdot}21\,\kappa}{10^{13}\log_{10}\left(\frac{2D}{d}-1\right)} \text{ farads per cm. length}$$

or, the capacitance per mile of two such parallel cylinders in air is

$$\frac{1{\cdot}95}{10^{8}\log_{10}\left(\frac{2D}{d}-1\right)} \text{ farads} \qquad (91)$$

If D is great compared with d,

$$C \doteqdot \frac{1\cdot95}{2D} \text{ farads per mile}$$

Case (*b*). When the cylinders are comparatively close together the treatment of the problem differs from that of Case (*a*), owing to the fact that the charges of $+Q$ and $-Q$ cannot now be assumed to be concentrated at the axes of the cylinders. The charges must now be taken as concentrated along other axes, parallel to and in the same plane as the axes of the cylinders, but displaced so that the distance apart of the axes along which the charges are assumed to be concentrated is now less than the distance D. To derive an expression for the capacitance in this case the distribution of the electrostatic field between the cylinders must first be considered.

When the cylinders are at a great distance apart, as in Case (*a*), the lines of force of the electrostatic field radiate from the cylinders uniformly in all directions, each line cutting the surfaces of the cylinders perpendicularly. Since the potential of a point along any one line of force decreases as the distance of the point from cylinder A is increased, a number of equipotential surfaces exist which are in the form of cylinders concentric with the cylindrical conductors, the lines of force cutting all of these cylinders perpendicularly.

If the cylindrical conductors are comparatively close together these equipotential surfaces are still cylinders, but they are not concentric with the surfaces of the cylindrical conductors whose capacitance is to be determined, nor are they concentric with one another.

It can be shown* that the equations of the traces of these cylindrical equipotential surfaces in the plane of the paper are $r_1 = Mr$, where r_1 and r are the distances of any point on one of the circular traces from the traces X and Y of the axes along which the charges $+Q$ and $-Q$ may be assumed to be concentrated and from which the lines of electrostatic force radiate (these lines of force being circles, as in Fig. 68), and M is a constant which differs for different traces. By giving M different values a series of circular traces is obtained, as shown in the figure. When $M = 1$ the trace is a straight line, this being the trace of a plane the potential of all points on which is zero.

Now, since the surfaces of the cylindrical conductors are equipotential surfaces, the equations of whose traces in the plane of the paper are given by the above relationship ($r_1 = Mr$), it follows that the traces X and Y are not coincident with the axes of the conducting cylinders, but are displaced as shown in Fig. 68.

Calculation of Capacitance. To calculate the positions of the axes whose traces are X and Y, proceed as below.

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* See T. F. Wall's *Electrical Engineering*, p. 46.

Let the points X and Y be displaced inwards from the centres of the two circles which are the traces of the cylindrical conductors A and B by a distance m in each case, and let their distance apart be l. Then $l = D - 2m$.

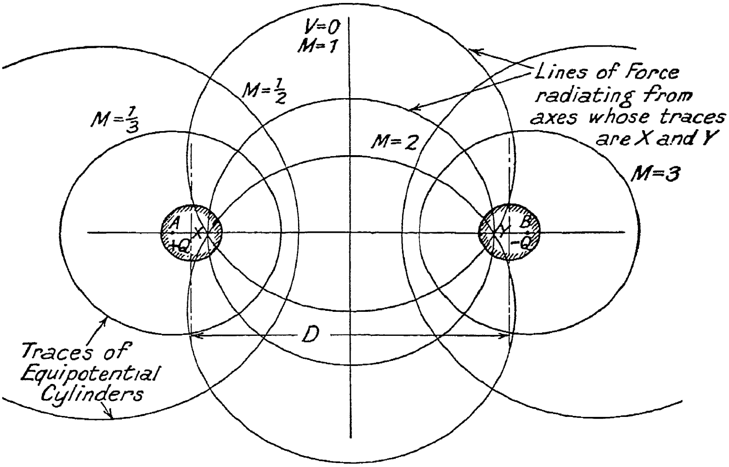

V=0
M=1
M=½
M=⅓
M=2
M=3
Lines of Force radiating from axes whose traces are X and Y
A
+Q
X
Y
B
-Q
D
Traces of Equipotential Cylinders

FIG. 68. ELECTROSTATIC FIELD BETWEEN CHARGED PARALLEL CYLINDERS WHICH ARE NEAR TOGETHER

Since the surfaces of the cylindrical conductors are equipotential surfaces, the equation $r_1 = Mr$ holds for their traces. Consider the point P (Fig. 69) on the trace of cylinder A on a line through X perpendicular to the line XY.

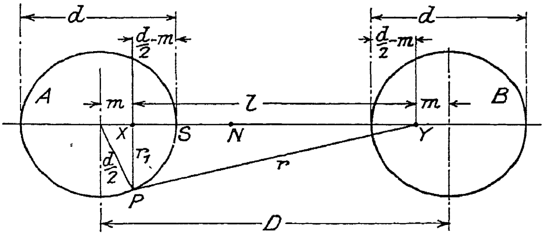


FIG. 69

Then $\left(\frac{d}{2}\right)^2 = m^2 + XP^2$, and $XP = M(PY)$, since for point P, $r_1 = XP$ and $r = PY$.

Also $l^2 + XP^2 = PY^2$

and $$m = \frac{D-l}{2}$$

For the point S,

$$r_1 = XS = \frac{d}{2} - m \text{ and } r = SY = l - \left(\frac{d}{2} - m\right)$$

Since for all points on the circular trace of A

$$r_1 = Mr$$

we have $\dfrac{d}{2} - m = M\left[l - \left(\dfrac{d}{2} - m\right)\right]$ for point S

$$\therefore M = \frac{\dfrac{d}{2} - m}{l - \left(\dfrac{d}{2} - m\right)}$$

Now $l^2 + XP^2 = PY^2 = \left(\dfrac{XP}{M}\right)^2 = \dfrac{XP^2}{\left[\dfrac{\frac{d}{2} - m}{l - \left(\frac{d}{2} - m\right)}\right]^2}$

and $\left(\dfrac{d}{2}\right)^2 = m^2 + XP^2$

$$\therefore l^2 + \left(\frac{d}{2}\right)^2 - m^2 = \frac{\left(\dfrac{d}{2}\right)^2 - m^2}{\left[\dfrac{\frac{d}{2} - m}{l - \frac{d}{2} + m}\right]^2}$$

Substituting $m = \dfrac{D - l}{2}$ and solving for l we have the solution

$$l = \sqrt{D^2 - d^2}$$

If d is small compared with D, we have $l = D$, as in Case (a).

Thus, to calculate the capacitance between the cylinders, the treatment is exactly the same as that of Case (a), except that the charges $+Q$ and $-Q$ per centimetre axial length are considered concentrated along parallel axes whose distance apart is now l instead of D.

We have then for the intensity of field at a point such as N (Fig. 69) distant x from X

$$E = \frac{2Q}{x} + \frac{2Q}{l - x}$$

and the potential difference between the cylinders

$$V = \int_{x = \frac{d}{2} - m}^{x = l - \left(\frac{d}{2} - m\right)} \left(\frac{2Q}{x} + \frac{2Q}{l - x}\right) dx$$

$$= 4Q \log_e \frac{l - \left(\dfrac{d}{2} - m\right)}{\dfrac{d}{2} - m}$$

or, since $l = \sqrt{D^2 - d^2}$ and $2m = D - l$

$$V = 4Q \log_e \left[\frac{\sqrt{D^2 - d^2} - (d - D)}{\sqrt{D^2 - d^2} + (d - D)}\right] \quad . \quad . \quad . \quad (92)$$

Thus capacitance per centimetre axial length is

$$C = \frac{Q}{V} = \frac{1}{4\log_e\left[\dfrac{\sqrt{D^2-d^2} - (d-D)}{\sqrt{D^2-d^2} + (d-D)}\right]}$$

Rationalizing and simplifying, we have

$$C = \frac{1}{4\log_e\left[\dfrac{D+\sqrt{D^2-d^2}}{d}\right]}$$

If the permittivity of the medium between the cylinders is κ, we have

$$C = \frac{\kappa}{4\log_e \dfrac{(D+\sqrt{D^2-d^2}}{d}} \quad . \quad . \quad . \quad . \quad (93)$$

or $$C = \frac{1{\cdot}95}{10^8 \log_{10}\left(\dfrac{D+\sqrt{D^2-d^2}}{d}\right)} \text{ farads per mile of double conductor in air}$$

These capacitances are given in farads *per mile*, since the arrangement of two long parallel conducting cylinders is chiefly met with in overhead transmission lines where the most useful unit of length is the mile. Formulae for the general case of two parallel cylindrical conductors have been given by Russell (Ref. (13)).

6. CAPACITANCE BETWEEN TWO COAXIAL CYLINDERS. An important case of this arrangement in practice is, of course, a concentric cable.

Consider two long conducting concentric cylinders, the diameter of the inner one being d cm. and the inner diameter of the outer one being D cm. Let $+Q$ and $-Q$ units be their charges per centimetre axial length. The lines of force of the electrostatic field will be radial, and the equipotential surfaces will be cylindrical and coaxial with the two conducting cylinders. The intensity of the field at some point at a radial distance of x cm. from the common axis of the cylinders will be $\frac{2Q}{x}$ if the dielectric separating the cylinders is air.

Thus the potential difference between the cylinders is

$$\int_{\frac{d}{2}}^{\frac{D}{2}} \frac{2Q}{x}\,dx = 2Q\left[\log_e \frac{D}{2} - \log_e \frac{d}{2}\right]$$

$$V = 2Q \log_e \frac{D}{d}$$

The capacitance per centimetre length is

$$C = \frac{Q}{V} = \frac{1}{2 \log_e \frac{D}{d}}$$

The general expression for a length l cm., the dielectric having a permittivity κ is

$$C = \frac{\kappa l}{2 \log_e \frac{D}{d}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

or

$$C = \frac{3{\cdot}89\kappa}{10^8 \log_{10} \frac{D}{d}} \text{ farads per mile}$$

7. CAPACITANCE OF A SINGLE STRAIGHT CONDUCTOR PARALLEL TO EARTH. *Method of Electric Images.* This method is based upon the imagination of an "image" of a conductor placed above the earth's surface, this image being of the same size and shape as the conductor considered and lying as far beneath the surface of the earth as the conductor considered is above the surface. The earth's surface is thus in the plane of zero potential for these two conductors—considering the image as being in actual fact a conductor placed at a distance $2H$ from the original one, H being the height of this original conductor above the earth.

Since the earth's surface is at zero potential, the electrostatic field from the charged conductor above the earth, to the surface of the earth, has the same distribution as the field which would exist between the conductor and the zero potential plane, in the case of two conductors placed at a distance of $2H$ apart.

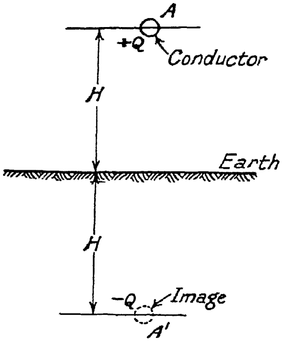


FIG. 70. CYLINDRICAL CONDUCTOR PARALLEL TO EARTH

Fig. 70 shows the trace of a cylindrical conductor A lying parallel to the earth's surface, and at a height H cm. above the earth; A' is its image. If conductor A has a charge of $+Q$ units per cm. axial length, then the potential difference between it and conductor A', which is supposed to have $-Q$ units per cm. axial length, is from Equation (90)

$$2V = 4Q \log_e \frac{4H - d}{d}$$

where d is the diameter of the conductors and is assumed small compared with H. V is the potential of A above that of the earth, and is also the potential of A' below earth potential.

Thus $$V = 2Q \log_e \frac{4H - d}{d}$$

and the capacitance per centimetre length of one conductor to earth is

$$C = \frac{\kappa}{2 \log_e \dfrac{4H - d}{d}} \quad . \quad . \quad . \quad . \quad . \quad (95)$$

where the dielectric has permittivity κ.

The capacitance per mile of one conductor to earth in air is therefore

$$C = \frac{3{\cdot}89}{10^8 \log_{10} \dfrac{4H - d}{d}} \text{ farads per mile}$$

If d is small compared with H (as is usually the case when an overhead line is considered) then

$$C = \frac{3{\cdot}89}{10^8 \log_{10} \dfrac{4H}{d}} = \frac{3{\cdot}89}{10^8 \log_{10} \dfrac{2H}{r}} \text{ farads per mile}$$

where r is the radius of the conductor in centimetres.

If d is not small compared with H, the calculation of capacitance must be based upon Equation (92) instead of Equation (90) as above.

8. CAPACITANCE BETWEEN TWO LONG, STRAIGHT CONDUCTORS, PARALLEL TO THE EARTH AND TO ONE ANOTHER. Consider two long cylindrical conductors M and N parallel to earth and to one another, their diameters being d cm. and their distance apart being D cm. Let H be their height above earth and let M' and N' be their images (Fig. 71). Suppose d small compared with H.

Let M and N have charges of $+ Q$ and $- Q$ per centimetre axial length respectively and M' and N' charges of $- Q$ and $+ Q$ units per cm. length respectively.

Consider a point P on the horizontal line joining the centres of M and N and distance x cm. from M. The intensity at P is due to all four conductors M, N, M', and N'. Thus intensity at P *in the direction* MN is—

Due to M . . $\left(\frac{2Q}{x}\right)$

Due to N . . $\left(\frac{2Q}{D - x}\right)$

Due to M'. . $$\left(-\frac{2Q}{PM'} \cos \alpha\right) = \frac{-2Q}{\sqrt{4H^2 + x^2}} \cdot \frac{x}{\sqrt{4H^2 + x^2}} = \frac{-2Qx}{4H^2 + x^2} \cdot$$

Due to N' . . $\left(\dfrac{-2Q}{\sqrt{(D-x)^2+4H^2}}\cos\beta\right)$

$$= \frac{-2Q\,(D-x)}{(D-x)^2+4H^2}$$

Resultant intensity at P is

$$\frac{2Q}{x}+\frac{2Q}{D-x}-\frac{2Qx}{4H^2+x^2}-\frac{2Q\,(D-x)}{(D-x)^2+4H^2}$$

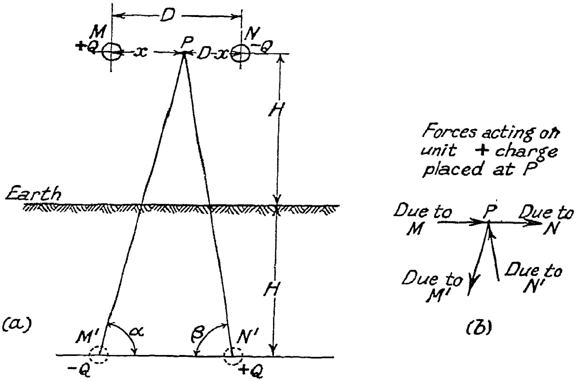


FIG. 71. TWO CHARGED PARALLEL CONDUCTORS NEAR TO EARTH

and the potential difference between M and N is

$$V=\int_{r}^{D-r}\left(\frac{2Q}{x}+\frac{2Q}{D-x}-\frac{2Qx}{4H^2+x^2}-\frac{2Q\,(D-x)}{(D-x)^2+4H^2}\right)dx$$

where r is the radius of the conductors.

Integrating, we have

$$V=2Q\left[2\log_e\frac{D-r}{r}+\log_e\frac{4H^2+r^2}{4H^2+(D-r)^2}\right] \quad . \quad . \quad (96)$$

If D is great compared with r,

$$V=4Q\log_e\frac{D}{r}+2Q\log_e\frac{4H^2}{4H^2+D^2}$$

The capacitance between the conductors is

$$C=\frac{Q}{V}=\frac{1}{4\log_e\dfrac{D}{r}+2\log_e\dfrac{4H^2}{4H^2+D^2}} \quad . \quad . \quad . \quad (97)$$

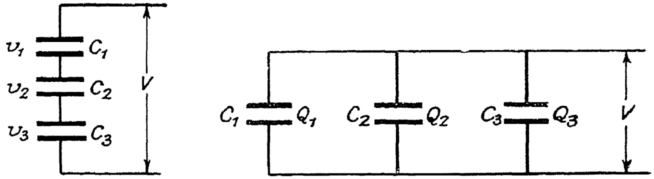
$$= \frac{1}{4 \log_e \frac{D}{r} \left(\frac{2H}{\sqrt{4H^2 + D^2}} \right)} \text{ per cm. length in air}$$

or
$$C = \frac{1{\cdot}95}{10^8 \log_{10} \frac{D}{r} \left(\frac{2H}{\sqrt{4H^2 + D^2}} \right)} \text{ farads per mile}$$

The capacitance of two parallel cylinders which are at a great distance from earth was previously found to be

$$\frac{1{\cdot}95}{10^8 \log_{10} \frac{2D}{d}} = \frac{1{\cdot}95}{10^8 \log_{10} \frac{D}{r}} \text{ farads per mile}$$

D being great compared with r.



(a) (b)

FIG. 72. CAPACITORS IN SERIES AND IN PARALLEL

Thus the proximity of the earth introduces the term $\frac{2H}{\sqrt{4H^2 + D^2}}$ in the denominator, as shown above.

The capacitance of a system of three or more conductors, parallel and near to the earth, can be found by similar methods (Refs. (1), (5), (8)).

Capacitors in Series and Parallel. (*a*) SERIES. If a number of capacitors are connected in series, as in Fig. 72 (*a*), a potential difference of V being applied between the outer terminals, there will be potential differences v_1, v_2, v_3, etc., between the different pairs of plates.

Let the capacitances of the capacitors (neglecting earth capacitances) be C_1, C_2, C_3, etc. If a quantity of electricity Q units is given to the system of capacitors by means of a current which flows for a short time through them until they are charged to the total potential difference V, then

$$v_1 = \frac{Q}{C_1}, \; v_2 = \frac{Q}{C_2} \; v_3 = \frac{Q}{C_3}$$

and so on.

If C is the capacitance of the whole system, the potential difference for which is V, then

$$C = \frac{Q}{V} \text{ or } V = \frac{Q}{C}$$

Thus, since

$$V = v_1 + v_2 + v_3$$

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \ldots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \quad . \quad . \quad . \quad (98)$$

(*b*) PARALLEL. If a potential difference V is applied to a number of capacitors connected in parallel (Fig. 72 (*b*)), then the potential difference across the plates of such capacitors is, in each case, V, but the quantities of electricity given to the capacitors are now different for the different capacitors. If these quantities are Q_1, Q_2, Q_3, etc., then

$$v_1 = \frac{Q_1}{C_1} \text{ or } Q_1 = v_1 C_1$$

$$v_2 = \frac{Q_2}{C_2} \text{ or } Q_2 = v_2 C_2$$

and so on.

But

$$v_1 = v_2 = v_3 = \ldots \quad = V$$

and the total quantity of electricity

$$Q = Q_1 + Q_2 + Q_3 + \ldots \quad = CV$$

where C is the total capacitance.

Thus

$$CV = C_1 v_1 + C_2 v_2 + C_3 v_3 + \ldots$$
$$= V(C_1 + C_2 + C_3 + \ldots)$$
$$\therefore C = C_1 + C_2 + C_3 + \ldots \quad . \quad . \quad . \quad (99)$$

Two-core Cable. In the case of multi-core cables generally, the earth capacitances of the cores cannot be neglected. A two-core cable consists essentially of two long parallel conductors embedded in some insulating material, the whole being enclosed by an earthed, conducting cylinder, as in Fig. 73(*a*).

This arrangement is equivalent to the system of capacitors shown in Fig. 73 (*b*). If the cores are represented by A and B, then C_{AB} is the capacitance between cores and C_A and C_B the earth capacitances of the two conductors. We thus have C_A and C_B in series with one another this series circuit being in parallel with C_{AB}, the equivalent arrangement being represented in Fig. 73 (*c*). The capacitance of C_A and C_B in series is $\frac{C_A C_B}{C_A + C_B}$, and when this is connected in parallel with C_{AB} the total, or working, capacitance is $C_{AB} + \frac{C_A C_B}{C_A + C_B}$.

Three-core Cable. The capacitances which exist in the case of a three-core cable are shown in Fig. 74 (*a*), in which C_1 is the intercore capacitance, and C_0 the earth capacitance. Diagram (*b*) shows the equivalent circuit of such a cable when used on a three-phase system of line voltage E.

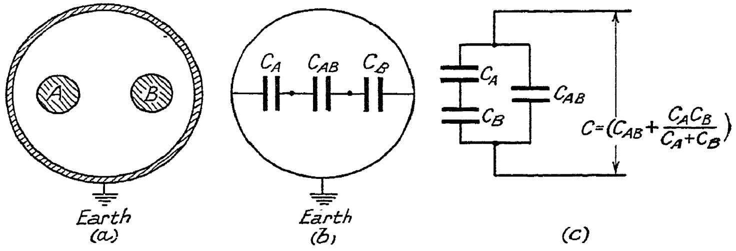


FIG. 73. CAPACITANCE OF A TWO-CORE CABLE

To facilitate calculations of the charging current per line it is usual to resolve the system shown in diagram (*b*) into either an *equivalent mesh* system, as in diagram (*c*), or an *equivalent star* system as in diagram (*d*). In the first case, the three capacitances C_0 are replaced by three imaginary capacitances C_m, connected in mesh,

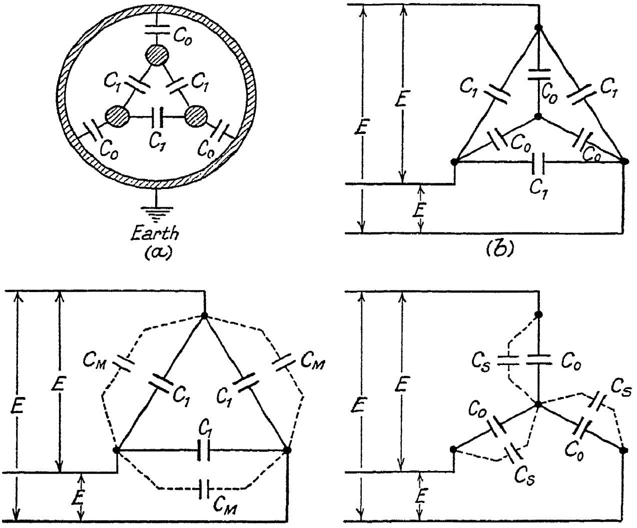


FIG. 74. CAPACITANCE OF A THREE-CORE CABLE

in parallel with the inter-core capacitances C_1, and having such values that the charging current per line is the same as that for the actual cable. The magnitude of C_m is thus determined as follows—

The voltage to neutral (i.e. the voltage across each capacitor C_0) is $E/\sqrt{3}$ and the charging current taken by each C_0 is $(E/\sqrt{3})\,.\,\omega\,C_0$.

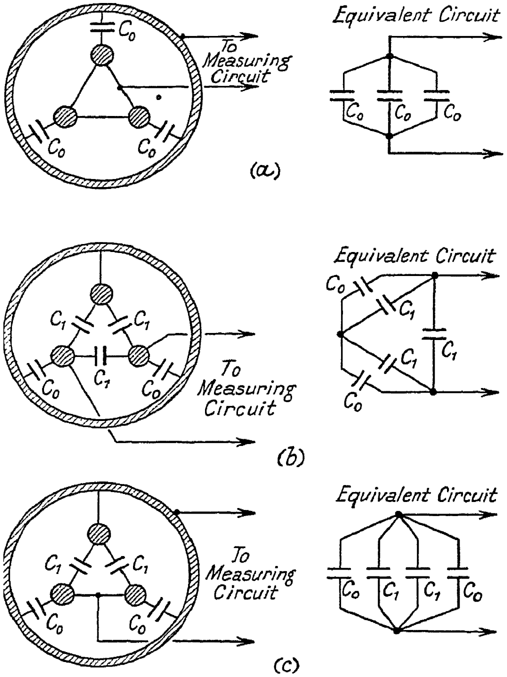


FIG. 75. CABLE CAPICITANCE MEASUREMENTS

In diagram (*c*) the current taken by each capacitor C_m is $E\,.\,\omega\,C_m$ and the *line* current on this account is thus $\sqrt{3}.\ E\,.\,\omega\,C_m$.

For equivalence this line current must be equal to $(E/\sqrt{3})\,.\,\omega\,C_0$.

Thus,
$$\sqrt{3}\,.\,E\,.\,\omega C_m = \frac{E}{\sqrt{3}}\,.\,\omega C_0$$

or
$$C_m = \frac{C_0}{3}$$

Hence the total equivalent mesh system consists of three groups of C_1 each in parallel with C_m, i.e. three capacitances $C_1 + \frac{C_0}{3}$ connected in mesh.

Diagram (*d*) shows the equivalent star system, in which the capacitors C_1 are replaced by three capacitors C_3, each in parallel with C_0 and of such values that the line currents are the same as for the actual cable.

To determine the value of C_s—

$$\text{Current taken by each capacitance } C_s = \frac{E}{\sqrt{3}} \cdot \omega C_s$$

Now, current taken by each capacitance C_1 (diagram (*b*)) is $E\omega C_1$, and the line current on this account $= \sqrt{3} \,.\, E\omega C_1$.

For equivalence

$$\frac{E}{\sqrt{3}} \cdot \omega C_s = \sqrt{3}\, E\omega C_1 \text{ or } C_s = 3C_1$$

So that the total equivalent star system consists of three groups of capacitors in star, each consisting of C_0 and C_3 in parallel, i.e. three capacitances of $C_0 + 3C_1$.

Measurements of Three-core Cable Capacitances. The values of the capacitances C_0 and C_1 for a given length of cable may be determined by means of two tests. First, the three cores are connected together and the capacitance between them and the sheath measured (see Fig. 75 (*a*)). The measured capacitance is obviously $3C_0$.

The second test may be of the capacitance between two cores, the third being connected to the sheath (Fig. 75 (*b*)) or between two cores connected together, and the sheath and third core connected together (Fig. 75 (*c*)).

In the former case the capacitance obtained by the measurement is

$$\frac{C_0 + C_1}{2} + C_1 = \frac{3}{2} C_1 + \frac{C_0}{2}$$

In the latter case the measured value is $2C_0 + 2C_1$.

The first test obviously enables C_0 to be determined and this value, substituted in either of the expressions obtained above for the two alternative methods of carrying out the second test, renders C_1 calculable.

Distributed Capacitance. In the foregoing paragraphs it has been assumed in all cases that the surfaces of the conductors considered have been assumed to be equipotential surfaces.

There are many important cases in practice when this is not so, and in these cases the calculation of capacitance cannot be carried out by the simple methods used above. In wire-wound solenoids we have capacitance between adjacent turns, and layers, and all the conductors in one layer are obviously not at the same potential. The earth capacitances of the turns in the coil also are not all the same. In such coils we have what is referred to as "distributed capacitance."

The effect of such distributed capacitance is, in many cases, small for low-frequency work, and an equivalent circuit, which represents such a coil sufficiently accurately for most purposes, can then be obtained by assuming the coil itself to be free from capacitance but as having a simple capacitor connected in parallel with it, and also having simple capacitors connected between parts of the coil and earth. The latter represent the distributed earth capacitance, while the former represents the distributed inter-turn capacitance.

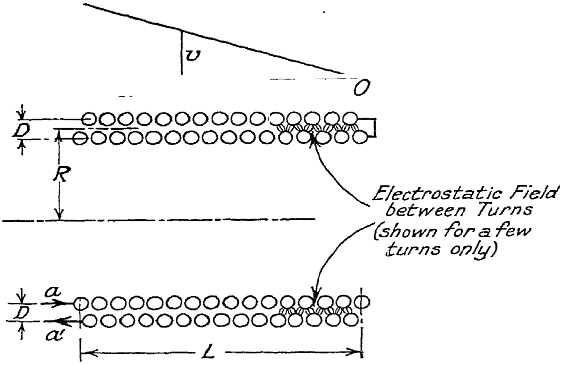

v
O
D
R
Electrostatic Field between Turns (shown for a few turns only)
a
D
a'
L

FIG. 76. CAPACITANCE OF A TWO-LAYER SOLENOID

If such a coil is to be used for very high frequency work, e.g. radio frequency work, such approximate methods of representation are not justifiable, since the distributed capacitance of the coil may, at such frequencies, become of more importance than its inductance.

Capacitance of a Two-layer Solenoid. Fig. 76 represents a solenoid of circular section, having two layers of insulated wire wound continuously so that, in effect, the layers are connected together at one end as shown. If a steady potential difference V is applied to the terminals (aa') of the coil, then the potential difference between layers will vary from V at the left-hand end of the coil to zero at the right-hand end, and the electrostatic field between adjacent turns will thus decrease from a maximum to zero, moving from left to right. Morecroft (*Principles of Radio Communication,* Chap. II) calculates the internal capacitance of such a coil by treating it as, essentially, two coaxial conducting cylinders, whose capacitance, if the layers of wire are close together compared with the diameter of the coil, is given by the formula for flat plates, assuming at first that the cylinders are equipotential surfaces.

Thus, $C = \frac{\kappa A}{4\pi D}$ where C is the capacitance when the potential difference is the same throughout the axial length of the cylinders, A being the area of each cylindrical surface, κ the permittivity of the medium, and D the distance between the layers.

If R is the radius of the section of the solenoid (assumed the same for both layers, since their distance apart is small) and L is their axial length, then $A = 2\pi RL$ and $C = \frac{\kappa RL}{2D}$ or capacitance per centimetre axial length is $\frac{\kappa R}{2D}$.

Actually the potential difference between layers varies along the axial length from V to zero. Assuming this variation to be according to a straight line law, we have

Energy stored in axial length dx

$$dW = \frac{c \, . \, v^2}{2} = \frac{\kappa R}{2D} \cdot \frac{v^2}{2} \, . \, dx$$

where v is the potential difference between layers at any point of axial distance x from the left-hand end (Fig. 76). Since $\frac{V}{L} = \frac{v}{L - x}$ we have $v = \left(1 - \frac{x}{L}\right) V$ and

$$dW = \frac{\kappa \, . \, R}{2D} \frac{V^2}{2} \left(1 - \frac{x}{L}\right)^2 dx$$

$\therefore$ Total energy stored is

$$W = \int_0^L \frac{\kappa R V^2}{4D} \left(1 - \frac{x}{L}\right)^2 dx$$

$$W = \frac{\kappa R V^2}{4D} \left[- \frac{L}{3} \left(1 - \frac{x}{L}\right)^3\right]_0^L = \frac{\kappa R V^2 L}{12D}$$

Thus, if C' is the distributed capacitance (in e.s.u.) to be calculated

$$W = C' \frac{V^2}{2}$$

$$\therefore C' \frac{V^2}{2} = \frac{\kappa R V^2 L}{12D}$$

or

$$C' = \frac{\kappa RL}{6\,D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

Morecroft (*loc. cit.*) gives the distributed capacitance for a solenoid of N layers as

$$C' = C_0 \times \frac{4}{3}\left(\frac{N-1}{N}\right)^2 \quad . \quad . \quad . \quad . \quad (101)$$

where C_0 is the capacitance between the outermost and innermost layers.

Breit (*Physical Review*, XVIII, p. 133 (1921)) gives the capacitance for a short single-layer solenoid in air as approximately $0{\cdot}07l$ e.s.u., where l is the length of one turn of wire on the solenoid.

Shielding and Guard Rings. In making measurements involving the use of capacitors it is often desirable—and in some cases absolutely necessary—to shield pieces of apparatus from the effect of electrostatic fields which are external to the apparatus itself. This is done by surrounding the apparatus by an earthed metal screen which may be of thin aluminium or copper sheet, or in the form of a wire mesh. Charges which may be induced in this screen pass to earth and have no effect upon the apparatus inside.

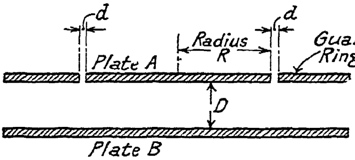


FIG. 77. GUARD RING

Guard rings are used in order to overcome the difficulty of calculating accurately the capacitance of a capacitor which has a fringing electrostatic field at its edges. The distribution of such fringing fields is somewhat uncertain and this renders exact calculations of capacitance difficult.

In calculating the capacitance of a parallel plate capacitor in a previous paragraph it was assumed that the effect of the field at the edge of the plate could be neglected. The simple formula obtained is rendered much more accurate by the use of a guard ring as shown in Fig. 77. The guard ring consists of a metal plate of the same thickness as the plate A which it surrounds, and from which it is separated by a narrow and uniform air gap. This ring is usually of the same outside dimensions as the opposing plate B of the capacitor, and is, in use, at the same potential as the plate A which it surrounds. Under these conditions the electrostatic field between the plates is perpendicular to the plates even up to the extreme edge of plate A, the fringing field being now transferred to the edges of the guard ring. The effective area of the plates to be used in the capacitance formula is now taken, of course, as the area of plate A.

A formula which corrects for the width of the air gap between plate A and the guard ring (which gap should be of zero length if no correction is to be used) has been given by Maxwell and is

$$C \doteqdot \frac{R^2}{4D} + \frac{1}{4} \cdot \frac{Rd}{D + 0{\cdot}22d}\left(1 + \frac{d}{2R}\right) \text{ e.s. units} \quad . \quad . \quad (102)$$

where the plate A (assumed circular) has a radius R cm., D being the distance between the plates in centimetres, and d being the width of the air gap, the dielectric being air.

When no guard ring is used, the edge effect can be taken into account in the calculation of capacitance by a formula due to Kirchhoff. This formula is

$$C = \frac{R^2}{4D} + \frac{R}{4\pi D}\left[D\left\{\log_e \frac{16\pi R(D+t)}{D^2} - 1\right\} + t\log_e\left(1 + \frac{D}{t}\right)\right] \quad . \quad . \quad . \quad . \quad (103)$$

where R is the radius of the circular plates of the capacitor, t being the thickness of the plates and D the distance between them, the dielectric being air.

In cylindrical capacitors the guard ring takes the form of two cylinders, of the same diameter as the cylindrical electrode to which they are adjacent, and placed one at each end of, and coaxial with, this electrode. They are connected together and are, in use, charged to the same potential as the electrode between them. Their use was described in Chapter II in connection with high voltage air capacitors.

Dielectrics. The broadest definition of a dielectric is, simply, "an insulator." More precisely, a dielectric is some medium in which a constant electrostatic field can be maintained without involving the supply of any appreciable amount of energy from outside sources. The term "dielectric" is applied when an insulating material is used to separate two neighbouring conductors such as the plates of a capacitor. As will be seen later, dielectrics increase the capacitance of a system of conductors as compared with the capacitance of the same system of conductors existing in vacuo. No dielectrics are at present known which, when placed between two conductors, decrease the capacitance between them.

Three very important quantities in connection with any dielectric are—

(*a*) Its "dielectric strength."
(*b*) Its "permittivity" or "dielectric constant."
(*c*) Its "dielectric loss angle" or power factor.

(*a*) DIELECTRIC STRENGTH. This may be defined as the ability of a dielectric to withstand breakdown when a voltage is applied to it. All insulating materials should, of course, have a very high resistivity, so that only an extremely small current flows through them when a voltage is applied. This is, however, an entirely different property from dielectric strength. If a gradually increasing voltage is applied between, say, the opposite faces of a slab of an insulating material, the material becomes electrically strained, the electrostatic field in it increasing in intensity with increasing voltage. Eventually a value of the field intensity is reached at which the material "breaks down," i.e. the material is punctured and is rendered useless for insulation purposes. This effect is observed in the case of all insulating materials, although the magnitude of the field intensity, or "potential gradient," for which it occurs differs for different materials. In liquid or gaseous dielectrics the breakdown is only temporary.

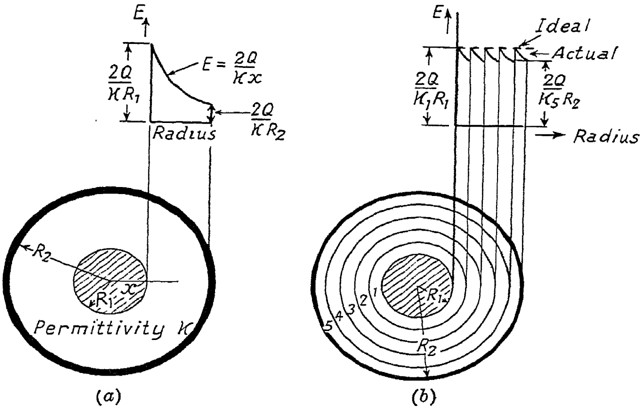
The dielectric strength is expressed in volts per millimetre or per centimetre, or in kilovolts per centimetre, etc.

The true or intrinsic dielectric strength of solid materials can be measured only if all discharges in the ambient medium are eliminated and if the heating effect of the applied field is negligible. Such intrinsic strengths are difficult to measure, but have been obtained for a few good dielectrics and lie in the region of 5×10^6 V/cm. When the dielectric strength is measured in the conventional manner between disc or sphere electrodes the breakdown is due to intense local concentration of stress at the end of ionic discharges outside the material, and values from 5 to 50 times lower than the intrinsic value are obtained. It is these lower values which are quoted in Table VII. The dielectric strength so measured is dependent on the geometry of the electrodes, upon the nature of the ambient medium (air or oil) and upon the thickness of the specimen, but no exact laws can be quoted. If the time of a test is prolonged to days or weeks in order to represent the useful life of the material, still lower values of breakdown strength are obtained which depend either on the erosion of microscopic holes through the material by ionic bombardment or on electro-chemical changes in the structure of the insulation. In low grade materials failure may be due to thermal instability, resulting from the heat liberated by dielectric losses.

When the applied voltage is alternating, the frequency of the supply affects the dielectric strength and, also, since the maximum value of the voltage is responsible for the breakdown, the wave-form of the voltage, as well as its r.m.s. value, is important. The shape of the electrodes by means of which the voltage is applied is important, since the distribution of the electrostatic field depends upon this shape, which therefore affects the dielectric strength. The true dielectric strength is the strength at breakdown when the electrostatic field is uniform.

Potential Gradient. In practice the potential gradient is an important matter. Consider the case of a single-core cable with a conducting outer sheath. From page 6 we have for the field intensity at a point, between two coaxial cylinders, and at a distance x from their common axis $E = \frac{2Q}{\kappa x}$ where Q is the charge on the inner conductor per centimetre axial length. Since the potential between two points is given by $\int E dx$, E is the potential gradient at any point. If R_1 is the radius of the core, and R_2 the internal radius of the sheath, the potential gradient at the surface of the core is $\frac{2Q}{\kappa R_1}$, and at the internal surface of the sheath $\frac{2Q}{\kappa R}$; the gradient in between these points varying as shown in Fig. 78 (*a*). Now, if the dielectric between the core and sheath consists of only one material, of permittivity κ, which is capable of withstanding, without breakdown, the maximum stress $\frac{2Q}{\kappa R_1}$ at the core surface, then the outer layers of dielectric, approaching the sheath, will not be economically used.

Graded Cables. To effect a more economical utilization of the dielectric between the core and sheath, several different dielectrics,



(*a*) (*b*)

FIGS. 78. POTENTIAL GRADIENT IN SINGLE-CORE CABLES

of permittivities κ_1, κ_2, κ_3, etc., are used, these being arranged so that their permittivities are in descending order as the radius increases. Cables insulated in this way are referred to as "graded" cables. Obviously, if the dielectric used could be varied continuously so that κ varied inversely as the radius x, an absolutely uniform potential gradient could be obtained, between core and sheath, as shown in the dotted line in Fig. 78 (*b*). Actually the potential gradient varies in the manner shown in the full-line curve.

In the previous work the potential difference between two coaxial cylinders of radii R_1 and R_2 was found to be

$$V = \frac{2Q}{\kappa} \log_e \frac{R_2}{R_1}$$

from which

$$Q = \frac{V\kappa}{2 \log_e \frac{R_2}{R_1}} \qquad . \quad . \quad . \quad . \quad . \quad (104)$$

Substituting this value for Q, we have for the potential gradient at any radius x

$$E = \frac{2}{\kappa x} \cdot \frac{V\kappa}{2 \log_e \frac{R_2}{R_1}} = \frac{V}{x \log_e \frac{R_2}{R_1}} \quad . \quad . \quad (105)$$

when only one dielectric, of permittivity κ, is used.

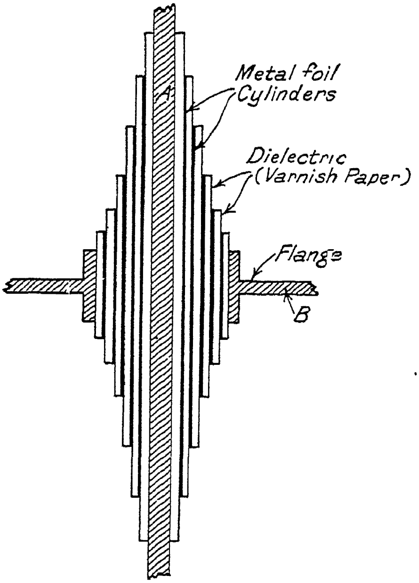
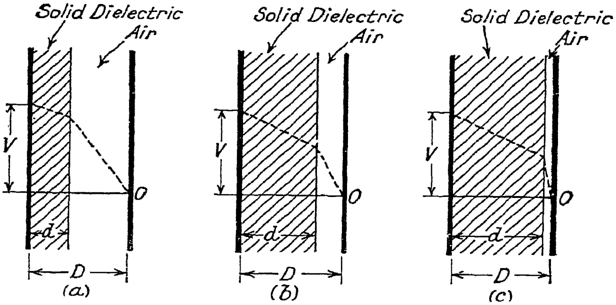


FIG. 79. CONDENSER BUSHING

Another method of obtaining a uniform potential gradient between two coaxial cylinders is by the interposition of metal intersheaths (consisting of cylindrical sheets of metal foil coaxial with the two conductors) in the dielectric, between the charged conductors. As an example of the use of such intersheaths, a "Condenser Bushing" will be considered.

Condenser Bushing. This is a type of bushing which is commonly used for the terminals of high voltage transformers and switchgear. Fig. 79 shows a conductor A which is charged to some high voltage V. This conductor is insulated from the flange B (at earth potential, say), by a condenser bushing consisting of some dielectric material with metal-foil cylindrical sheaths of different lengths and radii embedded in it, thus splitting up what is essentially a capacitor, having the high tension conductor and flange as its plates, into a number of capacitors in series. The capacitances of the capacitors formed by the metal-foil cylinders are given by the equation

l being the axial length of the capacitor and R_1 and R_2 the radii of its cylindrical plates (assumed to be of negligible thickness in the case of the metal foil). If these capacitors all have the same capacitance, since Q is the same for all (being the charge per centimetre axial length of the high tension conductor), the potential differences between their plates will be equal. They can be made to have the same capacitance by suitably choosing the axial lengths of successive sheets of foil together with the ratios of their radii $\frac{R_2}{R_1}$. If the radial spaces between successive sheets of foil are made equal and the lengths adjusted to make the capacitances equal, the potential gradient in the dielectric is uniform, but the edges of foil sheets lie on a curve, thus giving unequal surfaces of dielectric between the edges of successive sheets. This is undesirable from the point of view of flashover by "creeping" along the surface. If the differences between the lengths of successive sheets are made equal, the radial potential gradient is not uniform. A compromise between the two conditions is usually adopted.



(a) (b) (c)

FIG. 80. EFFECT OF DIELECTRIC THICKNESS UPON POTENTIAL GRADIENT IN A PLATE CAPACITOR

Effect of Varying Thicknesses of Solid Dielectric upon the Potential Gradient Between Parallel Plates. Fig. 80 shows the effect upon the potential gradient of varying the thickness of a slab of solid dielectric which is situated between the plates of a parallel plate capacitor, one plate being charged to a potential V volts and the other being at earth potential. The remaining space is air.

If σ is the surface density of charge on the plates and κ the permittivity of the solid dielectric, we have—

$$\text{Intensity in solid dielectric} = \frac{4\pi\sigma}{\kappa} = E_D$$

$$\text{Intensity in air space} = 4\pi\sigma = E_A$$

Thus $\kappa E_D = E_A$

Also, if d is the thickness of solid dielectric

$$E_D \cdot d + E_A(D - d) = V$$

Substituting for E_D we have

$$\frac{E_A}{\kappa} d + E_A(D - d) = V$$

or

$$E_A = \frac{V}{D - d\left(1 - \frac{1}{\kappa}\right)} \qquad . \quad . \quad . \quad . \quad (106)$$

Thus, increase of d increases the potential gradient in the air space, as is shown in Fig. 80. Also, if κ is much greater than 1, the potential gradient in the air space approaches the value $\frac{V}{D - d}$, which means that, in this case, the whole of the potential drop is across the air space.

The high potential gradient so produced is very likely to cause breakdown of the air in the case of a thin film of air included between a solid dielectric and a conducting plate. The air then becomes "ionized," and the insulation will ultimately fail due to damage by ionic bombardment.

The dielectric strengths of the most important insulating materials are given in Table VII under the conditions of the conventional one-minute dielectric strength test. The electrodes used in carrying out tests of dielectric strength are usually flat plates with rounded edges or smooth spheres of large diameter. In either case a fairly uniform electrostatic field is obtained.

✓(b) PERMITTIVITY. This quantity is defined as the ratio

$$\frac{\text{The capacitance of a capacitor having the material considered as its dielectric}}{\text{The capacitance of the same capacitor with air as the dielectric}} = \kappa$$

Strictly, the capacitance in the denominator should be that obtained when a vacuum exists between the plates, since the permittivity of a vacuum is unity, while that of air is about

1·0006. Most gaseous dielectrics have permittivity of the same order as that of air, while solid and liquid dielectrics have values of κ varying from about 2 upwards, as shown in Table VII.

TABLE VII

PROPERTIES OF DIELECTRICS

Dielectric	Approx. Dielectric Strength Volts/mm.	Permittivity	Power Factor (f = 50 c/s. except where noted)
Bakelite	20,000–25,000	5–6	
Bitumen (Vulcanized)	14,000	4·5	
Cotton Cloth (varnished)	3,000–4,000	4·5–5·5	0·2
Ebonite	10,000–40,000	2·8	0·01
Empire cloth	10,000–20,000	2	
Fibre	5,000	4–6	
Glass (plate)	5,000–12,000	6–7	0·006 (f = 800–1,000)
Guttapercha	10,000–20,000	3–5	
Hard rubber (loaded)	10,000–25,000	3·5–4·5	0·016
Marble	6,000	8	
Mica (Muscovite)	40,000–150,000	4·5–7	0·0003
Mycalex		6–7	0·002–0·005
Paper (dry)	4,000–10,000	1·9–2·9	0·005
Paraffin wax	8,000	2·2	0·0003 (f = 800–1,000)
Polystyrene		2·5–2·7	0·0002
Polythene		2·3	0·0001
Porcelain	9,000–20,000	5·5–6·5	0·005–0·01
Shellac	5,000–20,000	2·3–3·8	0·008
Silica (fused transparent)		3·8	0·0001–0·0003
Slate	3,000	6–7·5	
Steatite		4·1–6·5	0·002
Mineral insulating oil	25,000–30,000	2–2·5	0·0002
Water	—	40–90 (decreases with increase of temperature)	

NOTE. Owing to the different qualities of the various materials and to the variations in results according to the conditions of the test (e.g. frequency, and temperature) the above figures must be regarded as approximations only. The properties of dielectrics, including many of the recently introduced plastic materials are given in Refs. (39) to (44).

✓(*c*) DIELECTRIC LOSS AND POWER FACTOR. If a steady voltage V is applied to the plates of a perfect capacitor a "charging current" flows from the supply for a short time and gives to the capacitor a certain quantity Q of electricity, which is sufficient to produce a potential difference between the capacitor plates of V volts. When this potential difference has been attained, the current ceases to flow, the quantity of electricity Q, which has been supplied, being given by $Q = CV$ where C is the capacitance and is, of course, dependent upon the permittivity of the dielectric. In a perfect capacitor, therefore, the dielectric has only one electrical property, namely that of permittivity. It is found that with all practical dielectrics the current d not cease ve, lly over a long period of time as shown in Fig. 81. This means that dielectrics have other properties beyond that of permittivity.

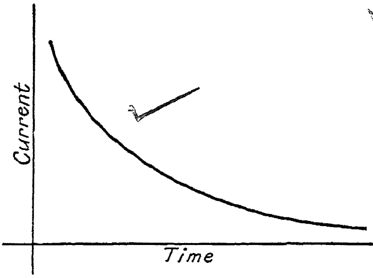


FIG. 81. CHARGING CURRENT IN AN IMPERFECT CAPACITOR

A very small "conduction" current will, of course, flow through the dielectric because of the fact that the resistance of the dielectric, though very high, is not infinite. This does not explain, however, the phenomena observed in most dielectrics, since the current is at first larger than that due to plain conduction and also it is not a constant current, but dies away gradually.

This second phenomenon is referred to as "absorption" and dielectrics in which it occurs are said to be "absorptive." All dielectrics are absorptive to some degree. If an absorptive capacitor after being charged, is discharged, the discharging connection being removed after a short time, it is found that the potential difference between the plates gradually rises again, i.e. the capacitor charges itself. This is known as the "residual" effect. Absorption is explained by assuming that there is a viscous movement of the molecules or ions of a dielectric when the plates between which it is situated are charged. In charging such a capacitor there are rapid electronic and molecular movements which correspond to the initial charging current. Thereafter there are slower molecular and ionic movements which correspond to the absorption current. Finally, there is a steady flow of ions which corresponds to the true conduction current.

The capacitance of a capacitor may thus be divided into two components, viz. the "geometric capacitance" and the "absorptive capacitance." In measuring the capacitance of a capacitor on direct current, the time of charging is thus very important. The shorter the charging time (provided this is long enough to charge the capacitor to the potential difference applied), the nearer the measured capacitance approaches the "geometric" capacitance. Fig. 82 shows the variation of the quantity of charge with time in an absorptive capacitor. The measurement of resistance of dielectrics must also be carried out, having regard to the time of application of the p.d., since the current for a given applied voltage varies with time as shown above.

Dunsheath (Ref. (10)) represents an absorptive capacitor symbolically, as in Fig. 83. The capacitor C_1 represents the geometric capacitance, the resistance R_1 represents the pure conduction effect, and C_2 and R_2 in series represents the absorption effect. In real materials the behaviour can rarely be represented by a single circuit R_2C_2. There is instead a whole spectrum of similar RC circuits in parallel, with different values of RC.

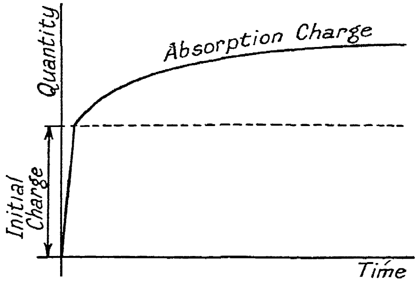
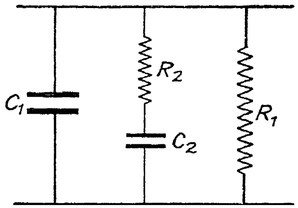
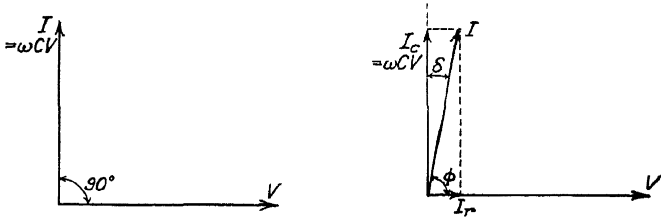


FIG. 82. ABSORPTION IN AN ABSORPTIVE CAPACITOR



(From "*High Voltage Cables*," Dunsheath.)

FIG. 83. SYMBOLIC COMBINATION TO IMITATE PRACTICAL DIELECTRIC



(a) (b)

FIGS. 84. CAPACITOR VECTOR DIAGRAMS

With alternating currents the absorption of the dielectric is intimately connected with the loss of power in the dielectric. In the case of air and most other gases, the losses are very small, and such dielectrics may be regarded as almost perfect.

If a sinusoidal voltage is applied to a perfect capacitor, the current which flows into the capacitor leads the voltage in phase by 90°, as shown in the vector diagram in Fig. 84 (*a*). If the voltage is

$$v = V_{max} \sin \omega t$$

the current in a perfect capacitor of capacitance C farads is

$$i = \omega C \,.\, V_{max} \cos \omega t$$

Its r.m.s. value is $\omega C.V$ amps. where V is the r.m.s. value of the applied voltage. Owing to the dielectric loss, the current in capacitors used in practice leads the voltage by some angle which is slightly less than 90°, as in Fig. 84 (*b*). The angle ϕ is the "phase angle" of the capacitor, the power factor being $\cos \phi$. The angle δ, which equals $90 - \phi$, is called the "loss angle." Obviously the power factor may also be expressed as $\sin \delta$.

In a perfect capacitor $\phi = 90°$, and therefore $\delta = 0$. The dielectric loss in an imperfect capacitor is given by $IV \cos \phi$ or $IV \sin \delta$ where I and V are r.m.s. values of current and voltage. Thus the loss in a perfect capacitor is

$$IV \sin \delta = 0, \text{ since } \delta = 0$$

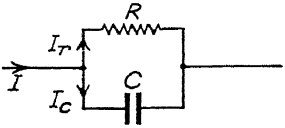


FIG. 85. SYMBOLIC REPRESENTATION OF AN IMPERFECT CAPACITOR

A capacitor having dielectric loss can be represented, at any single frequency, by a perfect capacitor in parallel with a resistance as in Fig. 85, but the value of the equivalent resistance in general varies with frequency. The current I in the capacitor can be split up into a current I_r in the resistance branch, in phase with the voltage, and a current I_c in the capacitor branch, leading the voltage by 90°. These components are shown in Fig. 84 (*b*). Then

$$I_c = \omega CV = I \cos \delta$$

where C is the effective capacitance of the capacitor,

$$\therefore C = \frac{I}{\omega V} \cos \delta$$

The dielectric loss $P = IV \sin \delta$

$$= V \sin \delta \times \frac{V\omega C}{\cos \delta}$$

$$= V^2 \omega C \tan \delta \quad \text{watts} \quad . \quad . \quad . \quad (107)$$

if C is in farads and V in volts.

The works referred to at the end of the chapter should be consulted by those who wish to carry the study of dielectric loss further. Refs. (15), (16), and (40) give the effect of frequency and of temperature upon dielectric loss. W. H. F. Griffiths* has investigated the question of losses in variable air capacitors.

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* *Experimental Wireless and The Wireless Engineer,* Vol. VIII, No. 90, March, 1931.

Measurement of Dielectric Loss and Power Factor. The two groups of methods of measuring dielectric losses which have been used are—

(*a*) Wattmeter methods,
(*b*) Bridge methods.

The cathode-ray oscillograph has also been applied to such measurements and is still used to investigate the dielectric properties of non-linear materials which would give no balance in a bridge circuit. One example of such a material is barium titanate.

(*a*) WATTMETER METHODS. These are now very seldom used and will be described here only briefly.

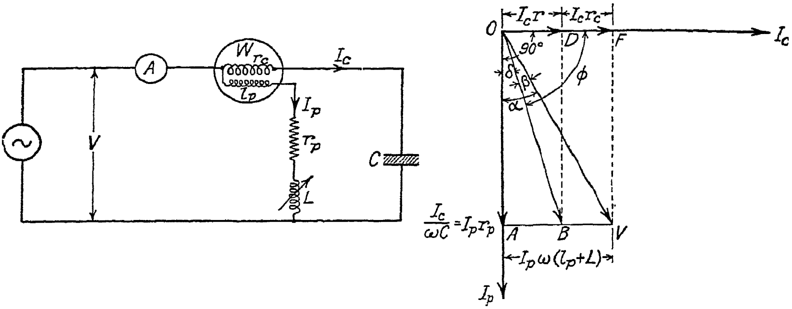


FIG. 86. WATTMETER METHOD OF MEASURING DIELECTRIC LOSS AND POWER FACTOR

Fig. 86 shows the connection diagram for a dynamometer wattmeter when used for this purpose. Owing to the very small power loss and low power factor (usually less than 0·01) the wattmeter must be very sensitive. A "null" method of use is preferable, the wattmeter reading being made zero by adjustment of the variable inductance L in the pressure coil circuit; this brings about a 90° phase difference between I_c and I_p.

Since the loss angle δ, of the capacitor C under test, is very small, as is also the angle β, we may write

$$\tan \beta = \frac{BV}{OB} = \frac{BV}{OA} \text{ approx.}$$

$$= \frac{I_c r_c}{I_c/\omega C} = \omega C r_c \text{ approx.}$$

Thus $$\beta = \tan^{-1} \omega C r_c$$

Again $$\alpha = \tan^{-1} \frac{\omega(l_p + L)}{r_p}, \text{ so that}$$

$$\phi = 90 - \alpha + \beta$$

$$= 90 - \tan^{-1} \frac{\omega(l_p + L)}{r_p} + \tan^{-1} \omega C r_c$$

The power factor of the capacitor is $\cos\phi$ and its loss angle $\delta = \alpha - \beta$.

Obviously, in addition to the value of the variable inductance L, the values of the inductance l_p and resistance r_p of the pressure coil circuit, of the resistance r_c of the current coil and C and ω must be known.

Rosa (Ref. (18)) has described several null methods of measurement of dielectric loss using wattmeters.

Electrostatic Wattmeter Method. This method has been used by many investigators. Fig. 87 (*a*) shows the connections for the method as used by Rayner (Ref. (20)). Fig. 87 (*b*) gives the equivalent diagram showing the instantaneous potentials v_1, v_2, etc., at various points; r is a non-inductive resistance.

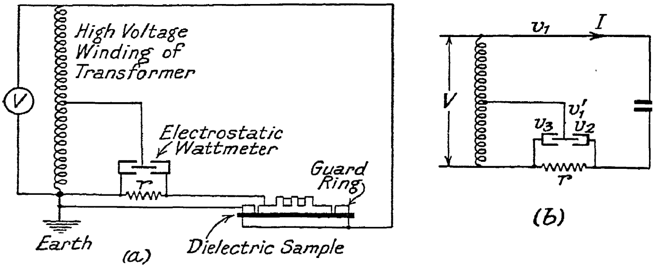


FIG. 87. DIELECTRIC LOSS MEASUREMENT BY ELECTROSTATIC WATTMETER

The moving vane of the electrostatic instrument is connected to a tapping point on the high voltage winding of a transformer from which the supply is obtained.

The sample of insulating material whose dielectric loss is to be measured, is connected as shown and is provided with a guard ring which is earthed.

From the theory of the electrostatic wattmeter given in Chap. XX, it can be shown that the mean torque of the wattmeter is proportional to

$$\frac{r}{n}(P + rI^2) - \frac{r^2I^2}{2}$$

where P = dielectric power loss

I = r.m.s. value of the current

Then, if K is the constant of the instrument and D is the deflection, we have

$$\frac{r}{n}(P + rI^2) - \frac{r^2I^2}{2} = KD$$

from which

$$P = \frac{nKD}{r} + \frac{n-2}{2}rI^2 \quad . \quad . \quad (108)$$

If the tapping point on the transformer winding is adjusted so that $n = 2$, the second term becomes zero, and we have

$$P = \frac{2KD}{r}$$

This avoids the correction for the power-loss in the resistance r.

The voltage used by Rayner in his measurements was 10,000 volts.

(b) BRIDGE METHODS. The Schering bridge method is now the most widely used of all methods of measuring dielectric loss and power factor. All bridge methods consist essentially of a Wheatstone bridge network, the battery supply being replaced by an a.c. supply at either power frequency or some higher frequency. The detector used depends upon the frequency, a vibration galvanometer being used for power frequency work and telephones for work at higher frequencies, the latter being often of the order of 800 to 1,000 cycles per second.

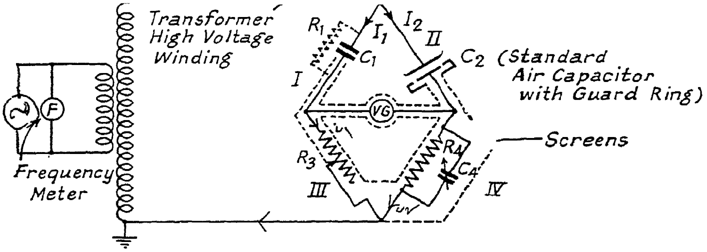


FIG. 88. CONNECTIONS OF SCHERING BRIDGE

Fig. 88 gives the connections of the Schering bridge, which can be used with high or low voltages. C_1 is the capacitor whose power factor is to be measured, R_1 being an imaginary resistance representing its dielectric loss component. C_2 is a standard air capacitor. of the type described in Chapter II. R_3 and R_4 are non-inductive resistors, the former being variable. C_4 is a variable capacitor Earthed screens are provided in order to avoid errors due to intercapacitance between the high and low voltage arms of the bridge. Instead of earthing one point on the network as shown in the figure, the earth capacitance effect on the galvanometer and leads is eliminated by means of a "Wagner earth" device (Ref. (22)), which will be described in a later chapter. V.G. is a vibration galvanometer of a special design suited to the purpose. This must have a high current sensitivity, since the impedances of arms 1 and 2 of the bridge are usually very high. For the same reason, this method of measurement involves only a small power loss. Since the impedances of branches 3 and 4 are usually small compared with those of arms 1 and 2, the galvanometer and the resistances are at a potential of only a few volts above earth even when a high voltage supply (of the order of 100 kilovolts) is used, except in the case of breakdown of one of the capacitor arms I and II.

In use, the bridge is balanced by successive variation of R_3 and C_4 until the vibration galvanometer indicates zero deflection. Then, at balance,

$$C_1 = C_2 \cdot \frac{R_4}{R_3} \cos^2 \delta = C_2 \cdot \frac{R_4}{R_3} \text{ approx.} \quad . \quad (109)$$

since δ is small, and

$$\tan \delta = R_4 \omega \cdot C_4 \quad . \quad . \quad . \quad . \quad . \quad (110)$$

where $\omega = 2\pi \times$ frequency

$\delta =$ the "loss angle" of the capacitor, sin δ giving the power factor

$C_1 =$ the effective parallel capacitance of the test capacitor

$C_2 =$ the capacitance of the standard capacitor

Theory. Consider first the impedances of the four arms of the bridge numbered I, II, III, and IV in Fig. 88.

Arm I. Consider this arm as consisting of the effective parallel capacitance of the capacitor whose power factor is to be obtained, in parallel with a resistance R_1, as shown, the latter representing its loss component.

$$\text{Total admittance of arm I} = \frac{1}{R_1} + \frac{1}{\frac{-j}{\omega C_1}}$$

$$= \frac{1}{R_1} + j\omega C_1$$

$$\therefore \text{Impedance of arm I} = \frac{1}{\frac{1}{R_1} + j\omega C} = \frac{R_1}{1 + j\omega C_1 R_1} = z_1$$

Arm II. Impedance $= \dfrac{-j}{\omega C_2} = z_2$

Arm III. Impedance $= R_3 = z_3$

Arm IV. Impedance $= \dfrac{R_4}{1 + j\omega C_4 R_4} = z_4$

Under balance conditions

$$\frac{z_1}{z_3} = \frac{z_2}{z_4}$$

i.e.

$$\frac{R_1}{R_3(1 + j\omega C_1 R_1)} = \frac{\frac{-j}{\omega C_2}}{\frac{R_4}{1 + j\omega C_4 R_4}} = \frac{-j}{\omega C_2 R_4}(1 + j\omega C_4 R_4)$$

Rationalizing, we have

$$\frac{R_1(1 - j\omega C_1 R_1)}{R_3(1 + \omega^2 C_1^2 R_1^2)} = \frac{-j}{\omega C_2 R_4}(1 + j\omega C_4 R_4)$$

Equating real terms

$$\frac{R_1}{1 + \omega^2 C_1^2 R_1^2} = \frac{C_4 R_3}{C_2}$$

Now, from Fig. 89, which shows the vector diagram for the capacitor (C_1 and R_1 in parallel) when a voltage E is applied to it,

$$\cos\delta = \frac{E\omega C_1}{E\sqrt{\dfrac{1}{R_1^2} + \omega^2 C_1^2}} = \frac{\omega C_1 R_1}{\sqrt{1 + \omega^2 C_1^2 R_1^2}}$$

$$\cos^2\delta = \frac{\omega^2 C_1^2 R_1^2}{1 + \omega^2 C_1^2 R_1^2}$$

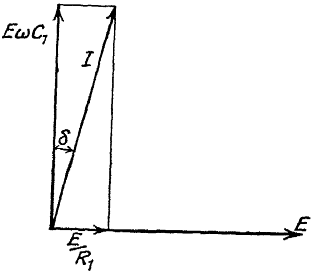


FIG. 89. VECTOR DIAGRAM FOR C_1 AND R_1 IN PARALLEL

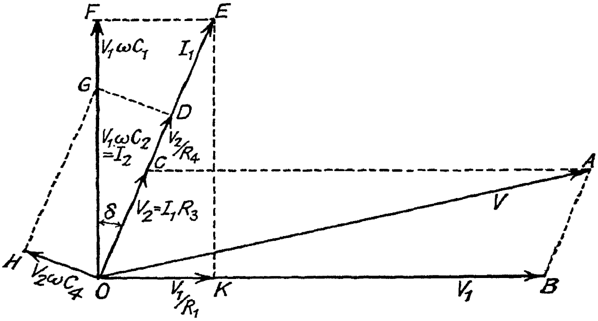


FIG. 90. VECTOR DIAGRAM FOR SCHERING BRIDGE BALANCE CONDITIONS

Substituting $\cos^2\delta$ in the equation of real terms obtained above, we have

$$\frac{\cos^2\delta}{\omega^2 C_1^2 R_1} = \frac{C_4 R_3}{C_2}$$

$$\therefore C_1 = \frac{C_2 \cos^2\delta}{\omega^2 C_1 C_4 R_1 R_3}$$

From Fig. 90, showing the complete vector diagram for the bridge network under balance conditions,

$$\tan\delta = \frac{\omega C_4}{\dfrac{1}{R_4}} = \omega C_4 R_4$$

(which is the expression previously stated), and also

$$\tan\delta = \frac{\frac{1}{R_1}}{\omega C_1} = \frac{1}{\omega C_1 R_1}$$

$$\therefore \omega C_4 R_4 = \frac{1}{\omega C_1 R_1}$$

or

$$R_4 = \frac{1}{\omega^2 C_1 C_4 R_1}$$

Substituting R_4 in the expression for C_1 gives

$$C_1 = \frac{C_2 R_4}{R_3} \cos^2\delta$$

as previously stated.

Imperfect Capacitor as a Series Circuit. An alternative method to that of representing an imperfect capacitor diagrammatically as a perfect capacitor C_1, in parallel with a resistance R_1, is to represent it as a perfect capacitor C_s in series with a resistance R_s.

The impedances in the two cases are

$$\frac{R_1}{1 + j\omega C_1 R_1} = \frac{R_1(1 - j\omega C_1 R_1)}{1 + \omega^2 C_1^2 R_1^2} \text{ in the parallel representation}$$

and

$$R_s - \frac{j}{\omega C_s} \text{ in the series representation.}$$

By equating the real and imaginary terms in the two impedances we obtain the relationships

$$R_s = \frac{R_1}{1 + \omega^2 C_1^2 R_1^2}$$

and

$$C_s = \frac{1 + \omega^2 C_1^2 R_1^2}{\omega^2 C_1 R_1^2}$$

The vector diagram of Fig. 90 needs, perhaps, some explanation. Vector *OA* represents the voltage applied to the bridge from the supply transformer. *OB* is the volt drop V_1 across arm II which, when no current flows in the vibration galvanometer branch (i.e. under balance conditions), is equal in magnitude and phase to the volt drop across arm I. Vector *OC* is the volt drop V_2 across arm III, which is equal in magnitude and phase to that across arm IV. The vector sum of *OB* and *OC* obviously gives the total bridge voltage *OA*. The current I_1 flowing in arms I and III is represented by vector *OE*, while *OG* represents the current I_2 flowing in branches II and IV. *OF* and *OK* represent the component parts of current I_1 when split up between the capacitance C_1 and resistance R_1. In the same way *OD* and *OH* represent the components of the current I_2 when similarly split up between R_4 and C_4.

The magnitudes of some of the vectors, e.g. *OC*, are exaggerated for the sake of clearness. V_2 will, in reality, be very small compared with V_1 and V.

A direct-reading Schering bridge for the measurement of permittivity and power factor of solid dielectrics at 1,600 cycles per sec.

and voltages of 100–200 is manufactured by Messrs. H. W. Sullivan, Ltd. This covers a range of capacitance up to 1,000 $\mu\mu$F.

The Cambridge Instrument Co. manufacture both low- and high-tension Schering bridges.

Muirhead and Co., Ltd. make a Schering Bridge, with a Wagner earth attachment (see p. 231) which is intended for the measurement of power factor and permittivity of insulating materials in accordance with the recommendations of *British Standard Specification No.* 234.

A portable high-voltage Schering bridge made by H. Tinsley and Co. has with it a screened, loss-free air capacitor of 100 $\mu\mu$F (within $\pm \frac{1}{2}$ per cent) and is for use at 11 kV. It may be used up to 150 kV using a compressed-air capacitor of nominal capacitance 100 $\mu\mu$F having a power factor $\not>$ 0·0001 at 50 cycles per sec. This requires an air pressure of 250 to 300 lb. per sq. in.

L. Hartshorn (Ref. (45)) adopted the Schering bridge to the measurement of very small capacitances (below 1 $\mu\mu$F) and the Hartshorn form of the bridge is the best method of measuring the permittivity and dielectric loss of sheet materials. B.S. 234 and B.S. 903 give detailed specifications for its use for this purpose.

A very full discussion of the Schering bridge, in its various forms, is given in Hague's *Alternating Current Bridge Methods.*

Dielectric Loss Measurement by Cathode-ray Oscillograph. The construction of the cathode-ray oscillograph is dealt with in Chap. XV. For the present purpose it is sufficient to know that it consists of a vacuum tube having, at one end, a filament which gives off a stream of electrons in a thin beam, or pencil, when the tube is in use. This beam passes two pairs of parallel plates, set at right angles to one another, and is deflected by potential differences applied to these pairs. A continuous path will be traced out by the beam on the fluorescent screen of the tube if the p.d.'s are alternating. This path will be a straight line if the p.d.'s are sinusoidal and are in phase but will be an ellipse if they are not in phase. The area of this ellipse is maximum—for any given maximum values of the two potential differences—when they are 90° out of phase with one another. Under these conditions, the semi-axes of the ellipse give the maximum values of the two potential differences to scale. The electron beam, having negligible inertia, can immediately take up a deflected position which is proportional, at any given time, to the deflecting force.

When used for dielectric loss measurements, a potential difference proportional to the applied voltage is applied to one pair of plates and one proportional to the integral of current through the dielectric to the other pair. This is obtained in the form of the p.d. across a relatively larger capacitor in series with the sample.

It will be shown below that the area of the ellipse traced out by the electron beam is then proportional to the power loss in the dielectric. If there is no power loss—as in the case of an air capacitor—the p.d.'s applied to the plates are in phase with one another and the path traced out is a straight line.

A record of the ellipse traced out in power loss measurements can be obtained photographically.

J. P. Minton (Ref. (23)) used a cathode-ray oscillograph for dielectric loss and power factor measurements. The full circuit arrangements are given by Hartshorn (Ref. (16)).

Fig. 91 shows a simpler arrangement than that of Minton which, nevertheless, will serve to illustrate this method of dielectric loss measurement.

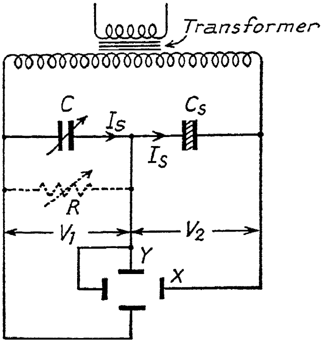


FIG. 91

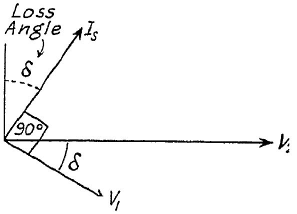


Fig. 92

OF DIELECTRIC LOSS BY C.R. OSCILLOGRAPH

C_s is the dielectric sample and C a loss-free capacitor of much greater capacitance than C_s. The resistor R, shown dotted, may be used, if desired, for compensating the loss angle of C_s. The C.R. oscillograph plates X and Y are connected as shown. (If the voltage on C_s is low, an amplifier will be needed between C and the Y plates since the method is inaccurate unless the voltage on C is much less than that on C_s.)

In the theory of the method which is given below it is assumed that the resistor R is omitted. The vector diagram may then be drawn as in Fig. 92 which shows the voltage V_2 across C_s, the current I_s through both C_s and C and the voltage V_1 across the latter.

The power loss in C_s is $V_2 I_s \sin \delta$. From the vector diagram, if $v_2 = V_{2\,max} \sin \omega t$ then $v_1 = V_{1max} \sin (\omega t - \delta)$.

The deflection produced by the Y plates is proportional to v_1 and we may write

$$y \text{ deflection} = \alpha \,.\, V_{1\,max} \sin (\omega t - \delta)$$

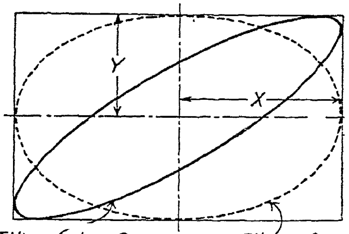
$$= \alpha \,.\, \frac{I_{s\,max}}{\omega c} \sin (\omega t - \delta)$$

Also, the x deflection $= \beta \,.\, V_{2\,max} \sin \omega t$
where α and β are proportionality constants.

Now the area of the ellipse traced out on the oscillograph screen is

$$A = \int y \,.\, dx$$

$$= \int_0^T \alpha \,.\, \frac{I_{s\,max}}{\omega C} \,.\, \sin (\omega t - \delta) \,.\, \beta \,.\, V_{2\,max} \omega \cos \omega t \,.\, dt$$


Y
X
Ellipse when Power Factor = cos θ
Ellipse for Unity Power Factor

G. 93. FORMS OF OSCILLOGRAMS OBTAINED WITH CATHODE RAY OSCILLOGRAPH

(where T is the periodic time)

$$= \frac{\alpha\beta \,.\, I_{s\,max} V_{2\,max}}{C} \int_0^T \sin (\omega t - \delta) \cos \omega t \,.\, dt$$

$$= \frac{\alpha\beta}{C} \,.\, \frac{I_{s\,max} V_{2\,max}}{2} \int_0^T [\sin (2\omega t - \delta) + \sin \delta]\, dt$$

i.e. $$A = \frac{\alpha\beta}{C} \,.\, I_s V_2 \,.\, \sin \delta \,.\, \frac{2\pi}{\omega}$$

since $$T = \frac{2\pi}{\omega}$$

Thus the area of the ellipse is proportional to $V_2 I_s \sin \delta$ which is the dielectric loss.

BIBLIOGRAPHY AND REFERENCES

(1) *Alternating Currents*, Vol. I, A. Russell.
(2) *Electrical Engineering*, T. F. Wall.
(3) *Electrical Condensers*, P. R. Coursey.
(4) *Principles of Radio Communication*, J. H. Morecroft.
(5) *The Calculation and Measurement of Inductance and Capacity*, W. H. Nottage.
(6) *Electricity and Magnetism*, Vol. I, Maxwell.

(7) *Alternating Current Bridge Methods*, B. Hague.

(8) *Principles of Electric Power Transmission*, H. Waddicor.

(9) *Electrical Insulating Materials*, A. Monkhouse.

(10) *High Voltage Cables*, P. Dunsheath.

(11) *Kapazität und Induktivität*, E. Orlich.

(12) "The Electrostatic Problem of Two Conducting Spheres," A. Russell, *Jour. I.E.E.*, Vol. LXV, p. 517.

(13) "Problems in Connection with Two Parallel Electrified Cylindrical Conductors," A. Russell, *Jour. I.E.E.*, Vol. LXIV, p. 238.

(14) "The Effect of Curved Boundaries on the Distribution of Electrical Stress Round Conductors," J. D. Cockcroft, *Jour. I.E.E.*, Vol. LXVI, p. 385.

(15) "Dielectric Problems in High Voltage Cables," P. Dunsheath, *Jour. I.E.E.*, Vol. LXIV, p. 97.

(16) "A Critical Résumé of Recent Work on Dielectrics," L. Hartshorn, *Jour. I.E.E.*, Vol. LXIV, p. 1152.

(17) "Low Power Factor Measurements at High Voltages," Rayner, Standing, Davis, and Bowdler, *Jour. I.E.E.*. Vol. LXVIII, p. 1132.

(18) Rosa, *Bulletin of the Bureau of Standards* (1905), Vol. I, p. 383.

(19) "Cables," C. J. Beaver, *Jour. I.E.E.*, Vol. LIII, p. 57.

(20) "High Voltage Tests and Energy Losses in Insulating Materials," E. H. Rayner, *Jour. I.E.E.*, Vol. XLIX, p. 3.

(21) "The Use of the Schering Bridge at 150 kilovolts," B. G. Churcher and C. Dannatt, *World Power*, Vol. V, p. 244.

(22) "The Permittivity and Power Factor of Micas," C. Dannatt and S. E. Goodall, *Jour. I.E.E.*, Vol. LXIX, p. 490.

(23) J. P. Minton, *Trans. Am. I.E.E.*, Vol. XXXIV, p. 1627.

(24) Miles Walker, *Trans. Am. I.E.E.* (1902), Vol. XIX, p. 1035.

(25) Alexanderson, *Proc. Inst. Rad. Eng.* (1914), Vol. II, p. 137.

(26) Schering, *Zeits. für Instr.* (1920), Vol. XL, p. 124, and (1924), Vol. XLIV, p. 98.

(27) W. M. Thornton, *Proc. Phys. Soc.* (1912), Vol. XXIV, p. 301.

(28) "Contact Effects between Electrodes and Dielectrics," B. G. Churcher, C. Dannatt, and J. W. Dalgleish, *Jour. I.E.E.*, Vol. LXVII, p. 271.

(29) "The Effect of Heat upon the Electric Strength of Some Commercial Insulating Materials," W. S. Flight, *Jour. I.E.E.*, Vol. LX, p. 218.

(30) "The Prevention of Ionization in Impregnated Paper Dielectrics," S. G. Brown and P. A. Sporing, *Jour. I.E.E.*, Vol. LXVII, p. 968.

(31) "Oil-filled and Condenser Bushings," W. H. Thompson, *Metropolitan-Vickers Gazette* (February and March, 1929).

(32) Kirchoff, *Berlin Akad. Monatsberichte* (1877), p. 144.

(33) Breit, *Physical Review*, XVIII, p. 133 (1921).

(34) "Ionization in Cable Dielectrics," P. Dunsheath, *Jour. I.E.E.*, Vol. LXXIII, p. 321.

(35) "The Capacitance of a Guard-Ring Sphere-Gap," G. Yoganandam, *Jour. I.E.E.*, Vol. LXXI, p. 830.

(36) "The Properties of a Dielectric containing Semi-conducting Particles of Various Shapes," R. W. Sillars, *Jour. I.E.E.*, Vol. LXXX, p. 378.

(37) "Measuring Equipment for Oil Power Factor," L. J. Berberich, *Trans. A.I.E.E.*, Vol. LV, p. 264.

(38) "A.C. Characteristics of Dielectrics," A. Baños, *Trans. A.I.E.E.*, Vol. LV, p. 1329.

(39) "The Properties of Insulating Materials used in Instruments," C. G. Garton, *Jour. I.E.E.*, Vol. XCVIII, Pt. II, p. 728.

(40) "Symposium of Papers on Insulating Materials," 16th to 18th March, 1953, *Jour. I.E.E.*, Vol. C, Pt. IIA, p. 1. (Extensive reference lists are given in these papers.)

(41) *Theory of Dielectrics*, H. Fröhlich.

(42) *Dielectric Breakdown in Solids*, S. Whitehead.

(43) "The Measurement of the Permittivity and Power Factor of Dielectrics

at Frequencies from 10^4 to 10^8 cycles per second," Hartshorn, L. and Ward, W. H., *Jour. I.E.E.*, Vol. 79, p. 597.

(44) "Plastics and Electrical Insulation," Hartshorn, L., Megson, N. J. L. and Rushton, E., *Jour. I.E.E.*, Vol. 83, p. 474.

(45) "A Method of Measuring Very Small Capacities," L. Hartshorn, *Proc. Phys. Soc.*, Vol. 36, p. 399.

(46) *Capacitors*, M. Brotherton.

(47) "The Power Factor and Capacitance of Mica Capacitors at Low Frequencies," P. R. Bray, *Jour. Sci. Insts.*, Vol. 30, No. 2, p. 49.

(48) "The Temperature Compensation of Condensers," *The Wireless Engineer*, Vol. XIX, No. 222, pp. 101–11, 2nd No. 223, pp. 148–57.

(49) "Law Linearity of Semi-circular plate Variable Air Condensers," *The Wireless Engineer*, Vol. XXII, No. 258, pp. 107–18.

(50) "The Direct Reading of the Frequency of Resonant Circuits," *The Wireless Engineer*, Vol. XX, No. 242, pp. 524–38.

(51) "Wide Range Variable Condenser for Special Laws," *The Wireless Engineer*, Vol. XI, No. 131, pp. 415–8.

(52) "Standards and Standardization," *The Wireless Engineer*, Vol. XX, No. 234, pp. 109–26.

(53) "The Losses in Variable Air Condensers," *The Wireless Engineer*, Vol. VIII, No. 90, pp. 124–6.

(54) "Notes on the Laws of Variable Air Condensers," *The Wireless Engineer*, Vol. III, No. 28, pp. 3–14.

(55) "Further Notes on the Laws of Variable Air Condensers," *The Wireless Engineer*, Vol. III, No. 39, pp. 743–55.

(56) "The Accuracy of Calibration Permanence of Variable Air Condensers for Precision Wavemeters," *The Wireless Engineer*, Vol. V, No. 52, pp. 17–24, 2nd No. 53, pp. 63–74.

(57) "Further Notes on the Calibration Permanence and Overall Accuracy of the Series-Gap Precision Variable Air Condenser," *The Wireless Engineer*, Vol. VI, No. 64, pp. 23–30 and No. 65, pp. 77–80.

(58) "Notes on the Accuracy of Variable Air Condensers for Wavemeters," *The Wireless Engineer*, Vol. IV, No. 51, pp. 754–7.

(59) "Electrical Standards for Research and Industry," a catalogue published by Messrs. H. W. Sullivan, Ltd. in 1954.

CHAPTER V

INDUCTANCE

Self-inductance. Whenever the number of lines of magnetic force linking with a circuit changes, an e.m.f. is induced in the circuit; this e.m.f. is given by

$$e = -N\frac{d\phi}{dt}$$

where e = induced e.m.f.

N = number of turns with which the flux links

$\frac{d\phi}{dt}$ = rate of change of the interlinking flux in lines per second

The negative sign indicates that the direction of the e.m.f. is such as to oppose the change in the flux.

If, now, the change in the flux is due to a change in the current flowing in the circuit itself (by which current the inter-linking magnetic flux is produced) and if, also, the reluctance of the path of the magnetic flux is constant, then

$$\phi = ki$$

where i is the current in the circuit.

Thus, $$e = -Nk\frac{di}{dt}$$

or $$e = -Nk\frac{di}{dt} \times 10^{-8} \text{ volts}$$

Since $k = \frac{\phi}{i}$, the above expression can be written

$$e = -\left(N\frac{\phi}{i} \times 10^{-8}\right)\frac{di}{dt} \text{ volts}$$

or $$e = -L\frac{di}{dt} \text{ volts}$$

where L is the "coefficient of self-induction" or, simply, the "inductance" of the circuit. Obviously L is constant for any given circuit, only if k is constant—i.e. when no magnetic material is present. If i is expressed in amperes, $\frac{di}{dt}$ is in amperes per second, and k is the flux produced by 1 amp. flowing in the circuit. Then L is in henries.

It follows from the above that the inductance of a circuit, in henries, can be expressed in words as

$$\frac{\text{Number of turns} \times \text{flux produced per ampere}}{10^8}$$

(When, as in an a.c. circuit, the flux per ampere is not constant, a better definition for the inductance is given by the induced voltage divided by the rate of change of current.)

Mutual-inductance. If two coils are close together and unit current flows in one of them, then the number of "linkages" with the other coil, of the magnetic flux due to this current, is called "the coefficient of mutual-induction," or simply the "mutual-inductance" between the coils. By "linkages" is meant the product of lines of force and the number of turns on the coil.

If the current i_1 in coil 1 varies, its rate of change being $\frac{di_1}{dt}$, then the e.m.f., e_2, induced in the second coil is given by

$$e_2 = - M\frac{di_1}{dt}$$

where M is the mutual-inductance.

If i_1 is in amperes, and M is the number of linkages with coil 2, per ampere in coil 1, divided by 10^8, then

$$e_2 = - M\frac{di_1}{dt} \text{ volts} \quad . \quad . \quad . \quad (111)$$

If the current i_2 flows in coil 2 instead of coil 1, then the e.m.f. induced in coil 1 when the rate of change of current in coil 2 is $\frac{di_2}{dt}$ is given by

$$e_1 = - M\frac{di_2}{dt} \text{ volts}$$

it being assumed that M is the same in each case.

To determine the *direction of the induced e.m.f.* in coil 1 consider the current in coil 2 to be increasing; then a self-induced e.m.f. will be produced in coil 2, the direction of which is in opposition to the direction of the current. Since the same flux which induces this self-induced e.m.f. is also inducing the e.m.f. in coil 1, this latter e.m.f. will also be in a direction opposing that of the current in coil 2. If the circuit of coil 1 is closed, a current will flow, due to the induced e.m.f. and in the same direction. This current reduces the interlinking flux and thus reduces the self-inductance of coil 2. Hence there is a mutual action between the coils.

Mutual-inductance is measured, like self-inductance, in henries.

A mutual-inductance of 1 *henry exists between two circuits when* *a rate of change of current of* 1 *amp. per second in one circuit induces an e.m.f. of* 1 *volt in the other circuit.*

Relations Between Self- and Mutual-inductance. Suppose that two coils, having respectively N_1 and N_2 turns, are so close together that the whole of the flux produced by a current in one coil links with the other. Let this flux be ϕ when the current in coil 1 is i_1. Then the self-inductance of coil 1 is $L_1 = N_1 \frac{\phi}{i_1}$ and the mutual inductance is $M = N_2 \frac{\phi}{i_1} = \frac{N_2}{N_1} L_1$.

Similarly, if i_2 flows in coil 2, its self-inductance $L_2 = \frac{N_2 \phi'}{i_2}$ and $M = N \frac{\phi'}{i_2} = \frac{N_1}{N_2} L_2$.

$$\therefore \frac{N_2}{N_1} L_1 = \frac{N_1}{N_2} L_2 = M$$

or

$$M^2 = L_1 L_2$$

$$M = \sqrt{L_1 L_2} \quad . \quad . \quad . \quad . \quad . \quad (112)$$

As stated above, this relationship is true only when the whole of the flux from one coil links with the other. In practice this condition is not fulfilled, although if the coils are very close together it is very nearly so. The ratio $\frac{M}{\sqrt{L_1 L_2}}$ is called the *coefficient of coupling*, and is of importance, especially in radio work. If this ratio is nearly unity, the circuits are said to be "close coupled," while if it is considerably less than unity they are said to be "loosely coupled."

Self-inductances in Series. If two coils, of self-inductances L_1 and L_2 henries, are connected in series and the mutual-inductance between the coils is M, then if the flux, produced by coil 2, linking with coil 1, is in the same direction, at any instant, as the self-produced flux of coil 1, then the effective self-inductance of coil 1 is $L_1 + M$. In the same way the effective self-inductance of coil 2 is $L_2 + M$, provided the self and mutual fluxes are in the same direction at any instant. Thus the total self-inductance of the circuit is

$$L = L_1 + L_2 + 2M$$

Expressing this generally, to include the case when the mutual and self-produced fluxes are in opposition at any instant, we have

$$L = L_1 + L_2 \pm 2M \quad . \quad . \quad . \quad (113)$$

Figs. 94 (*a*) and (*b*) show two coils connected in series, and with the directions of current in them such that their magnetic effects are (*a*) cumulative, (*b*) in opposition.

In the first case, $L = L_1 + L_2 + 2M$, and in the second case $L = L_1 + L_2 - 2M$.

1. Inductance of Two Long, Parallel Cylinders. Consider two long, parallel cylinders X and Y, each of radius of cross-section R cm. and carrying currents of I amp., in opposite directions as shown in Fig. 95 (*a*). Let the distance between the axes of the cylinders be D cm., and the surrounding medium be air. Suppose, also, that the material of which the cylinders are made is non-magnetic.

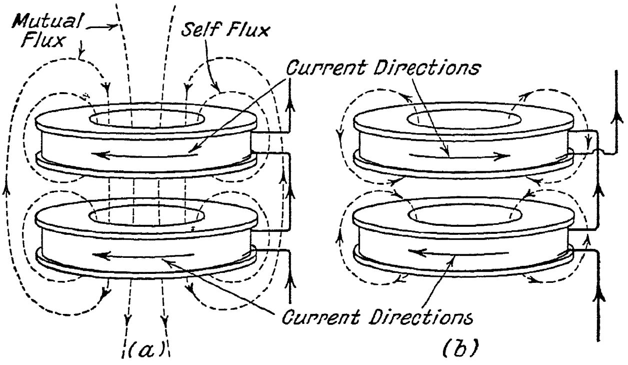


FIG. 94 SELF-INDUCTANCES IN SERIES

The flux, produced by the currents, and to which the inductance is due, is composed of two parts which must be treated separately.

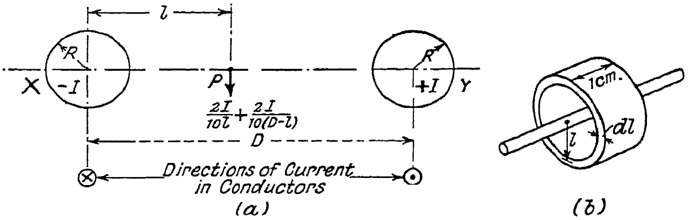

l
R
X
-I
P
$\frac{2I}{10l}+\frac{2I}{10(D-l)}$
D
R
+I
Y
Directions of Current in Conductors
(a)
1cm.
l
dl
(b)

FIG. 95. INDUCTANCE OF TWO LONG PARALLEL CYLINDERS

These are (*a*) the flux surrounding the two conductors, and (*b*) the flux which exists inside the conductors themselves. These will be considered in order.

(*a*) The magnetic intensity at a distance r cm. from a conductor carrying I amperes is $\frac{2I}{10r}$.

Thus, the total intensity of field at a point P distant l cm. from cylinder X and $(D-l)$ cm. from Y is $\frac{2I}{10l}+\frac{2I}{10(D-l)}$, the addition of the intensities due to the two cylinders separately being because the currents in them are in opposite directions. The resultant intensity is downwards, as shown in the figure. Since the medium between the cylinders is air, the flux density B at P is also equal to $\frac{2I}{10l} + \frac{2I}{10(D-l)}$.

Thus, the flux in a ring of very small radial width dl and axial length 1 cm., the ring being of mean radius l cm. and concentric with cylinder X (see Fig. 95 (*b*)) is

$$B \times dl \times 1 = \frac{2I}{10}\left[\frac{1}{l} + \frac{1}{D-l}\right] dl$$

The total flux between the cylinders per centimetre axial length is

$$\begin{aligned} \int_R^{D-R} B dl &= \frac{2I}{10} \int_R^{D-R} \left[\frac{1}{l} + \frac{1}{D-l}\right] dl \\ &= \frac{2I}{10}\Big[\log_e l - \log_e (D-l)\Big]_R^D \\ &= \frac{4I}{10} \log_e \frac{D-R}{R} \end{aligned}$$

or $\frac{4I}{10} \log_e \frac{D}{R}$ if R is small compared with D.

The flux surrounding each wire is one-half the total flux, i.e.

$$\frac{2I}{10} \log_e \frac{D}{R} \text{ lines per cm. axial length}$$

Since inductance in henries $= \frac{\text{No. of turns} \times \text{flux per amp.}}{10^8}$ the inductance of one conductor per centimetre axial length due to its external flux alone is

$$\frac{2}{10^9} \log_e \frac{D}{R} \text{ henries}$$

(*b*) In considering the flux existing inside each conductor, assume that the current is distributed uniformly over the cross-section of the conductor. This assumption is justified if the supply frequency is low. At high frequencies, the current flows almost entirely in the outside "skin" of the conductor and in this case the flux inside the conductor is negligibly small. The expression for inductance derived below, together with most of the succeeding expressions, gives therefore the "low-frequency" inductance. Slight modifications, due to the negligible internal flux, are required to convert them to expressions for "high-frequency" inductance.

It is convenient to consider the conductor as being made up of a very large number of filaments, all parallel to the axis and each carrying a small fraction of the total conductor current I (see Fig. 96).

Consider an elemental ring of radius r and radial width dr as shown.

Then the current enclosed within this ring

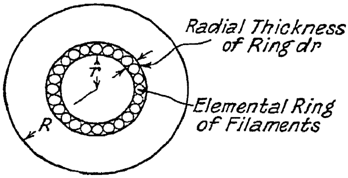


FIG. 96. CURRENT DISTRIBUTION IN A CYLINDRICAL CONDUCTOR

$$I_r = \frac{\pi r^2}{\pi R^2} \times I = \frac{r^2}{R^2} I$$

and the intensity of the magnetic field at radius r within the conductor is

$$B_r = \frac{2I_r}{10r}$$

the permeability of the conductor material being unity.

The flux in the elemental ring, of axial length 1 cm., radius r, and radial width dr, is $B_r dr \times 1$

$$= \frac{2I_r}{10r} dr \text{ lines}$$

This flux does not surround the whole conductor current, but only the current I_r. Referring it to the whole conductor, we have

$$d\phi = \frac{2I_r}{10r} dr \times \frac{r^2}{R^2}$$

by multiplying by $\frac{r^2}{R^2}$—the inverse ratio of the numbers of filaments.

Thus $$d\phi = \frac{2Ir^2}{10rR^2} \times \frac{r^2}{R^2} dr = \frac{2Ir^3}{10R^4} dr$$

The total flux of the inside of the conductor, referred to the whole conductor,

$$= \int_0^R \frac{2Ir^3}{10R^4} dr = \frac{2IR^4}{4 \times 10R^4} = \frac{I}{20} \text{ lines}$$

For *one conductor* the inductance L is $\frac{\text{flux}}{\text{amp.} \times 10^8}$

$$= \frac{\frac{2I}{10} \log_e \frac{D}{R} + \frac{I}{20}}{I \times 10^8} = \frac{\frac{2}{10} \log_e \frac{D}{R} + \frac{1}{20}}{10^8}$$

$$= \left(2 \log_e \frac{D}{R} + \frac{1}{2}\right) \times 10^{-9} \text{ henries per cm. axial length}$$

The inductance per mile of one conductor is thus

$$L = 0{\cdot}0804 + 0{\cdot}740 \log_{10} \frac{D}{R} \text{ millihenries per mile} \quad (114)$$

2. Single Straight Wire Parallel to Earth

Let the axial length of the wire = l cm.
,, ,, radius of wire = R cm.
,, ,, height of wire above earth = H cm.

Assume the wire to be of non-magnetic material and that the radius of the wire is small compared with its length. Then, using the method of images, imagine that an exactly similar conductor, running parallel to the overhead one, is embedded in the earth at a depth H cm. immediately below the latter. If the embedded conductor carries the same current as the overhead one, but in the opposite direction, then the distribution of the magnetic field will be the same as that of the single overhead conductor existing alone.

The distance between the overhead and imaginary embedded conductor is $2H$. We may, therefore, from the results of the previous paragraph, state the inductance of the overhead conductor as

$$L = 0{\cdot}0804 + 0{\cdot}740 \log_{10} \frac{2H}{R} \text{ millihenries per mile}$$

replacing D by $2H$.

The inductance of a single straight cylindrical conductor, distant from earth and other conductors, is given by

$$L = 2l\left(\log_e \frac{2l}{R} - 0{\cdot}75\right) \text{millihenries}$$

where l = length of wire in centimetres
R = radius of wire in centimetres

it being assumed that the material of the wire is non-magnetic and that the surrounding medium is also non-magnetic.

If the wire is of magnetic material the inductance is given by

$$L = 2l\left(\log_e \frac{2l}{R} - 1 + \frac{\mu}{4}\right) \text{millihenries} \quad . \quad . \quad (115)$$

where μ = permeability of the material of the wire.

3. A Single Circular Turn of Round Wire. The inductance for continuous current and low frequencies is given by Rayleigh and Niven's Formula (Ref. (1)), i.e.—

$$L = 2\pi D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9} \text{ henries} \quad (116)$$

where D = mean diameter of the turn in centimetres
d = diameter of cross-section of the wire in centimetres

It is assumed in this formula that the circle is complete, i.e. there is no gap in it, and that the wire is of non-magnetic material.

If a gap of length g cm. is left in the circle, then

$$L = \left(1 - \frac{g}{\pi D}\right) 2\pi D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9}\ \text{henries} \quad . \quad (117)$$

If, instead of one circular turn, we have a circular coil of circular cross-section and N turns, the self-inductance of the coil is

$$L = 2\pi N^2 D\left[\left(1 + \frac{d^2}{8D^2}\right)\log_e \frac{8D}{d} + \frac{d^2}{24D^2} - 1{\cdot}75\right] \times 10^{-9}\ \text{henries} \quad . \quad . \quad . \quad (118)$$

where d is now the diameter of the section of the coil. The formula for a single turn is obviously a special case of the coil when $N = 1$.

For high frequencies the formula, given by Grover (Ref (2)) for a single turn, is

$$L = 2\pi D\left[\left(1 - \frac{d^2}{4D^2}\right)\log_e \frac{8D}{d} - 2\right] \times 10^{-9}\ \text{henries} \quad . \quad . \quad (119)$$

4. **Mutual-inductance Between Two Concentric Circles.** The mutual-inductance between two concentric circles can be calculated by integration, using the equation

$$H = \frac{idl}{r^2}\sin\theta$$

Fig. 97 (a) shows two concentric circular wires of radii r_1 and r_2, the outer of which carries a current of i e.m. units. If the flux threading the inner circle, due to the current in an element dl of the outer circle, is calculated, then the total flux threading the inner circle, when i units of current flow in the outer, can be found by integrating over the whole circumference of the latter. The mutual inductance is then given by $\frac{\phi}{i}$ where ϕ is the total flux linking with the inner circle.

It can be shown by integration* that the flux linking with the inner circle when a current of i e.m. units flows in the outer is

$$4\pi r_1 i\left\{\log_e \frac{8r_1}{r_1 - r_2} - 2\right\}$$

if $r_1 - r_2$ is small and assuming the medium to have unit permeability. Thus, the mutual inductance between the circles is given by

$$M = \frac{4\pi r_1}{10^9}\left\{\log_e \frac{8r_1}{r_1 - r_2} - 2\right\}\ \text{henries}$$

the radii r_1 and r_2 being expressed in centimetres.

If, instead of being circles of one turn only, the coils had a number of turns, these turns being assumed to be coincident in space, then

---

* See Drysdale and Jolley, *Electrical Measuring Instruments*, Vol. I.

$$M = \frac{4\pi r_1}{10^9} N_1 N_2 \left\{ \log_e \frac{8r_1}{r_1 - r_2} - 2 \right\} \text{ henries} \quad . \quad (120)$$

where N_1 = No. of turns on outer coil

N_2 = " " inner coil

The length $r_1 - r_2$ may be expressed as the distance between the circumferences of the coils. Since the assumption that the turns on the coils are coincident in space is not usually justified even as an approximation, a length R called the "geometrical mean distance,"

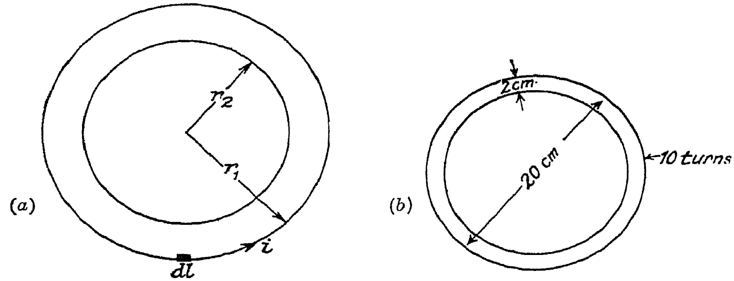


FIG. 97

first introduced by Maxwell, is used instead of $r_1 - r_2$. The mutual inductance is then given by

$$M = \frac{4\pi N_1 N_2 r_1}{10^9} \left\{ \log_e \frac{8r_1}{R} - 2 \right\} \text{ henries} \quad . \quad (121)$$

"*Geometrical Mean Distance*" may be defined as follows. Consider a point P external to a circuit. Let d_1, d_2, d_3, etc., be distances from P to various points on the circuit. Then, if an infinite number of these distances be taken, the "geometrical mean distance" R is given by

$$R = \sqrt[n]{d_1 d_2 d_3} \text{ etc., where } n_{\rightarrow \infty}$$

or

$$\log_e R = \text{limit}_{n \rightarrow \infty} \frac{1}{n} \Sigma \log_e d$$

The factor R is used in many of the formulae for the calculation of both mutual- and self-inductance. In the case of self-inductance, R is the G.M.D. of the circuit from itself, or, in the case of a multi-turn coil, of the turns from each other.

To find the self-inductance of the coil of N_1 turns, using Equation (121), with the correct value of R we have

$$L = \frac{4\pi N_1^2}{10^9} r_1 \left\{ \log_e \frac{8r_1}{0{\cdot}7788\rho} - 2 \right\} \text{ henries} \quad . \quad (122)$$

The G.M.D. of a circular area from itself is 0·7788 ρ where, in this case, ρ is the radius of the cross-section of the coil (assumed circular).

This expression, if compared with the expression given previously (Equation (118)) for the self-inductance of a similarly-shaped coil, will be found to give the same result in any particular case, provided ρ is small compared with r.

Example. Calculate the self-inductance of a coil of mean diameter 20 cm., having 10 turns, whose cross-section *is circular* and of radius 1 cm. (Fig. 97(*b*)). Then, using Equation (122), we have

(i) $$L = \frac{4\pi \times 100 \times 10}{10^9}\left[\log_e \frac{80}{0{\cdot}7788 \times 1} - 2\right]$$

$$= 0{\cdot}000033080 \text{ henries}$$

(ii) Using Equation (118),

$$L = \frac{2\pi \times 100 \times 20}{10^9}\left[\left(1 + \frac{4}{8 \times 400}\right)\log_e \frac{8 \times 20}{2} + \frac{4}{24 \times 400} - 1{\cdot}75\right] \text{ henries}$$

$$= \frac{4000\pi}{10^9}\left[\frac{801}{800} \times 4{\cdot}3828 + \frac{1}{2400} - \frac{7}{4}\right]$$

$$= 0{\cdot}000033150 \text{ henries}$$

Table V gives some of the more important geometrical mean distances.

Some exact expressions for the geometrical mean distance in several cases are given by Butterworth (*Dictionary of Applied Physics*, Vol. II, p. 391). For the calculation of geometrical mean distances, see Refs. (3), (4), and (12) at the end of the chapter.

If two circles are coaxial, but not concentric, and if the difference $r_1 - r_2$ between their radii is not small compared with their radii then the formula (Ref. (11)) is

$$M = \frac{4\pi\sqrt{r_1r_2}}{10^9}\left\{\log_e \frac{8\sqrt{r_1r_2}}{D_1}\left[1 + \frac{3}{16}\alpha - \frac{15}{1024}\alpha^2 + \frac{35}{128^2}\alpha^3 \dots\right] - \left[2 + \frac{1}{16}\alpha - \frac{31}{2048}\alpha^2 + \frac{247}{6(128)^2}\alpha^3 \dots\right]\right\} \text{ henries} \quad . \quad (123)$$

where $\alpha = D_1\sqrt{\dfrac{1}{r_1r_2}}$

This equation may be written

$$M = M_0\sqrt{r_1r_2}$$

where M_0 equals $\frac{4\pi}{10^9}$ multiplied by the expression in brackets.

Nottage (Ref. (5)) gives a table of values of M_0 for different values of $\frac{D_1}{D_2}$ where D_1 and D_2 are the least and greatest distances between the circles (see Fig. 98). M_0 varies from zero, when $\frac{D_1}{D_2} = 1$, to 50·16, when $\frac{D_1}{D_2} = 0{\cdot}01$.

If the circles have approximately equal radii, and the distance between them is small compared with their radius

$$M = \frac{4\pi r_2}{10^9}\left(\log_e \frac{8r_2}{d} - 2\right) \text{ henries} \quad . \quad . \quad . \quad . \quad . \quad (124)$$

where d is the distance (in centimetres) between the circles.

5. **Mutual Inductance Between Two Coaxial Circular Coils of Rectangular Cross-section of Winding.** From the previous paragraph an approximate formula can be derived for the mutual inductance between two coaxial circular coils of rectangular cross-section, i.e.

$$M = \frac{N_1 N_2 M_o}{10^9} \text{ henries} \quad . \quad . \quad . \quad (125)$$

where M_o is the mutual inductance between the two central turns of the two coils and can be obtained from Equation (123). N_1 and N_2 are the numbers of turns on the two coils.

TABLE VIII

GEOMETRICAL MEAN DISTANCES

Shape of Circuit	Geometrical Mean Distance (R)	Interpretation of Symbols Used
Line from itself	$R = 0{\cdot}22313l$	l = length of line
Rectangular area from itself	$R = 0{\cdot}2235\,(a + b)$ (approx. expression)	a and b = sides of rectangle
Circular area from itself	$R = 0{\cdot}7788r$	r = radius of circle
Annular ring from itself	$\log_e R = \log_e r_1 - \log_e \frac{m}{(m^2 - 1)^2} + \frac{(3 - m^2)}{4(m^2 - 1)}$	r_1 = external radius r_2 = internal radius $m = \frac{r_1}{r_2}$
Ellipse from itself	$\log_e R = \log_e \frac{a + b}{2} - 0{\cdot}25$	a and b = semi-axes of ellipse
Two parallel straight lines	$\log_e R = \frac{D^2}{l^2} \log_e D + \frac{1}{2}\left(1 - \frac{D^2}{l^2}\right) \log_e (D^2 + l^2) + 2\frac{D}{l} \tan^{-1}\frac{l}{D} - \frac{3}{2}$	l = length of lines D = distance between lines

The accuracy of this formula is of the order of 1 per cent in most practical cases.

Rayleigh's Formula. This is a more exact formula than the above, since it takes into account the dimensions of the cross-sections of the coil windings to a greater degree.

Referring to Fig. 99, let the mutual-inductance between a circle of radius r_1 with centre X, passing through point a_1, and a circle of radius r_2, centre Y, passing through o_2, be given by Mo_2a_1. There will be, in all, eight such mutual-inductances—four referred to coil 2 and four referred to coil 1. a_1, b_1, c_1, and d_1 are the mid-points of the sides of the section of coil 1 of which section o_1 is the centre point. The points a_2, b_2, c_2, d_2, and o_2 are similarly situated on the section of coil 2.

Then, by Rayleigh's formula, the mutual-inductance between the coils is given by

$$M = \frac{1}{6}\Big(Mo_1a_2 + Mo_1b_2 + Mo_1c_2 + Mo_1d_2 + Mo_2a_1 + Mo_2b_1 + Mo_2c_1 + Mo_2d_1 - 2M_o\Big) \quad . \quad . \quad . \quad (126)$$

where M_o is the mutual-inductance between the two central circles of the coils (through points o_1 and o_2). The mutual-inductances Mo_1a_2, etc., can be calculated as indicated in the previous paragraph.*

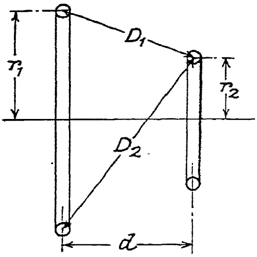


FIG. 98

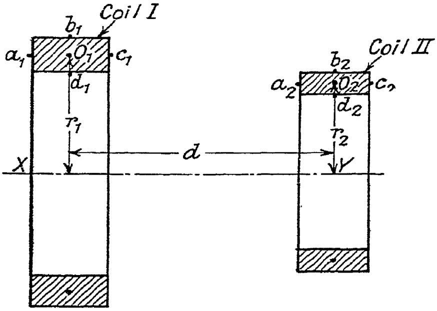


FIG. 99. MUTUAL-INDUCTANCE BETWEEN COAXIAL COILS

If instead of one of the coils being, as above, external to the other and displaced axially from it, one of the coils is inside the other at its centre, the coils being still coaxial, the mutual-inductance can be calculated as below. This case refers particularly to the mutual-inductance used in ballistic galvanometer work for calibration purposes, where a small coil is fixed inside a long circular solenoid as in Fig. 100.

Let l = length of long solenoid in centimetres
R = radius of long solenoid in centimetres
r = radius of internal short solenoid
N_1 = No. of turns on outer solenoid
N_2 = No. of turns on inner solenoid

If a current of I amp. flows in the outer solenoid, the magnetic field strength at its centre is given by

$$H = \frac{4\pi}{10}\,\frac{N_1 I}{l}\cos\theta$$

---

* Other methods of calculation of the mutual-inductance between two such coils, due to Lyle, and Nottage are given by the latter (Ref. (5)).

$$= \frac{4\pi}{10} \frac{N_1 I}{l} \frac{\frac{l}{2}}{\sqrt{R^2 + \frac{l^2}{4}}} = \frac{2\pi N_1 I}{10\sqrt{R^2 + \frac{l^2}{4}}}$$

This also gives the value of the flux density at the centre, since the core is air ($\mu = 1$). Thus the flux threading the small solenoid is

$$\frac{2\pi N_1 I}{10\sqrt{R^2 + \frac{l^2}{4}}} \times \pi r^2 = \phi$$

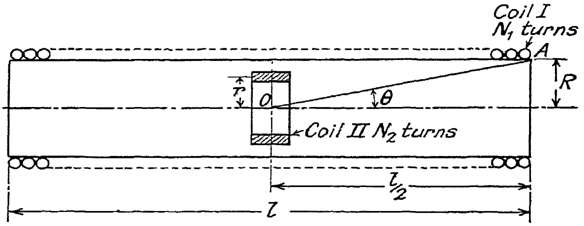


FIG. 100. MUTUAL-INDUCTANCE BETWEEN CONCENTRIC COILS

The mutual-inductance is thus $\frac{\phi N_2}{10^8 I}$ henries

or
$$M = \frac{2\pi^2 N_1 N_2 r^2}{10^9 \sqrt{R^2 + \frac{l^2}{4}}} \quad . \quad . \quad . \quad . \quad . \quad (127)$$

This is, however, only an approximate expression for the mutual inductance, since the strength of field H only refers to the centre point O of the solenoid and its intensity varies both axially and radially.

Corrections. If the internal coil had negligible axial length, the mutual inductance, corrected for radial variation of field strength, would be obtained by multiplying M (above) by the expression (Ref. (12))

$$1 + \frac{3}{8} \frac{R^2 r^2}{\left(R^2 + \frac{l^2}{4}\right)^2} + \frac{5}{64} \frac{R^4 r^2}{\left(R^2 + \frac{l^2}{4}\right)^4} \left(3 - \frac{l^2}{R^2}\right)$$

Correction for the axial length of the internal coil is obtained by subtracting a quantity, given by the following expression, from the mutual-inductance (Ref. (12))

$$-\frac{5}{9}\sqrt{\frac{3}{5}}\,\frac{S}{l} N_1 N_2 \left\{ M_{\left(\frac{l}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}\frac{S}{2}\right)} - M_{\left(\frac{l}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}\frac{S}{2}\right)} \right\}$$

where S = length of internal short coil

and $M_{\left(\frac{l}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right)}$ and $M_{\left(\frac{l}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right)}$ indicate the mutual-inductances between two circles, of radii R and r, at distances apart of

$$\left(\frac{l}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right) \text{ and } \left(\frac{l}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\cdot\frac{S}{2}\right) \text{ respectively.}$$

As the above two corrections (for axial and radial variation of field) tend to neutralize one another, the difference between the final value of the mutual-inductance and the approximate value originally obtained is usually quite small—of the order of 1 in 1,000 for usual dimensions of a mutual-inductance for ballistic galvanometer calibration.

If both $\frac{l}{R}$ and $\frac{S}{r}$ are equal to $\sqrt{3}$, and if the ratio is small, the corrections become unnecessary (Ref. (17)).

The mutual-inductance when the short coil, instead of being situated *within* the long coil at its centre, is situated at the centre but *outside*, is obtained by the same method as above, but R and r are interchanged.

Example. Calculate the mutual-inductance between two coaxial circular coils, given that—

Length of long coil = 80 cm.
Radius of long coil = 4 cm.
No. of turns of long coil = 500
Length of short coil = 6 cm.
Radius of short coil = 3 cm.
No. of turns of short coil = 150

Small coil placed inside, and at the centre of, the larger coil.

Then
$$M = \frac{2\pi^2 \times 500 \times 150 \times 3^2}{10^9\sqrt{4^2+\frac{80^2}{4}}}$$

$$= \frac{332}{10^6} \text{ henries, or 332 microhenries (very nearly)}$$

6. **Self-inductance of Circular Coils of Rectangular Cross-section of Winding.** Consider a single-layer coil of axial length l cm. and radius of cross-section r cm., having N turns, with a current of I amp. flowing in it. If $\frac{l}{r}$ is great, the magnetic intensity within the coil is $\frac{4\pi}{10}\cdot\frac{NI}{l}$. If no magnetic material is present, this is also the flux density within the coil. Thus the flux inside the solenoid is $\frac{4\pi}{10}\cdot\frac{NI}{l}\times\pi r^2$, and the inductance is

$$L = \frac{4\pi^2N^2r^2}{10^9 l} \text{ henries (approx.)} \quad . \quad . \quad . \quad (128)$$

This may also be written

$$L = \frac{S^2}{10^9 l} \text{ (approx.)}$$

where S = the total length of wire on the coil = $2\pi Nr$.

These formulae must be regarded as approximate only, since the expression of the magnetic intensity is only true for an infinitely long solenoid. In practice, the *whole* of the flux produced does not link with *all* the turns, and this reduces the inductance of the solenoid. Nagaoka (Ref. (7)) has given the values of a factor by which the above expression may be multiplied in order to take into account the dispersion of the lines of force. This factor varies according to the ratio of length to diameter of the coil.

Equation (128) may be written

$$L = \frac{\pi^2 N^2 d^2}{10^9 l}$$

d being the diameter of the coil. Introducing Nagaoka's factor K, we have

$$L = \frac{\pi^2 N^2 d^2}{10^9 l} \cdot K \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

which is considerably more exact than the previous equation (128).

If the wire on the solenoid is closely wound, so that adjacent turns are touching, this expression gives results which are sufficiently accurate for most purposes. A correction is necessary if the turns are widely spaced. Fig. 101 gives the values of the factor K for different ratios of length to diameter of coil. The curve refers to a single layer coil or to a coil whose depth of winding is small compared with its diameter.

Coursey (Ref. (8)) has given values of a second factor K_1, for use when the depth of winding on a coil is appreciable. This factor varies with the ratio $\frac{\text{depth of winding}}{\text{mean diameter of coil}}$ and also with the ratio $\frac{\text{length of coil}}{\text{depth of winding}}$. The inductance formula, when K_1 is used, becomes

$$L = (K - K_1) \frac{\pi^2 . N^2 d^2}{10^9 . l} \text{ henries} \quad . \quad . \quad . \quad (130)$$

A full table of Nagaoka's factors is given by Nottage (Ref. 5), where other tables for the calculation of the inductance of special coils are also given. For Coursey's curves and further tables for such calculations, see Refs. (5), (6), (8), (9), (10).

Fig. 102 gives values of Coursey's factor $(K - K_1)$ for various ratios of depth of winding to mean diameter of coil, and of length of coil to depth of winding.

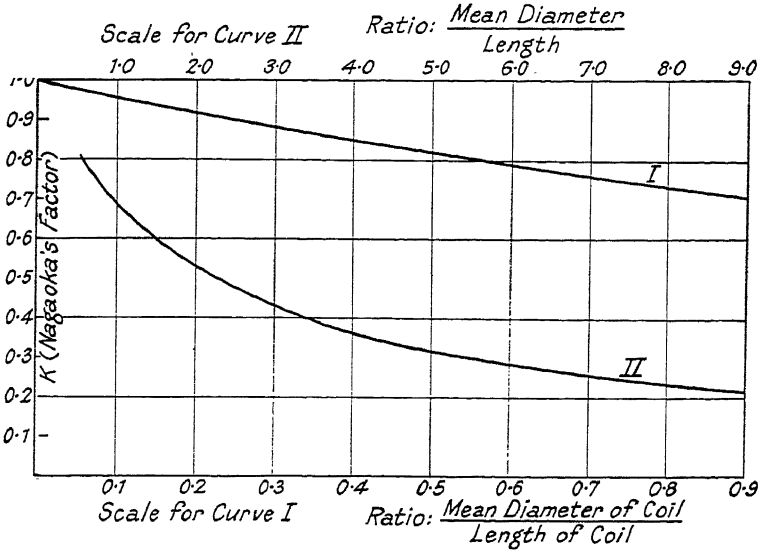


FIG. 101. CURVES OF NAGAOKA FACTORS

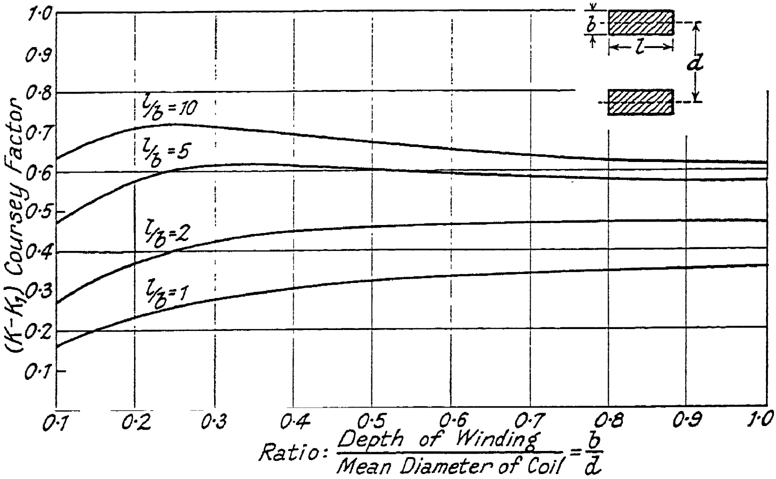


FIG. 102. CURVES OF COURSEY FACTORS

Equations (129) and (130) are especially suited to long, circular coils whose depth of winding is small compared with their mean diameter. The assumption is made, in these formulae, that the distribution of the current over the cross-section is uniform.

Based on formulae derived by Rayleigh and Niven, Lyle, and Spielrein, Grover (Ref. 9) gives the formula

$$L = \frac{N^2 dP}{2 \times 10^9}$$

d being the mean diameter of the coil. P is a factor depending upon the ratios of the various dimensions of the coil and the values of P for different coil dimension ratios are given by Grover (*loc. cit.*).

This formula is more suited to the calculation of the inductances of short circular coils of rectangular cross-section whose depth of winding is comparatively large compared with their mean diameter, although it can be used also for the calculation of inductance in the same cases as Equation (130) with very little error.

Example. Calculate the inductance of a circular coil, of 500 turns, having a rectangular cross-section of winding. Given—

Axial length of coil = 10 cm.
Mean diameter of coil = 5 cm.
Depth of winding of coil = 1 cm.

(i) Using the Coursey curve (Fig. 102) in conjunction with equation (130),

Ratio $\frac{l}{b} = 10$. Ratio $\frac{b}{d} = \frac{1}{5} = 0{\cdot}2$.

From the curve $(K - K_1) = 0{\cdot}701$

$$L = \frac{0{\cdot}701\pi^2 \times 500^2 \times 5^2}{10^9 \times 10}$$

$$= \frac{4{,}325}{10^6} \text{ henries or } 4{,}325 \text{ microhenries.}$$

(ii) Using Equation (131),

From Grover's table, the value of P corresponding to these ratios is 6·92. Hence

$$L = \frac{500^2 \times 5 \times 6{\cdot}92}{2 \times 10^9}$$

$$= \frac{4{,}325}{10^6} \text{ henries, or } 4{,}325 \text{ microhenries, as before.}$$

Correction for Thickness of Insulation. As mentioned above, the formulae given take no account of the insulation between turns on the coil. For accurate calculations a correction for this insulation must be applied, although it is usually quite small.

This correction is made by subtracting the quantity $\frac{6{\cdot}283}{10^9} dN(A+B)$ henries

(Ref. (5)) from the calculated inductance, where d and N are as above, and A and B are constants depending upon the relative thickness of insulation and number of turns on the coil respectively. Values of these constants are given by Nottage (*lot. cit.*).

(7) **Self-inductance of Flat Coils.** By "flat" coils are meant those whose axial length is small compared with their mean diameter and depth of winding.

Spielrein gave the formula for such flat or "disc" coils of circular form as

$$L = \frac{N^2 dQ}{2 \times 10^9} \text{ henries} \quad . \quad . \quad . \quad . \quad . \quad (132)$$

where N = No. of turns on the coil
d = mean diameter of the coil

and Q is a factor which can be calculated from the expression

$$Q = \frac{\left(1 + \frac{b}{d}\right)^3}{4\left(\frac{b}{d}\right)^2} \left[\begin{matrix} 6{\cdot}96957 - \beta^3 \; 30{\cdot}3008 \log_{10} \frac{1}{\beta} + 9{\cdot}08008 \\ + 1{\cdot}48044\beta^5 + 0{\cdot}33045\beta^7 + 0{\cdot}12494\beta^9 + \ldots \end{matrix} \right]$$

where b = depth of winding

$$\beta = \text{the ratio } \frac{\text{inner radius of coil}}{\text{outer radius of coil}}$$

A table of values of the factor Q are given by Grover (Ref. (9)) for different values of $\frac{b}{d}$. If the axial length of the coil is appreciable Equation (131) (previous paragraph) applies.

To Correct for Insulation Thickness. In the case of a flat spiral wound with metal strip or ribbon of rectangular cross-section, the quantity

$$\frac{12{\cdot}57}{10^9} Nr(A_1 + B_1) \text{ henries (Ref. Grover, } \textit{loc. cit.})$$

is added to the calculated inductance.

N = No. of turns on coil
r = mean radius of coil in centimetres

$$A_1 = \log_e \frac{\nu + 1}{\nu + \tau}$$

$$B_1 = -2\left[\frac{N-1}{N}\delta_{12} + \frac{N-2}{N}\delta_{13} + \frac{N-3}{N}\delta_{14} + \ldots \frac{1}{N}\delta_{1n}\right]$$

$\nu = \frac{w}{D}$ where w = axial length of strip
D = distance between adjacent turns

$\tau = \frac{t}{D}$ where t = thickness of strip

The factors δ_{12}, δ_{13}, etc., are given in tabular form by Grover for different values of τ and ν.

8. **Self-inductance in Other Cases.** (*a*) *Coils Wound on Polygonal Formers.* Grover (Ref. (10)), in a Bureau of Standards paper on this subject, gives a method of calculating the inductance of coils of this general form by obtaining, in each case, the "equivalent radius" of the coil, and then treating it as a circular coil having this radius. The formulae for calculation of the equivalent radii are somewhat complex, and reference should be made to the original paper for information on the subject.

(*b*) *Toroidal Coils.* These are coils whose axis and cross-section are either both circular or the former circular and the latter rectangular.

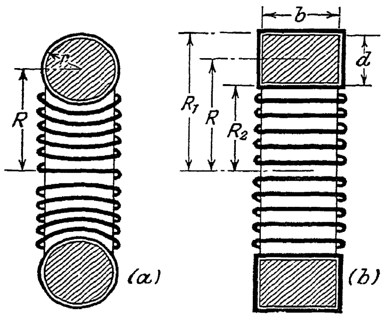


FIG. 103. SELF-INDUCTANCE OF TOROIDAL COILS

(i) Axis circular, cross-section circular (torus) (Fig. 103 (*a*)).

Russell (*Alternating Currents*, Vol. I, p. 50) shows that the flux inside such a coil, of N turns, when a current of I amp. flows in it, is

$$\phi = \frac{4\pi}{10} NI(R - \sqrt{R^2 - r^2})$$

where R = mean radius of axis of coil in centimetres
r = radius of the cross-section of the coil in centimetres.

Thus, the inductance is given by

$$L = \frac{4\pi}{10^9} N^2(R - \sqrt{R^2 - r^2}) \text{ henries} \quad . \quad . \quad (133)$$

(ii) Axis circular, cross-section rectangular (Fig. 103 (*b*)).

Again, from Russell's expression for the flux within such a coil, we have

$$\phi = \frac{2NbI}{10} \log_e \frac{R + \frac{d}{2}}{R - \frac{d}{2}}$$

where b = the breadth of the coil
d = radial depth of the coil, both in centimetres
R = mean radius of axis of coil

Thus,
$$L = \frac{2bN^2}{10^9} \log_e \frac{R + \frac{d}{2}}{R - \frac{d}{2}}$$

or
$$L = \frac{2bN^2}{10^9} \log_e \frac{R_1}{R_2} \text{ henries} \quad . \quad . \quad . \quad (134)$$

where R_1 and R_2 are the outer and inner radii of the ring respectively.

Design of Inductance Coils for Maximum Time-constant. The ratio $\frac{\text{inductance}}{\text{resistance}}$ for any coil is spoken of as its "Time-constant." It is usually desirable, in designing inductance coils, to make this ratio as great as possible. This means that the dimensions must be such that the greatest possible inductance is obtained with a given length of wire. Since the resistance of coils increases considerably at high frequencies, compared with the continuous current, or low-frequency, resistance, it is difficult to give rules for the most economical design of coil to suit different frequencies when these are high.

Referring to low-frequency conditions, the maximum inductance for a given length of wire, using a coil of rectangular cross-section, is obtained when the cross-section is square, i.e. when the axial length of the coil is equal to the depth of winding on the coil.

Maxwell showed, also, that with a square section coil the inductance is maximum when the mean diameter of the coil is made 3·7 times the axial length of the coil, but, as already pointed out on page 92, later work by Shawcross and Wells (Ref. (26)) has shown that 2·95 (or more conveniently 3) is a better value than 3·7 for the ratio of mean diameter to axial length.

The maximum inductance for a given length of wire, if the coil is not limited as to shape, is obtained by making the coil of circular cross-section, with a ratio of $\frac{\text{radius of circular axis of coil}}{\text{radius of cross-section of the coil}}$ = 2·575. (Ref. (25)).

Experimental work on the most efficient shape of coil has been carried out also by Brooks and Turner (Ref. (11)) and a valuable paper by H. B. Brooks on the design of inductance coils is mentioned in Ref. (27)).

Iron-cored Inductances. The formulae for inductance so far considered have all been for coils with air, or non-magnetic, cores. The inductance of iron-cored coils cannot be calculated easily with great accuracy owing to the fact that the permeability of the iron core is not constant, but varies with the magnetizing force producing the flux. If an expression is to be given for the permeability under these conditions it must be some mean value of the different permeabilities occurring at different times throughout the current cycle. The question is further complicated by the fact that the value of the magnetizing force is not the same for all parts of the iron core, even for a given value of current in the coil. Thus the effective value to be assigned to the permeability of the core of such a coil is largely a matter for experimental determination under a given set of conditions.

The foregoing remarks apply especially to coils with open iron cores—say in the form of a straight bar. If the core is nearly closed, having a comparatively narrow air gap, the calculation of inductance can be carried out approximately as follows—

Let S_i = reluctance of iron path
S_a = reluctance of air gap
N = No. of turns on the coil
I = r.m.s. value of the current in the coil
ϕ = r.m.s. value of the flux produced

Then,

$$\phi = \frac{\frac{4\pi}{10} . NI}{S_i + S_a}$$

Now, obviously the value of the flux per ampere, i.e. $\frac{\phi}{I}$, would be constant if $S_i + S_a$ were constant. But, although the reluctance of the air gap S_a is constant whatever the value of the magnetizing force, the iron path reluctance S_i varies with varying current as pointed out above. If, however, the reluctance S_a is made large compared with S_i, the variation in the latter is negligible, since S_i may then be entirely neglected with very little error.

$$L = \frac{\phi N}{I \times 10^8}$$

$$= \frac{4\pi N^2}{10^9 S_a} = \frac{4\pi N^2 A}{10^9 l} \text{ henries} \qquad . \quad . \quad . \quad (135)$$

$$\text{since } S_a = \frac{l}{A} \times \frac{1}{\mu} \text{ and } \mu = 1 \text{ for air}$$

Under these circumstances, the iron core provides a low reluctance path for the flux, thus increasing the latter for a given magnetizing force, and hence increasing the inductance of the coil.

Example. A coil of 500 turns is wound on a cylindrical former 10 cm. long and 1·5 cm. radius. This former is placed on a rectangular iron core of effective cross-section 2 sq. cm., and whose length of magnetic path is 30 cm. The core contains an air-gap 0·5 cm. long. A current of 0·1 amp. r.m.s. flows through the coil. Given that the mean permeability of the iron of the core under these conditions = 1,000, calculate the inductance of the coil.

Reluctance of air gap

$$S_a = \frac{0{\cdot}5}{2} = 0{\cdot}25$$

Reluctance of iron path

$$S_i = \frac{30}{2} \times \frac{1}{1{,}000} = 0{\cdot}015$$

$$\text{Flux} = \phi = \frac{\frac{4\pi}{10} \times 500 \times 0{\cdot}1}{0{\cdot}25 + 0{\cdot}015} = 226 \text{ lines}$$

$$\text{Inductance} = \frac{\phi \times N}{I \times 10^8} = \frac{226 \times 500}{0{\cdot}1 \times 10^8} = 0{\cdot}0113 \text{ henry}$$

Obviously, if the reluctance of the iron path had been entirely neglected, the calculated inductance would have been some 6 per cent larger than the above value. Thus, uncertainty as to the correct value of the permeability of the iron under working conditions causes a negligible error if the air gap is made comparatively large.

To illustrate the effect of the iron core in increasing the inductance, we will calculate the inductance of the same coil with an air core.

From the approximate equation (128) this inductance is

$$\frac{4\pi^2 \times 500^2 \times 1{\cdot}5^2}{10^9 \times 10} = 0{\cdot}0022 \text{ henry (approx.)}$$

Skin Effect. It was pointed out earlier in the chapter that there is internal flux inside a straight cylindrical conductor which is carrying current. Considering the conductor to be made up of an infinite number of small filaments, parallel to its axis, each carrying a small fraction of the total current, I amp., of the conductor, and assuming the current density to be uniform over the conductor cross-section (an assumption which is really justified only with unidirectional or low-frequency current), we have for the flux density at a radius r within the conductor

$$B_r = \frac{2I_r}{10r} \text{ where } I_r = \frac{r^2}{R^2} \cdot I$$

R being the radius of the conductor itself.

$$\therefore B_r = \frac{2r^2 I}{10rR^2} = \frac{2rI}{10R^2}$$

Thus, $B_r \propto r$. In Fig. 104 B_r is shown plotted against radius r. The total flux surrounding the filaments of the conductor (including the flux external to the entire conductor), when plotted against radius r, gives the dotted curve of Fig. 104. From this curve it can be seen that the flux surrounding the filaments near the centre of the conductor is greater than that surrounding the filaments near its surface. Thus the centre filaments have greater inductance than the surface filaments.

If P is the resistance of one filament and L its inductance, then its impedance is $\sqrt{P^2 + \omega^2 L^2}$ where $\omega = 2\pi \times$ frequency. At low frequencies the term $\omega^2 L^2$ is small compared with P, so that if a voltage V is applied to the two ends of the conductor the current carried by any one filament is $\dfrac{V}{\sqrt{P^2 + \omega^2 L^2}} = \dfrac{V}{P}$ (very nearly), and the current density within the conductor is uniform over its cross-section. At high frequencies P^2 is small compared with $\omega^2 L^2$ and the current carried by a filament $= \frac{V}{\omega L}$ (very nearly). Under these conditions, then, the difference in inductance between central and surface filaments becomes very important. The central filaments carry only a very small current, due to their greater inductance, and the current in the conductor is almost entirely carried by the surface filaments, i.e. by the outer "skin" of the conductor. Hence the name "*skin effect*" given to this phenomenon.

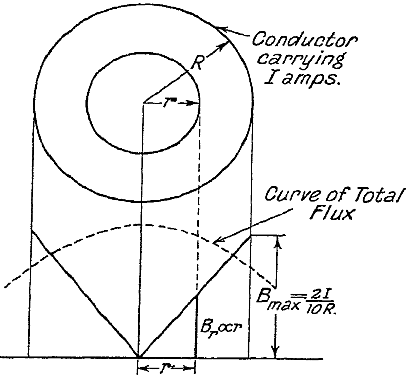


FIG. 104. DISTRIBUTION OF INTERNAL FLUX IN A CYLINDRICAL CONDUCTOR

The effective cross-section of the conductor at high frequencies is therefore only the area of an outer skin, and the resistance of the conductor is increased accordingly. Thus the "*high-frequency-resistance*" of a conductor is higher than its d.c. or low-frequency resistance, the difference depending upon the cross-section of the conductor, the frequency, and upon the permeability and resistivity of the material of the conductor. Since the material used for such conductors is usually non-magnetic, the permeability is almost always unity.

The high-frequency resistance of a conductor is given by

$$R_f = R\left[1 + \frac{1}{12}A^2 - \frac{1}{180}A^4 + \ldots\right] \qquad . \quad . \quad . \quad (136)$$

where R is the steady-current resistance and

$$A = \frac{2\pi f l \mu}{10^9 R} = \frac{2\pi f l \mu}{10^9 \times \frac{l \times \rho}{\pi r^2}} = \frac{2\pi^2 r^2 f \mu}{10^9 . \rho}$$

where f = frequency

l = length of conductor in centimetres

r = radius of conductor in centimetres

ρ = resistivity of conductor material in ohms per cm. cube

μ = permeability of the material of the conductor.

The inductance of the entire conductor is slightly reduced by the skin effect, since there is less internal flux.

The high frequency inductance is given by

$$L = \frac{l}{10^9}\left[M + \frac{1}{2} - \frac{1}{48}A^2 + \frac{13}{8640}A^4 - \ldots\right] \text{ henries} \quad . \quad (137)$$

M is a constant which depends upon the position of the return conductor of the circuit. The above two equations are due to Maxwell.

Reduction of Skin Effect. From Equation (136) it is obvious that the smaller the term A is made, the less the increase of resistance of the conductor with increasing frequency.

A can be kept small by making the radius r of the conductor as small as is consistent with current-carrying requirements and by using non-magnetic material (so that $\mu = 1$). If high-resistance material can be used, so that the resistivity ρ is large, this again will reduce A.

Other means which are adopted to reduce the effect are the employment of tubular conductors, or conductors consisting of two parallel discs with a number of parallel high-resistance rods, set at equal distances apart round their circumferences, joining them together, the whole forming a cage or barrel-shaped arrangement. In these cases the internal flux of the conductor is small and the conductors may be thought of as consisting merely of "skins" with hollow interiors.

Stranded conductors are used; these consist of a large number of fine strands, insulated from one another, and woven so that each strand lies as much at the centre of the conductor, and as much at the surface, as every other strand. In such conductors all the strands have the same surrounding flux and therefore have equal inductances.*

Skin Effect in Coils. Consider a cylindrical coil. The flux within such a coil is, of course, axial, and is distributed over the cross-section right up to the outer surface of the winding. Thus, in addition to the internal flux distribution previously considered, as affecting the inductance of the imaginary component filaments of the conductors, we have now a greater inductance of the radially outermost filaments of the coil, as compared with the filaments on the inner surface of the winding. This is due to the fact that they enclose the whole of the coil-flux, while the latter only enclose the flux within the winding (i.e. the flux in the core of the coil). A variation of the inductance between the various filaments is also caused by the proximity of other conductors.

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* For a number of curves relating to the high frequency resistance of straight conductors, the reader should consult Morecroft's *Principles of Radio Communication*, Chap. II.

Morecroft (Ref. (11)) has carried out a full investigation of the effect in various types of coils, and this work should be consulted for further information on the subject. The effect is usually negligibly small in coils when used at low frequencies for alternating-current measurement purposes.

In addition to the "skin effect" upon the actual self inductance of a coil, there is the effect of this current re-distribution upon the temperature coefficient of inductance. The extent of the re-distribution within a conductor (assuming unity permeability) depends upon its resistivity as well as upon the frequency. Therefore, as the resistivity changes with temperature, the inductance of a coil wound with that conductor varies with temperature to an extent which depends upon frequency. W. H. F. Griffiths* shows how to determine the more or less narrow band of frequencies in which this augmentation of temperature coefficient becomes appreciable.

Skin Effect in Iron Plates. In iron plates which are carrying alternating magnetic flux, the skin effect is of a different nature from that considered above. It is, in this case, the flux which is forced outwards so as to be carried almost entirely by the outer "skin" of the plate instead of being distributed uniformly over the cross-section.

The effect is due to the demagnetizing effect of "eddy currents" induced in the iron plates by the alternating flux. The plate itself acts as the short-circuited secondary winding of a transformer, and "eddy currents" flow in paths lying in a plane perpendicular to the axis of the plate, as shown in Fig. 105 (*a*), the currents being induced by the alternating flux.

The effect will be referred to later in Chapter XIV on "Eddy Currents."

Obviously the magnitude of the effect depends upon the thickness of the plate and upon the frequency, and it is for this reason that, in order to obtain uniform flux distribution—and hence economical utilization—of the iron cores of alternating current apparatus, it is necessary to limit the thickness of the laminations to be used, according to the supply frequency.

Fig. 105 (*b*) illustrates the diminution of flux density at the centre of an iron plate due to this demagnetizing effect. The figure refers to a fairly thick plate when used with a comparatively high frequency. In practice, with plates of the normal thickness (about

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* *The Wireless Engineer*, Vol. XIX, No. 221, pp. 56–63.

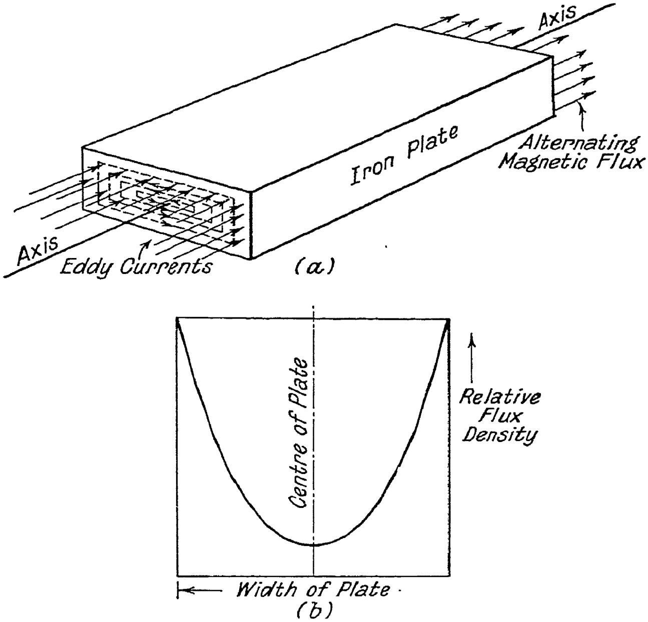


FIG. 105. FLUX DISTRIBUTION IN IRON PLATES

0·014 in.) and commercial frequencies, the variation in flux density over the cross-section is very much less than that shown.

BIBLIOGRAPHY AND REFERENCES

(1) Rayleigh, *Collected Papers*, II, 15.
(2) F. W. Grover, *Phys. Rev.*, XXX, 787 (1901).
(3) *Electricity and Magnetism*, J. C. Maxwell, Vol. II.
(4) E. B. Rosa, *Bull. Bur. Stands.* IV, 325 (1907).
(5) *The Calculation and Measurement of Inductance and Capacity*, W. H. Nottage.
(6) *Electrical Measuring Instruments*, Vol. I, Drysdale and Jolley.
(7) Nagaoka, *Journal College of Science, Tokyo*, Art. 6, p. 18 (1909).
(8) P. R. Coursey, *Proc. Phys. Soc., London*, XXXI, p. 159 (1919).
(9) F. W. Grover, *Bull. Bur. Stands.*, Sc. Paper No. 455, 1922.
(10) F. W. Grover, *Bull. Bur. Stands* Sc. Paper No. 468 (1923).
(11) *Principles of Radio Communication*, J. H. Morecroft.
(12) *Dict. App. Physics*, Vol. II, p. 388.
(13) *Electrical Power, Transmission, and Intercommunication*, C. Dannatt and J. W. Dalgleish.

(14) *Alternating Currents*, Vol. I, A. Russell.

(15) *Principles of Electric Power Transmission*, H. Waddicor.

(16) "The Design of Inductances for High-frequency Circuits," C. L. Fortescue, *Jour. I.E.E.*, Vol. LXI, p. 933.

(17) *Absolute Measurements in Electricity and Magnetism*, A. Gray, Vol. II, Pt. 1, p. 275.

(18) Lyle, *Bur. Stands.*, Sc. Paper No. 320.

(19) Weinstein, *Ann. der Phys.*, 257, p. 329 (1884).

(20) Stefan, *Ann. der. Phys.*, 258, p. 113 (1884).

(21) Butterworth, *Proc. Phys. Soc., London*, 27, p. 371 (1915).

(22) Dwight, *Trans. Am. Inst. Elec. Eng.*, 38, Pt. 2, p. 1675 (1919).

(23) Dwight, *Electrical World*, 71, p. 300 (1918).

(24) Butterworth, *Phil. Mag.*, 29, p. 578 (1915).

(25) A. Russell, *Electrician* (23rd April, 1915).

(26) "On the Form of Coils to Give Maximum Self Inductance for a Given Length and Thickness of Wire," Shawcross and Wells, *Electrician*, Vol. LXXV p. 64 (1915).

(27) "Design of Standards of Inductance and the Proposed Use of Models in the Design of Air-core and Iron-core Reactors," H. B. Brooks, *Bull. Bur. Stands.*, Research Paper, No. 342 (1931).

(28) "The Inductance of Linear Conductors of Rectangular Section," A. H. M. Arnold, *Journ. I.E.E.*, Vol. LXX, p. 579.

(29) "Calculation of Reactance in Parallel Conductors of Rectangular Cross Section," I. Herlitz, *A.S.E.A. Jour.*, August, 1934.

(30) "Inductance of Steel Reinforced Aluminium Cable," L. F. Woodruff, *Trans. A.I.E.E.*, Vol. 54, p. 296.

(31) "Copper Losses in Large Cables at Power Frequencies" (E.R.A. Report), *Jour. I.E.E.*, Vol. LXXVI, p. 299.

(32) "Measurements of the High Frequency Resistance of Single-layer Solenoids," W. Jackson, *Jour. I.E.E.*, Vol. LXXX, p. 440.

(33) "Some Series Formulas for Mutual Inductance of Solenoids," H. B. Dwight and F. W. Grover, *Trans. A.I.E.E.*, Vol. LVI, p. 347.

(34) "The Design of Inductances for Frequencies between 4 and 25 Megacycles," D. Pollack, *Trans. A.I.E.E.*, Vol. LVI, p. 1169.

(35) "The Dependence on Frequency of the Temperature Coefficient of Inductance of Coils," H. A. Thomas, *Journ. I.E.E.*, Vol. LXXXIV, p. 101.

(36) *The Theory and Design of Inductance Coils*, V. G. Welsby.

(37) "Recent Improvements in Air-cooled Inductances," W. H. F. Griffiths, *The Wireless Engineer*, January, 1942.

(38) *Inductance Calculations*, F. W. Grover.

(39) "Standards and Standardisation," *The Wireless Engineer*, Vol. XX, No. 234, pp. 109–26.

(40) "Electrical Standards for Research and Industry," a catalogue published by Messrs. H. W. Sullivan, Ltd., in 1954.

CHAPTER VI

MEASUREMENT OF INDUCTANCE AND CAPACITANCE

Self-inductance. Several approximate methods of measuring self-inductance are worthy of mention before the more precise methods—most of which are alternating current bridge methods—are described.

AMMETER AND VOLTMETER METHOD. Inductances of about 50 to 500 millihenries can be measured by this method. It is suitable for iron-cored coils, since the full normal current to be carried by the coil can be passed through it during the measurement.

A suitable current, of normal frequency, is passed through the coil, and this is measured by an a.c. ammeter while the voltage drop across the coil is measured by a high resistance voltmeter. The d.c. resistance R of the coil—which will be the same as the a.c. resistance, to a close approximation, if the frequency f is low—must also be measured. Then the inductance L of the coil is given by

$$L = \frac{\sqrt{Z^2 - R^2}}{2\pi f} \text{ henries}$$

where $Z = \dfrac{\text{voltmeter reading}}{\text{ammeter reading}}$ = the impedance of the coil

Application of the A.C. Potentiometer to the Method. An improvement upon this simple method is the introduction of an alternating current potentiometer (see Chapter VIII) for the more precise measurement of the current and voltage drop.

A non-inductive resistance is then connected in series with the coil under test and the voltage drop across this, as well as that across the coil, is measured. The phase of the voltage drop across the coil, as well as its magnitude, is measured.

Let R = the value of the non-inductive resistance
θ = the phase angle between the current and the voltage drop across the coil
V' = the voltage drop across the coil
I = the current (of frequency f)

Then $V' = I\sqrt{r^2 + (2\pi f L)^2}$

where r and L are the resistance and inductance of the coil under test.

Also $I = \dfrac{V}{R}$

where V = the voltage drop across the non-inductive resistance.

$$V' = \frac{V}{R}\sqrt{r^2 + (2\pi fL)^2}$$

or,
$$\sqrt{r^2 + (2\pi fL)^2} = \frac{V'R}{V}$$

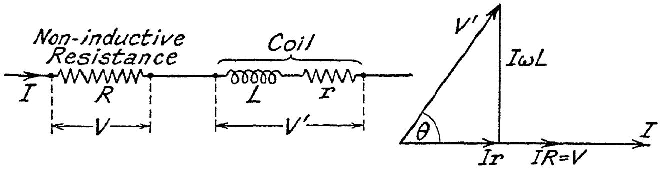


FIG. 106. VECTOR DIAGRAM FOR AMMETER AND VOLTMETER METHOD

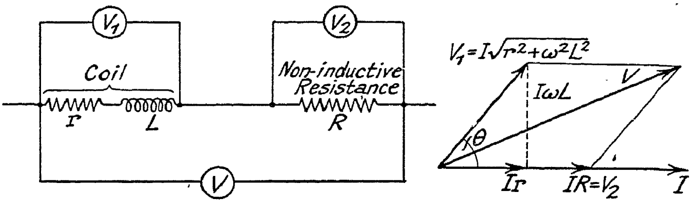


FIG. 107. VECTOR DIAGRAM FOR THREE-VOLTMETER METHOD

From the vector diagram of Fig. 106,

$$I \times 2\pi fL = V' \sin\theta = I\sqrt{r^2 + (2\pi fL)^2}\,\sin\theta$$

$$\therefore L = \frac{V'R}{V \times 2\pi f}\sin\theta = \frac{V'}{I} \times \frac{\sin\theta}{2\pi f}. \qquad . \qquad . \quad (138)$$

Similarly, $\frac{Ir}{V'} = \cos\theta$

from which $r = \frac{V'R}{V}\cos\theta$ . . . . . (139)

No measurement of the d.c. resistance is necessary, but the frequency and phase angle θ must be observed.

THREE-VOLTMETER METHOD. The connections of this method are shown in Fig. 107. A suitable current is passed through the coil, in series with a non-inductive resistance R, and the voltage drops across both parts of the circuit and across the whole circuit are measured as shown.

From the vector diagram in the figure

$$V^2 = V_1^2 + V_2^2 + 2V_1V_2\cos\theta$$

or
$$\cos\theta = \frac{V^2 - V_1^2 - V_2^2}{2V_1V_2}$$

But
$$\cos\theta = \frac{r}{\sqrt{r^2 + (2\pi fL)^2}}$$

where r and L are the resistance and inductance of the coil.

Thus
$$\sqrt{r^2 + (2\pi fL)^2} = \frac{2rV_1V_2}{V^2 - V_1^2 - V_2^2}$$

or
$$L = \frac{1}{2\pi f}\sqrt{\frac{4r^2V_1^2V_2^2}{(V^2 - V_1^2 - V_2^2)^2} - r^2} \quad . \quad . \quad (140)$$

The resistance r is measured on direct current.

THREE-AMMETER METHOD. The diagram of connections and vector diagram for this method are as shown in Fig. 108. In this case the non-inductive resistance R, together with an ammeter, is connected in parallel with the coil whose inductance is to be measured.

The theory of the method is exactly similar to that of the three-voltmeter method, but with currents I_1, I_2, and I replacing V_1, V_2, and V.

Thus
$$L = \frac{1}{2\pi f}\sqrt{\frac{4r^2I_1^2I_2^2}{(I^2 - I_1^2 - I_2^2)^2} - r^2} \quad . \quad . \quad . \quad (141)$$

ALTERNATING CURRENT BRIDGE METHODS. The best, and most usual methods for the precise measurement of self- and mutual-inductance and capacity are those employing a bridge network with an alternating current supply. The supply may be of commercial frequency—when a vibration galvanometer is used as the detector—or it may be of higher frequency (say 500 to 2,000 ~ per second), when telephones or thermionic detectors (see Ref. (1)) are employed.

These networks are all, in general, modifications of the original Wheatstone bridge network and their operation is also similar.

In the Wheatstone bridge method of measuring resistance with direct current, the bridge is balanced (i.e. zero galvanometer deflection is obtained) when the voltage drops across the two arms connecting one of the supply terminals to the two ends of the galvanometer branch of the network are equal in magnitude. With a.c. bridge networks these volt drops must also be alike in *phase* as well as in magnitude, and for this reason the introduction of inductances or capacitances in other arms of the network is necessary when (say) an inductance to be measured is connected in one of the arms.

The bridge network to be chosen for the measurement of a given self-inductance depends upon the magnitude of that inductance and upon its "time-constant"—i.e. the ratio of inductance to resistance. In the following the magnitudes of the inductances to the measurement of which the various methods are best suited will be stated.

√*Maxwell's Method.* In this method the unknown inductance is

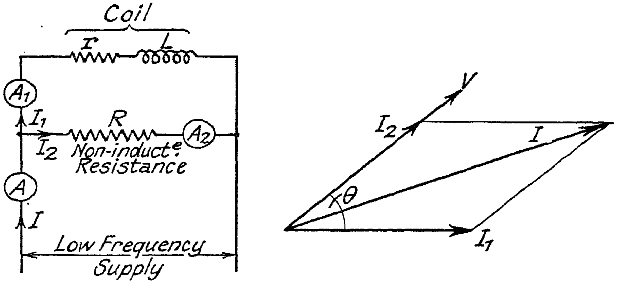
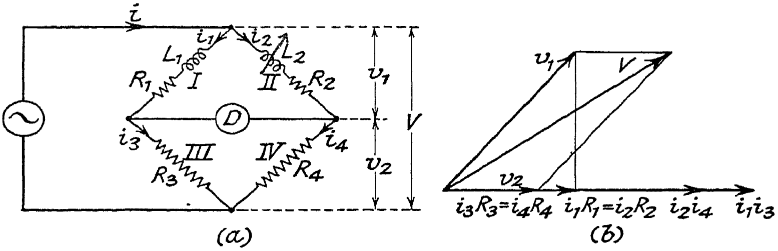


FIG. 108. THREE-AMMETER METHOD



(a) (b)

FIG. 109. MAXWELL'S METHOD FOR THE MEASUREMENT OF SELF-INDUCTANCE

compared with a known self-inductance. The connections for a.c. working, together with the vector diagram, are given in Fig. 109.

L_1 = unknown self-inductance of resistor R_1
L_2 = known self-inductance of resistor R_2
R_3 and R_4 = non-inductive resistors
D = detector

The resistances R_1, R_2, etc., include, of course, the resistances of the leads and contact resistances in the various arms. It is most convenient to use for the known inductance L_2, a variable self-inductance of constant resistance, its inductance being of the same order as that of L_1.

The bridge is balanced by varying L_2 and one of the resistors R_3 or R_4. Alternatively, R_3 and R_4 can be kept constant, and the resistance of one of the other two arms can be varied by connecting in the arm an additional resistor.

Theory. At balance the voltage drop v_1 across branch I = voltage drop across branch II, and the current i_1 in branch I = current i_3 in branch III. Similarly, volt drop v_2 across branch III = volt drop across branch IV, and $i_2 = i_4$, the volt drops being equal both in magnitude and phase.

Then, using the symbolic notation,

$$\frac{(R_1 + j\omega L_1)i_1}{R_3 i_3} = \frac{(R_2 + j\omega L_2)i_2}{R_4 i_4}$$

or

$$R_1R_4 + j\omega L_1R_4 = R_2R_3 + j\omega L_2R_3$$

Equating real and imaginary quantities, we have

$$R_1R_4 = R_2R_3$$

or

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

and also,

$$\frac{L_1}{L_2} = \frac{R_3}{R_4}$$

Thus

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} = \frac{L_1}{L_2} \qquad . \quad . \quad . \quad . \quad . \quad (142)$$

The inductances L_1 and L_2 should be placed at a distance from one another and the leads used in the arms should be carefully twisted together to avoid loops. It should be remembered in this connection that a loop having an enclosed area of 1 sq. ft. has an inductance of roughly 1 microhenry.

The vector diagram of Fig. 109 (*b*) is for balance conditions, and shows i_1 and i_3 in phase with i_2 and i_4. This is obviously brought about by adjusting the impedances of the various branches so that these currents lag by the same phase angle behind the applied voltage V.

This method is very suitable for the measurement of inductances of medium magnitudes and can be arranged to give results of considerable precision.

Anderson Bridge. This method requires a standard capacitor [illegible] lf-inductance is expressed. It is actually a [illegible] d of comparing an inductance with [illegible] plicable to the precise measurement of inductances over a wide range of values, and is one of the commonest and best bridge methods.

Fig. 110 gives the diagram of connections and the vector diagram for balanced conditions.

L = self inductance to be measured

C = standard capacitor

R_1 = resistance of arm 1 (including the resistance of the self inductance)

r, R_2, R_3, R_4 = known non-inductive resistances

In the original method a battery and key were used instead of an alternating current supply. R_2, R_3, and R_4 were adjusted to give a balance for steady currents, with the battery key closed. The resistance r was then adjusted (without altering the original resistance settings) to give a balance when the battery key was opened or closed, the two balances being quite independent of one another.

When used with alternating currents, it is still convenient to obtain a preliminary balance for steady currents, using an ordinary galvanometer as detector, the alternating-current balance being then obtained by varying r. Either telephones or a vibration galvanometer—according to the supply frequency—must be used for the detector when alternating currents are used.

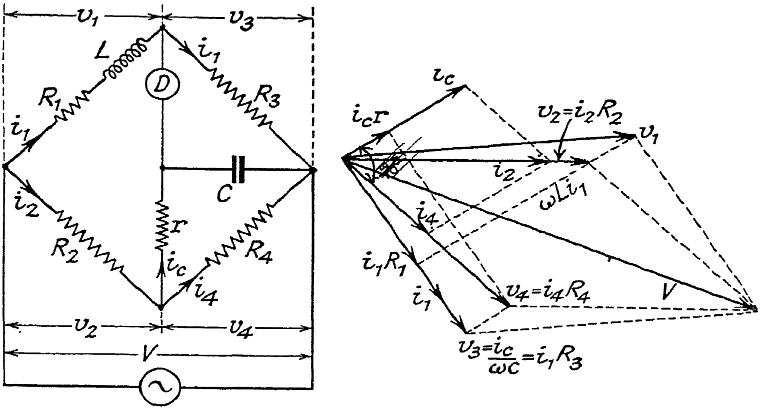


FIG. 110. ANDERSON BRIDGE FOR THE MEASUREMENT OF SELF-INDUCTANCE

When the bridge is finally balanced the self-inductance is given by

$$L = \frac{C \cdot R_3}{R_4}\left[r(R_2 + R_4) + R_2R_4\right] \quad . \quad . \quad . \quad (143)$$

Theory. Assume the capacitor C to be loss-free and the resistances completely non-inductive.

Referring to the simplified network diagram of Fig. 111, where the branch impedances are represented by Z_1, Z_2, etc., and the mesh currents by A, V, X, and $X + Y$, so that the detector current is Y, we have the mesh equations

Mesh I.

$$Z_1(X + Y) + Z_5(Y + X - X) + Z_6(X + Y - Y) + Z_2(X + Y - A) = 0$$

or $$X(Z_1 + Z_2 + Z_6) + Y(Z_1 + Z_2 + Z_5 + Z_6) - VZ_6 - AZ_2 = 0$$

Mesh II.

$$Z_3X + Z_7(X - V) + Z_5(X - X - Y) = 0$$

or $$X(Z_3 + Z_7) - Z_5Y - Z_7V = 0$$

Mesh III.

$$Z_7(V - X) + Z_4(V - A) + Z_6(V - X - Y) = 0$$
$$- X(Z_6 + Z_7) - YZ_6 + V(Z_4 + Z_6 + Z_7) - AZ_4 = 0$$

Solving algebraically for Y and equating Y to zero (which is the condition at balance), we have

$$0 = Z_2 + \frac{Z_6Z_4}{(Z_4 + Z_6 + Z_7) - Z_7\dfrac{(Z_6 + Z_7)}{Z_3 + Z_7}}$$
$$- \frac{Z_4(Z_1 + Z_2 + Z_6)Z_7}{(Z_7 + Z_3)\left[(Z_4 + Z_6 + Z_7) - \dfrac{Z_7(Z_6 + Z_7)}{Z_3 + Z_7}\right]}$$

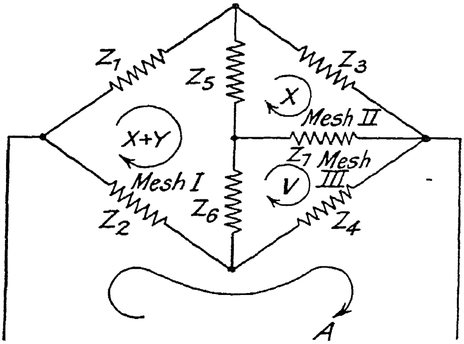


FIG. 111. SIMPLIFIED ANDERSON BRIDGE NETWORK

From which,

$$0 = Z_2Z_3Z_4 + Z_2Z_3Z_6 + Z_2Z_3Z_7 + Z_3Z_4Z_6 - Z_1Z_4Z_7$$

Expressing the impedances symbolically, we have

$[Z_1] = R_1 + j\omega L$
$[Z_2] = R_2$
$[Z_3] = R_3$
$[Z_4] = R_4$
$[Z_6] = r$
$[Z_7] = \dfrac{-j}{\omega C}$

Substituting in the impedance equation gives,

$$0 = R_2R_3R_4 + R_2R_3r - R_2R_3\frac{j}{\omega C} + R_3R_4r - (R_1 + j\omega L)R_4\left(\frac{-j}{\omega C}\right)$$

Equating real and imaginary quantities,

$$0 = R_2R_3R_4 + R_2R_3r + R_3R_4r - R_4\frac{\omega L}{\omega C}$$

or

$$L = \frac{CR_3}{R_4}(R_2R_4 + R_2r + R_4r)$$

and $$\frac{jR_2R_3}{\omega C} = \frac{jR_1R_4}{\omega C}$$

or $$R_2R_3 = R_1R_4$$

which is the balance condition for steady currents.

Thus, the effective resistance of the self-inductance under test is

$$R_1 = \frac{R_2R_3}{R_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

If the capacitor is not perfect, but has dielectric loss, the self-inductance value given by the above expression is unaltered, but the value of R_1 is affected.

If the self-inductance of the leads to the coil under test is appreciable, this may be measured by short-circuiting the coil and obtaining a second balance. The actual inductance of the coil may then be obtained by subtraction.

Theory at Balance. The above theory may be simplified by considering the case under balance conditions only.

Referring to Fig. 110 and taking $i_4 = i_2 - i_c$ (in symbolic notation), we have the equations

$$i_1(R_1 + j\omega L) = i_2R_2 + i_cr$$
$$i_1R_3 = i_c/j\omega C$$
$$i_cr + i_c/j\omega C = (i_2 - i_c)R_4$$

Eliminating the currents i_1, i_2, and i_c gives

$$R_1 + j\omega L - j \,.\, \omega C \,.\, \frac{R_2R_3r}{R_4} - \frac{R_2R_3}{R_4} - j\omega C \,.\, R_2R_3 - j\omega C \,.\, rR_3 = 0$$

The balance conditions, as obtained above, are then determined by equating real and imaginary terms.

The method can also be used to measure the capacitance of the capacitor C if a calibrated self-inductance is available.

Butterworth's Method. This method is especially suitable for the measurement of small inductances (e.g. a few microhenries).

The diagram of connections and the vector diagram are given in Fig. 112.

L = the self-inductance (of resistance R_1) to be measured

R_2 = a slide wire

C = fixed standard capacitor

R_3, R_4, r = non-inductive resistances.

The resistance balance, or the balance for steady current, can be obtained independently of the inductance balance by adjusting the resistances R_3 and R_4, while the inductance balance is obtained by adjustment of r and the slide wire setting.

At balance

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

and $$L = \frac{(R_2 - r_2)}{R_4}[(R_3 + R_4)r + R_3(R_4 + r_2)]C \qquad (146)$$

(Ref. (6)). These conditions may be obtained from the mesh equations by a similar method to that used when considering the Anderson bridge.

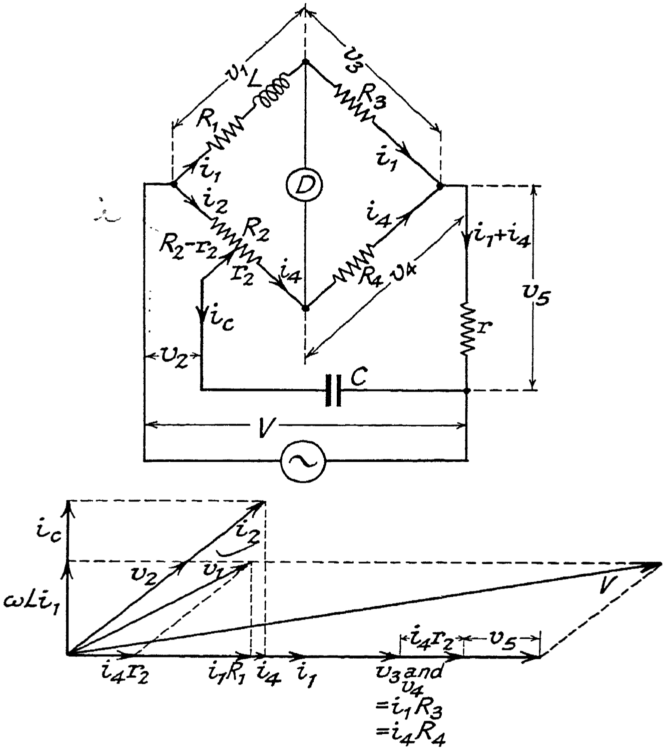


FIG. 112. BUTTERWORTH'S METHOD FOR THE MEASUREMENT OF SMALL

To obtain maximum sensitivity, Butterworth has shown that the following relationships should be fulfilled—

$$R_4 = \sqrt{S \, . \, D}, \qquad R_3 = \sqrt{R_1 D\left(\frac{R_1 + S}{R_1 + D}\right)}$$

$$R_2 = \sqrt{R_1 S\left(\frac{R_1 + D}{R_1 + S}\right)}$$

where S is the resistance of the alternator branch (including r) and D is the resistance of the detector branch. The resistance r, also should be small.

Hay's Bridge. This method of measurement is particularly suited to the measurement of large inductances having a comparatively low resistance (i.e. having a large time-constant). The diagram of connections and the vector diagram are given in Fig. 113.

L is the inductance to be measured. R_1 is its resistance, C is a variable standard capacitor and R_2 R_3 and R_4 non-inductive resistances. Balance may be obtained by variation of C, R_4, and R_2.

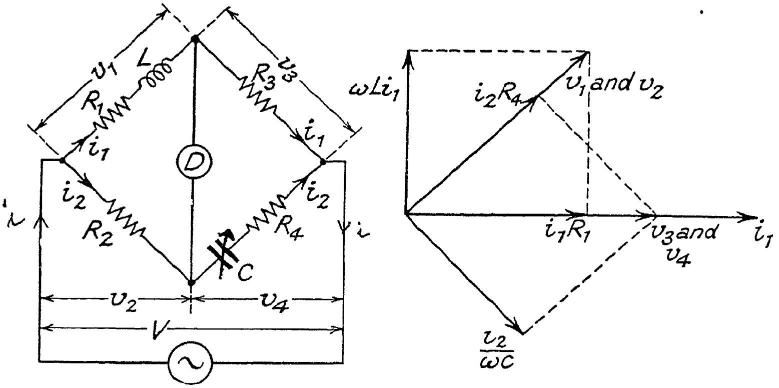


FIG. 113. HAY'S BRIDGE FOR THE MEASUREMENT OF LARGE SELF-INDUCTANCES

At balance, volt drop across arm I = volt drop across arm II, and volt drop across arm III = volt drop across arm IV.

Thus $$(R_1 + j\omega L)i_1 = R_2 i_2$$

$$R_3 i_1 = \left(R_4 - \frac{j}{\omega C}\right)i_2$$

From which $$L = \frac{R_2 R_3 C}{1 + \omega^2 R_4^2 C^2} \quad . \quad . \quad . \quad (147)$$

and the effective resistance R_1 of the coil is

$$R_1 = \frac{R_2 R_3 R_4 \omega^2 C^2}{1 + \omega^2 R_4^2 C^2} \quad . \quad . \quad . \quad (148)$$

Since the expressions for L and R_1 involve ω (= $2\pi f$), the frequency must be accurately measured.

Measurement with Superposed D.C. and A.C. It has been shown by Landon and by Hartshorn (see Ref. 1, Fourth Edition, p. 391) that Hay's bridge may be used for the measurement of self-inductance in the case of iron-cored coils, in which both direct and alternating currents are flowing. The arrangement of the bridge for such a measurement is as shown in Fig. 114.

The direct current, which may be adjusted to the required value by r_1, passes through R_1, L, and R_3 only, the capacitors preventing its passage through the other branches. The magnitude of the alternating current passing through the coil L (under test) is regulated by C_2 and r_2. This current is obtained from the reading, when the bridge is balanced, of an electrostatic voltmeter connected across R_2. This reading gives the potential difference across R_1L. The voltmeter is removed before final balance of the bridge is made. To obtain the requisite sensitivity, the detector D consists of a vibration galvanometer supplied through a step-up transformer, the capacitor C_3 being adjusted to resonate this transformer.

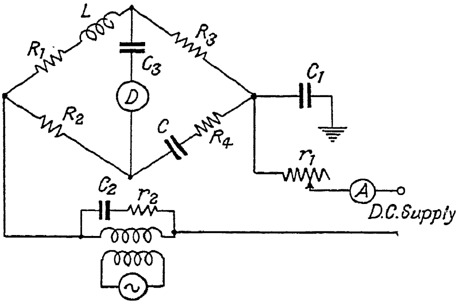


FIG. 114

C_1 is a large capacitor through which one corner of the bridge is earthed. Its use avoids direct earthing of one side of the d.c. supply. The bridge is balanced in the usual way after the direct alternating currents have been adjusted to the requisite values.

aviside-Campbell Bridge. This method employs a standard variable mutual-inductance, and can be used for the measurement of self-inductance over a very wide range. It is one of the best methods for general laboratory use. Fig. 115 shows the diagram of connections of Heaviside's bridge.

The primary of the mutual inductometer is in the supply circuit, and the secondary of self-inductance L_2 and resistance R_2 form arm II of the bridge. The inductance to be measured, of self-inductance L_1 and resistance R_1, is placed in arm I of the bridge. R_3 and R_4 are non-inductive resistances.

Balance may be obtained by varying the mutual-inductance and resistances, R_3 and R_4. At balance,

$$i_2(R_2 + j\omega L_2) + j\omega M i = (R_1 + j\omega L_1)i_1$$

and

$$i_2 R_4 = i_1 R_3$$

Since

$$i = i_1 + i_2$$

$$i_2\,[R_2 + j\omega(L_2 + M)] = i_1\,[R_1 + j\omega(L_1 - M)]$$

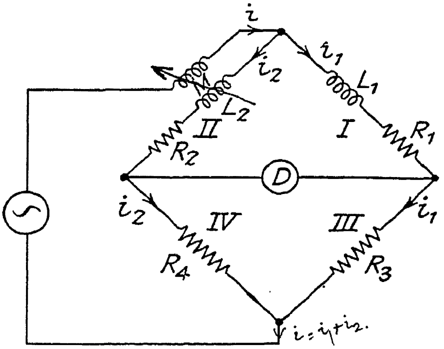


FIG. 115. HEAVISIDE BRIDGE

Thus

$$\frac{R_2 + j\omega(L_2 + M)}{R_4} = \frac{R_1 + j\omega(L_1 - M)}{R_3}$$

or

$$R_3\,[R_2 + j\omega(L_2 + M)] = R_4\,[R_1 + j\omega(L_1 - M)]$$

Equating real and imaginary quantities

$$R_2 R_3 = R_1 R_4 \quad (149)$$

and

$$R_3(L_2 + M) = R_4(L_1 - M) \quad (150)$$

If the resistances R_3 and R_4 are equal,

$$L_2 + M = L_1 - M$$

or

$$L_1 - L_2 = 2M$$

In *Campbell's Modification* of the bridge (Refs. (10) and (11)), the resistances R_3 and R_4 are made equal. A "balancing coil" of self-inductance equal to the self-inductance L_2 of the mutual-inductance secondary coil and of slightly greater resistance than the latter is introduced in arm I, in series with the inductance to be measured. A non-inductive resistance box and a "constant-inductance rheostat" are also introduced in arm II. These additions are shown in Fig. 116.

Balance is now obtained, by variation of the mutual inductometer and the variable resistance r, with the coil L_1, R_1, whose inductance and resistance are to be measured, in circuit. Suppose the readings of the mutual-inductance and resistance r are M_1 and r_1. The coil L_1R_1 is now removed, or short-circuited across its terminals, and balance is again obtained, giving, say, readings M_2 and r_2.

Then $$L_1 = 2(M_1 - M_2)$$
and $$R_1 = r_1 - r_2$$

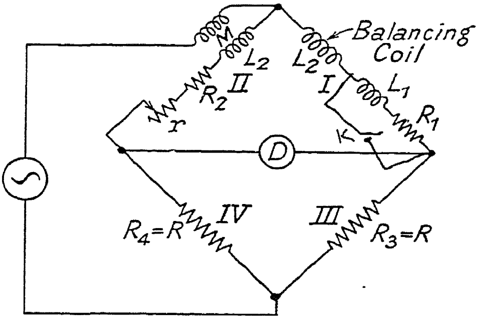


FIG. 116. CAMPBELL'S MODIFICATION OF THE HEAVISIDE BRIDGE

By this method of operation the self-inductance and resistance of the leads is eliminated and the inductance and resistance of the coil obtained directly.

The use of a balancing coil in the above arrangement reduces the sensitivity of the bridge. Fig. 117 shows a better arrangement, which improves the sensitivity and eliminates the balancing coil. For this arrangement the secondary fixed coil of the inductometer must be made up of two equal coils LL, the primary coil reacting with both of them as shown. L_1 is the coil whose self-inductance is to be measured. The resistances R_3 and R_4 are equal ($R_3 = R_4 = R$). When so arranged the bridge is known as the *Heaviside-Campbell Equal Ratio Bridge*.

At balance—obtained by varying the constant-inductance rheostat r, and the mutual-inductance $M_1 + M_2$—we have the relationships

$$R_1 = R_2$$
and $$L_1 = 2(M_1 + M_2)$$

where R_1 and R_2 are the total resistances of arms I and II and $M_1 + M_2$ is the reading of the inductometer. With equal ratio arms R_3 and R_4 it is obvious that the magnitude of the self-inductance L_1 which can be measured is limited to twice the inductometer range. If L_1 is greater than this value, unequal ratio arms are used with

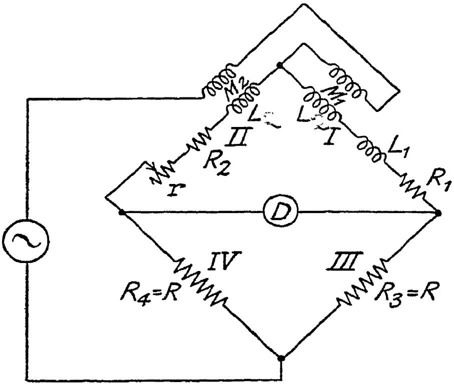


FIG. 117. HEAVISIDE-CAMPBELL EQUAL RATIO BRIDGE

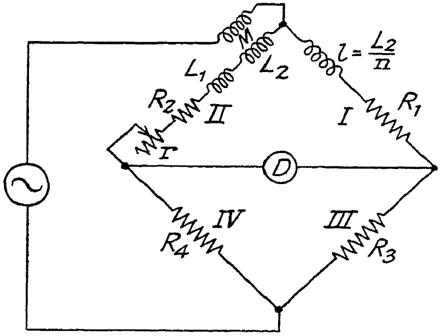


FIG. 118. HEAVISIDE-CAMPBELL BRIDGE WITH BALANCING COIL

a balancing coil l, the connections then being as shown (Fig. 118). Let the ratio

$$\frac{R_4}{R_3} = n$$

Then, if the inductance of the balancing coil is made equal to $\frac{L_2}{n}$

(where L_2 = inductance of inductometer secondary coil) the balance conditions are

$$\frac{R_2}{R_1} = n$$

and

$$L_1 = (n + 1)M$$

When used as described above, the bridge can be used for the measurement of inductances varying from very low values to medium values. D. W. Dye (Ref. (9)) has modified the arrangement in order to make it suitable also for the measurement of large inductances, and, when so modified, the method is a very good one for this purpose.

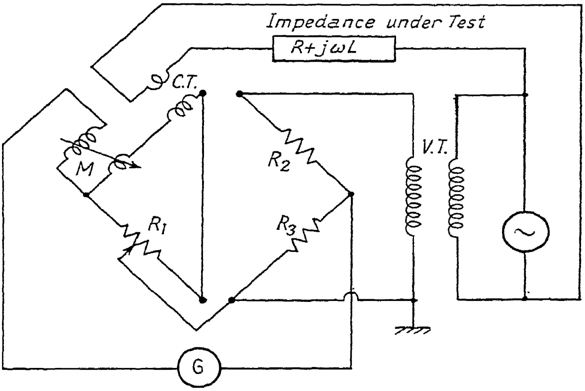


FIG. 119. KURIYAMA METHOD OF IMPEDANCE MEASUREMENT

Kuriyama Method. B. Hague (Ref. (1), 4th Edition, p. 433) has described an interesting method, due to M. Kuriyama, of measuring the value of an impedance which is carrying a large current at high voltages. Instrument transformers are used to isolate the bridge from the high-voltage circuit. The connections of the method are given in Fig. 119 in which *C.T.* and *V.T.* are the current and voltage transformers whose ratios and phase angles (see Chapter XIX) are K_c and β (for *C.T.*) and K_v and γ (for *V.T.*).

If the impedance of the primary of the current transformer is negligible compared with that under test, it can be shown that

$$R \fallingdotseq \frac{K_v}{K_c} \cdot \frac{R_2 + R_3}{R_3} [R_1 + \omega M(\beta - \gamma)]$$

$$\fallingdotseq \frac{K_v}{K_c} \cdot \frac{R_2 + R_3}{R_3} \cdot R_1$$

and also
$$L \fallingdotseq -\frac{K_v}{K_c}\cdot\frac{R_2+R_3}{R_3}\left[M-\frac{R_1}{\omega}(\beta-\gamma)\right]$$
$$\fallingdotseq -\frac{K_v}{K_c}\cdot\frac{R_2+R_3}{R_3}.M$$

✓**Measurement of Mutual Inductance.** The simplest method of measuring mutual inductance consists of passing an alternating current—measured by an ammeter—through the primary of the mutual inductance and observing the voltage induced in the secondary by means of an electrostatic voltmeter. It is important that the current shall have a purely sinusoidal wave-form, since harmonics may introduce serious errors.

FIG. 120. FELICI'S METHOD OF MEASURING MUTUAL INDUCTANCE

If the current in the primary is given by
$$i = I_{max}\sin\omega t$$
then the induced voltage in the secondary will be
$$e = M\frac{di}{dt} = MI_{max}\,\omega\cos\omega t$$
or, taking r.m.s. values of current and voltage,
$$E = \omega MI$$
from which
$$M = \frac{E}{\omega I}$$
Since $\omega = 2\pi \times$ frequency, the frequency of the supply should be accurately measured.

A *Method.* If a variable standard mutual inductometer is available this method is an improvement upon the above. It can, however, only be used for the measurement of mutual inductance within the range of the standard inductometer. The connections are shown in Fig. 120, where M_1 is the mutual inductance to be measured and M is a variable standard mutual inductance.

An alternating current is passed through the two primary coils in series, and the secondary coils are connected in opposition. The standard inductance M is varied until the vibration galvanometer D shows no deflection. Then, the reading of the standard gives the mutual inductance of M_1.

Some difficulty may be encountered—especially at high frequencies—in obtaining zero deflection of the detector D. The self-capacity of the mutual inductances, and eddy currents induced in circuits, or metal parts, in the vicinity, may render it impossible to obtain exact balance, only a minimum deflection being obtainable.

Campbell (Ref. (3)) has discussed these effects very fully and shows how they may be taken into account.

MEASUREMENT OF A MUTUAL-INDUCTANCE AS A SELF-INDUCTANCE. It was pointed out in Chapter V that if two coils, of self-inductances

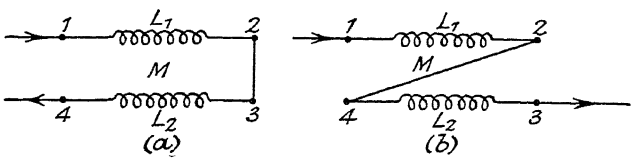


FIG. 121. MUTUAL INDUCTANCE CONNECTED AS A SELF-INDUCTANCE

L_1 and L_2, are connected in series, and if the mutual-inductance between them is M, then the self-inductance of the arrangement is given by $L = L_1 + L_2 \pm 2M$, the alternative sign depending upon the connections and on the relative positions of the two coils.

A simple method of obtaining the mutual-inductance M is to measure—by one of the methods already described—the self-inductance of the combination, first with the two coils connected in series, as in Fig. 121 (*a*), and then when connected as shown in Fig. 121 (*b*). In the first case $L = L_1 + L_2 + 2M$, and in the second $L' = L_1 + L_2 - 2M$, where L and L' are the measured self-inductances. By subtraction

$$L - L' = 4M \qquad . \quad . \quad . \quad . \quad . \quad (151)$$

or

$$M = \frac{L - L'}{4}$$

SUREMENT BY BALLISTIC GALVANOMETER. The secondary winding of the mutual-inductance is connected to a ballistic galvanometer, and a current of I amp. is passed through the primary winding. Upon reversal of the current I an average e.m.f. of $\frac{2MI}{t}$ volts is induced in the secondary winding, where M is the mutual-inductance in henries and t is the time in seconds taken for the reversal of the current I. The average current in the galvanometer circuit will be

$\frac{2MI}{tR}$., where R is the total resistance of the ballistic galvanometer circuit. Thus the quantity of electricity passed through the ballistic galvanometer during the reversal is $\frac{2MI}{R}$ coulombs. Now, the equation giving the deflection of a ballistic galvanometer when a quantity of electricity Q passes through it is

$$Q = \frac{T}{\pi} \cdot K \left(1 + \frac{\lambda}{2}\right) \sin \frac{\theta}{2} \text{ (see Chap. IX)}$$

where T = time in seconds of one complete vibration of the galvanometer moving system

K = the galvanometer constant

θ = the "throw" of the galvanometer

λ = the logarithmic decrement of the galvanometer vibration

Hence, the mutual-inductance M is given by

$$M = \frac{RT}{2\pi I} \cdot K \left(1 + \frac{\lambda}{2}\right) \sin \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (152)$$

Maxwell's Method. The connections for the comparison of two

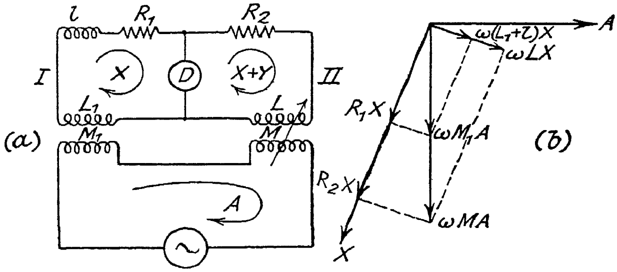


FIG. 122. MAXWELL'S METHOD FOR THE COMPARISON OF TWO MUTUAL-INDUCTANCES

unequal mutual-inductances are shown in Fig. 122 (*a*). M_1 is the mutual-inductance to be compared with the standard variable inductometer M. L_1 and L are the self-inductances of their secondary windings. R_1 and R_2 are the total resistances of the two branches, and l is a variable self-inductance inserted in either branch I or branch II to obtain exact balance. The final balance is obtained by successive adjustments of l, R_1, and R_2.

Theory. Using the mesh currents as in Fig. 122 (*a*), the mesh equations are—

Mesh I.

$$R_1X + j\omega(L_1 + l)X - DY + j\omega M_1A = 0$$

or

$$X[R_1 + j\omega(L_1 + l)] - DY + j\omega M_1A = 0$$

Mesh II.

$$(R_2 + j\omega L)(X + Y) + DY + j\omega MA = 0$$

or $$(R_2 + j\omega L)X + (R_2 + D + j\omega L)Y + j\omega MA = 0$$

where D = impedance of the detector circuit.

Since the detector current Y is zero at balance, we have

$$X[R_1 + j\omega(L_1 + l)] + j\omega M_1 A = 0 \quad . \quad . \quad . \quad \text{(i)}$$

$$X(R_2 + j\omega L) + j\omega MA = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

Substituting in (i) for A from (ii),

$$X[R_1 + j\omega(L_1 + l)] - M_1 \frac{(R_2 + j\omega L)X}{M} = 0$$

or $$R_1 + j\omega(L_1 + l) = \frac{M_1}{M}(R_2 + j\omega L)$$

Equating real and imaginary quantities, we have

$$R_1 = \frac{M_1}{M} R_2 \text{ or } \frac{R_1}{R_2} = \frac{M_1}{M}$$

and $$L_1 + l = \frac{M_1}{M} L \text{ or } \frac{L_1 + l}{L} = \frac{M_1}{M}$$

Hence $$\frac{M_1}{M} = \frac{R_1}{R_2} = \frac{L_1 + l}{L} \quad . \quad . \quad . \quad . \quad (153)$$

Fig. 122 (*b*) gives the vector diagram for the network under balance conditions. Then, since $Y = 0$, the current in both of the branches I and II will be X.

Campbell's Method of Comparing Two Unequal Mutual-inductances. Fig. 123 gives the connections of the network for this method. M is the unknown mutual-inductance whose primary winding has self-inductance L. M_1 is a standard mutual inductometer with self-inductance L_1 in its primary winding. Suppose that M is greater than the maximum value of M_1, then a variable self-inductance, in series with the primary of the unknown, is inserted to make $L + l$ greater than L_1. R_1, R_2, R_3, and R_4 are the resistances of the four arms.

The switches S_1 and S_2 are first thrown on to contacts *aa*, so as to exclude from the detector circuit the secondaries of the mutual inductances. The bridge is then balanced by varying the resistances and the self-inductance l. Then, as shown in connection with Maxwell's method for the comparison of self-inductances

$$\frac{R_1}{R_2} = \frac{R_3}{R_4} = \frac{L_1}{L + l} \quad . \quad . \quad . \quad (154)$$

The secondaries of the mutual-inductances are then connected in series with the detector, and in opposition to one another. This is done by throwing over switches S_1 and S_2 on to contacts *bb*. Balance is then again obtained by adjusting the variable standard M_1, the four branches I, II, III, and IV being left unaltered.

Theory. Taking mesh currents X, $X + Y$, and A, as in the figure, we have

Mesh I.

$$(R_1 + j\omega L_1)(X - A) + [R_2 + j\omega(l + L)]X - DY + j\omega MX - j\omega M_1(X - A) = 0$$

or $(R_1 + j\omega L_1 - j\omega M_1)(X - A) + [R_2 + j\omega(l + L) + j\omega M]X - DY = 0$

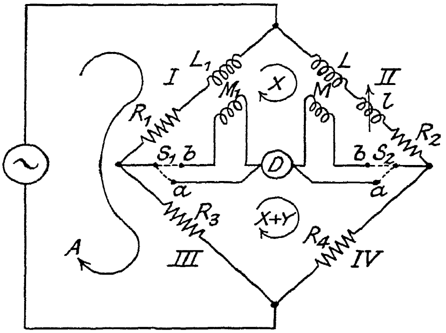


FIG. 123. CAMPBELL'S METHOD FOR THE COMPARISON OF TWO MUTUAL-INDUCTANCES

Mesh II.

$$R_4(X + Y) + R_3(X + Y - A) + YD - j\omega MX + j\omega M_1(X - A) = 0$$

or $(R_4 - j\omega M)X + (R_4 + R_3 + D)Y + (R_3 + j\omega M_1)(X - A) = 0$

where D = total impedance of the detector circuit.

When $Y = 0$,

$$(R_1 + j\omega L_1 - j\omega M_1)(X - A) = -X[R_2 + j\omega(l + L) + j\omega M]$$

and $(R_3 + j\omega M_1)(X - A) = -X(R_4 - j\omega M)$

By division

$$\frac{R_1 + j\omega L_1 - j\omega M_1}{R_3 + j\omega M_1} = \frac{R_2 + j\omega(l + L) + j\omega M}{R_4 - j\omega M}$$

Hence

$$R_1R_4 + j\omega L_1R_4 - j\omega M_1R_4 - j\omega MR_1 + \omega^2L_1M = R_2R_3 + j\omega R_3(l + L) + j\omega MR_3 + j\omega M_1R_2 - \omega^2M_1(l + L)$$

Equating real and imaginary quantities

$$R_1R_4 + \omega^2L_1M = R_2R_3 - \omega^2M_1(l + L) \quad . \quad . \quad \text{(i)}$$

$$L_1R_4 - M_1R_4 - MR_1 = R_3(l + L) + MR_3 + M_1R_2 \quad . \quad \text{(ii)}$$

Using the conditions for the preliminary balance, viz. $\frac{R_1}{R_2} = \frac{R_3}{R_4} = \frac{L_1}{l + L}$ we have, from Equation (i),

$$L_1M = -M_1(l + L) \text{ or } -\frac{M}{M_1} = \frac{l + L}{L_1} = \frac{R_2}{R_1} = \frac{R_4}{R_3}$$

and from Equation (ii),

$$-M_1(R_4 + R_2) = M(R_1 + R_3) \text{ or } -\frac{M}{M_1} = \frac{R_2 + R_4}{R_1 + R_3}$$

Hence, finally, the balance conditions are

$$-\frac{M}{M_1} = \frac{l + L}{L_1} = \frac{R_2 + R_4}{R_1 + R_3} = \frac{R_2}{R_1} = \frac{R_4}{R_3} \quad . \quad . \quad (155)$$

Heydweiller's Modification of Carey Foster's Method. The connections of the method are as in Fig. 124 (*a*). M is the mutual-inductance to be measured, having a self-inductance L in its secondary

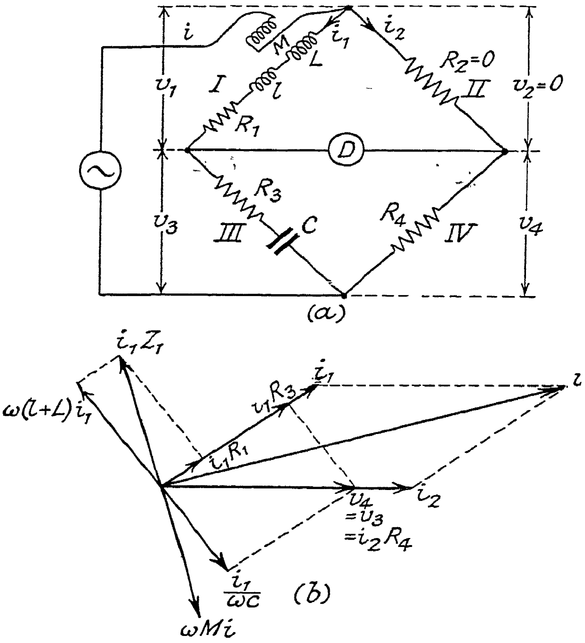


FIG. 124. HEYDWEILLER'S MODIFICATION OF CAREY FOSTER'S METHOD

winding; l is an additional self-inductance which may be necessary to obtain balance of the bridge. C is a standard capacitor. R_3 and R_4 are non-inductive resistances and R_1 the total resistance of arm I. The resistance R_2 is made zero in Heydweiller's modification of the original Carey Foster bridge.

When R_2 is zero the resistance R_4 is obviously connected directly across the supply (neglecting the primary of M). For this reason

R_4 is often a non-inductive, oil-cooled standard resistance. Balance is obtained by varying R_3, R_4, and C. The primary of the mutual-inductance must be connected so that the voltage induced by it in arm I neutralizes the volt drop due to the current i_1 in this branch, since, when $R_2 = 0$, the volt drop in branch I must be zero for balance of the bridge to be obtained.

Theory. At balance

$$i_1[R_1 + j\omega(l + L)] - j\omega Mi = i_2R_2$$

or, since $i = i_1 + i_2$

$$i_1[R_1 + j\omega(l + L) - j\omega M] = i_2[R_2 + j\omega l$$

Also,

$$i_1\left[R_3 - \frac{j}{\omega C}\right] = i_2R_4$$

Therefore,

$$\frac{R_1 + j\omega(l + L) - j\omega M}{R_3 - \dfrac{j}{\omega C}} = \frac{R_2 + j\omega M}{R_4}$$

or,

$$R_1R_4 = R_2R_3 + \frac{M}{C}$$

$$\omega(l + L)R_4 - \omega MR_4 = \omega MR_3 - \frac{R_2}{\omega C}$$

Thus,

$$M = C(R_1R_4 - R_2R_3)$$

and

$$(l + L) = M\left(1 + \frac{R_3}{R_4}\right) - \frac{R_2}{\omega^2CR_4}$$

When $R_2 = 0$, the expression for $(l + L)$ is made independent of frequency, since the second term is then zero and

$$(l + L) = M\left(1 + \frac{R_3}{R_4}\right) \quad . \quad . \quad . \quad . \quad (156)$$

Fig. 124 (*b*) gives the vector diagram for balance conditions, when $R_2 = 0$. The vector i_1z_1 representing the volt drop in the impedance (z_1) of branch I, is counterbalanced by the vector ωMi, representing the induced voltage in the secondary of the mutual-inductance, so that v_1 is zero.

Measurement of Capacitance. Although the commonest, and usually the best, methods of measuring capacitance are the alternating-current bridge methods, the apparatus required for such methods may not always be available. Under such circumstances one of the following methods might be used.

AMMETER AND VOLTMETER METHOD. If an alternating voltage of pure sine wave-form is applied to a capacitor of capacitance C farads, a current of ωCV amp. will flow, where V is the r.m.s. value of the applied voltage and $\omega = 2\pi \times$ frequency. If the current is measured by a low-reading ammeter and the voltage across the capacitor by an electrostatic voltmeter, the capacitance can be determined in terms of the readings of these instruments and of the frequency.

Instead of measuring the current by an ammeter, a non-inductive resistance of known value may be connected in series with the capacitor, and the volt drop across this resistance measured by the voltmeter. The current is then given by the voltmeter reading divided by the series resistance.

If the voltage wave-form contains harmonics of appreciable magnitude a correction may be made for this by multiplying the measured value of the capacitance, obtained as above, by the factor

$$\sqrt{\frac{V_1^2 + V_3^2 + V_5^2 + \ldots}{V_1^2 + 9V_3^2 + 25V_5^2 + \ldots}}$$

where V_1, V_3, V_5, etc., are the values of the various components of the voltage wave-form. It is important in measuring capacitance to bear in mind the fact that, since the capacitance reactance is $\frac{1}{J}$ the reactance to the harmonics is less than the reactance to the fundamental of the voltage wave, and thus the current wave is not of the same shape as the voltage wave, the harmonics being accentuated (see Chapter XV).

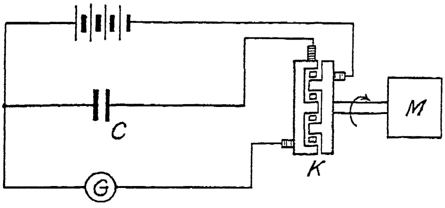


FIG. 125. FLEMING AND CLINTON'S COMMUTATOR METHOD FOR CAPACITANCE MEASUREMENTS

FLEMING AND CLINTON'S COMMUTATOR METHOD. The connections for this method—which is a method using direct current—are shown in Fig. 125. *C* is the capacitor whose capacitance is to be measured, *G* is a moving-coil galvanometer whose natural period of vibration is large compared with the time of charge and discharge of the capacitor.

Commutator Construction. The latter operation is performed by the commutator *K*, the connections to which are as shown. This commutator consists of three metal barrels, insulated from one another, and mounted on one shaft as shown. The two outer barrels, which are connected, through brushes pressing on them, to one terminal of the battery and one terminal of the galvanometer respectively, each have the same even number of lugs on their peripheries. The inner barrel has teeth projecting radially and fitting in between these lugs. The commutator is driven at a constant speed by a small motor direct-coupled to it, a counter being geared to the shaft for the purpose of speed measurement. A third brush, connected to one terminal of the capacitor, presses on the rim of the commutator as shown. As the commutator rotates this brush makes contact, first with the barrel connected to the battery—which charges the capacitor to the voltage V of the battery—and then with the barrel connected to one terminal of the galvanometer—which discharges the capacitor. The third inner barrel is provided to ensure smooth running of this brush.

A commutator of this type, driven by a phonic motor to ensure a very steady speed, is manufactured by Messrs. Muirhead & Co.

Since the time of charge and discharge of the capacitor is small compared with the period of the galvanometer, the latter is continuously deflected.

Let this deflection correspond to a current of I amp. in the galvanometer. Then, if Q is the charge (in coulombs) given to the capacitor at each charge, and N is the number of charges per second. the quantity of electricity discharged through the galvanometer is NQ coulombs per second.

Thus the current $I = NQ$

But $Q = CV$ where V is the battery voltage.

Therefore $I = NCV$

or
$$C = \frac{I}{NV} \text{ farads} \quad . \quad . \quad . \quad . \quad (157)$$

Leakage in the capacitor may be detected by connecting the galvanometer in series with the battery to measure the charging current, a short-circuiting wire replacing the galvanometer in the discharge circuit. The capacitance of the capacitor, determined from $\frac{I'}{NV}$, where I' is the charging current, should be the same as the previously determined value if leakage is negligible.

MAXWELL'S COMMUTATOR BRIDGE METHOD has already been described in Chapter II, page 67.

BALLISTIC GALVANOMETER METHOD. In this method the capacitor is charged to a known voltage V by means of a battery, and then discharged through a ballistic galvanometer, the connections being the same as those of Fig. 125, except that a key replaces the commutator. The quantity of electricity (in coulombs) discharged by the capacitor is then given by

$$Q = \frac{T}{\pi} . K \left(1 + \frac{\lambda}{2}\right) \sin \frac{\theta}{2}$$

Then
$$C = \frac{Q}{V}$$

This method can also be used for the comparison of an unknown capacitance with a standard by comparing the quantities of electricity discharged through the Ballistic galvanometer when charged to the same voltage in each case.

With direct-current methods of measurement, the time of charge, and of discharge, is important in the case of absorptive capacitors, since the measured value of the capacitance will depend to some extent upon these times (see Chapter IV).

C. BRIDGE METHODS. *De Sauty Method.* This method is the simplest way of comparing two capacitances. When used on a.c. the connections are as in Fig. 126.

C_1 = capacitor whose capacitance is to be measured
C_2 = a standard capacitor
R_1 and R_2 = non-inductive resistances

Balance is obtained by varying either R_1 or R_2.

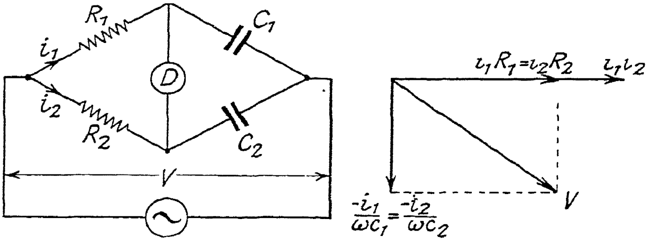


FIG. 126. DE SAUTY BRIDGE

At balance $\quad i_1R_1 = i_2R_2$

and $\quad -\frac{j}{\omega C_1}i_1 = -\frac{j}{\omega C_2}i_2$

Thus $\quad \frac{R_1}{R_2} = \frac{C_2}{C_1}$

or

$$C_1 = C_2\frac{R_2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

For maximum sensitivity, C_2 should be equal to C_1. The advantage of the simplicity of this method is largely nullified by the fact that it is impossible to obtain a perfect balance if the capacitors are not both free from dielectric loss. Only in the case of air capacitors can a perfect balance be obtained.

If two imperfect capacitors are to be compared, the bridge is modified by connecting resistances in series with them, as in Fig. 127 (*a*). R_3 and R_4 are the series resistances, while r_1 and r_2 are small resistances representing the loss components of the capacitors. Balance is obtained by variation of the resistances R_1, R_2, R_3, R_4. At balance

$$i_1R_1 = i_2R_2$$

$$i_1\left[R_3 + r_1 - \frac{j}{\omega C_1}\right] = i_2\left[R_4 + r_2 - \frac{j}{\omega C_2}\right]$$

from which it follows that

$$\frac{R_1}{R_2} = \frac{R_3 + r_1}{R_4 + r_2} = \frac{C_2}{C_1}$$

The vector diagram of Fig. 127 (*b*) shows the relative positions of the vector quantities under balance conditions. The angles δ_1 and δ_2 are the phase angles of capacitors C_1 and C_2 respectively. Obviously

$$\tan \delta_1 = \frac{r_1}{\dfrac{1}{\omega C_1}} = r_1 \omega C_1$$

and
$$\tan \delta_2 = r_2 \omega C_2$$

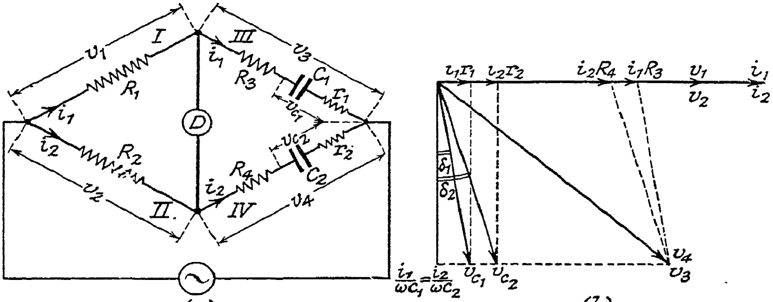


FIG. 127. MODIFICATION OF DE SAUTY BRIDGE

From the condition $\dfrac{C_2}{C_1} = \dfrac{R_3 + r_1}{R_4 + r_2}$

we have
$$C_2 r_2 - C_1 r_1 = C_1 R_3 - C_2 R_4$$
or
$$\omega C_2 r_2 - \omega C_1 r_1 = \omega (C_1 R_3 - C_2 R_4)$$
i.e.
$$\tan \delta_2 - \tan \delta_1 = \omega (C_1 R_3 - C_2 R_4)$$
Since
$$\frac{C_2}{C_1} = \frac{R_1}{R_2}$$

$$\tan \delta_2 - \tan \delta_1 = \omega C_1 \left(R_3 - \frac{R_1 R_4}{R_2} \right) \quad . \quad . \quad . \quad (159)$$

from which expression the phase angle of one capacitor can be found in terms of the phase angle of the other.

This method is due to Grover (Ref. (16)).

In the vector diagram the angles δ_2 and δ_1 are exaggerated for convenience in drawing. These angles are usually small, and thus it is usually a sufficiently good approximation to write

$$\tan (\delta_2 - \delta_1) = \omega C_1 \left(R_3 - \frac{R_1 R_4}{R_2} \right) \quad . \quad . \quad . \quad (160)$$

Grover's Series Inductance Method. This method is somewhat similar to the above, inductances being used instead of series resistances. It is a useful method for the determination of the capacitance and power factor of a small capacitor by comparison with a standard capacitor. The connections and vector diagram are given in Fig. 128.

L_1 and L_2 are variable standard inductances. R_1 and R_2 are the resistances of the arms in which these inductances are situated. Non-inductive resistances may be connected in series with L_1 and L_2, in which case R_1 and R_2 are the resistances of the arms, including the resistances of L_1 and L_2. C_1 is the unknown capacitance. C_2 is the standard capacitor, while r_1 and r_2 are resistances representing the loss components of these capacitors.

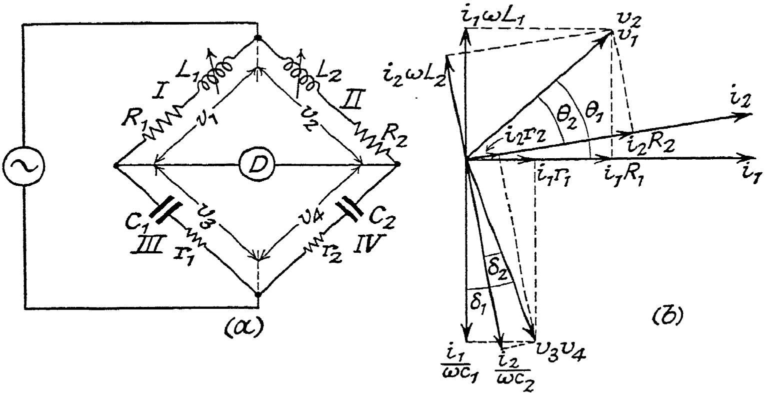


FIG. 128. GROVER'S SERIES INDUCTANCE METHOD

Balance is obtained by variation of the inductances L_1 and L_2, and of the series resistances in these arms if necessary.

At balance $(R_1 + j\omega L_1)i_1 = (R_2 + j\omega L_2)i_2$

and
$$\left(r_1 - \frac{j}{\omega C_1}\right) i_1 = \left(r_2 - \frac{j}{\omega C_2}\right) i_2$$

From which we have
$$\frac{C_1}{C_2} = \frac{R_2}{R_1} + \frac{\omega^2 C_1}{R_1}(L_1 r_2 - L_2 r_1) \quad . \quad . \quad (161)$$

In most cases, when L_1 and L_2 are not large, the second term may be neglected, whence
$$\frac{C_1}{C_2} \doteqdot \frac{R_2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (162)$$

The phase angles δ_1 and δ_2 of the two capacitors may be obtained from the expressions

$$\tan \delta_1 = r_1 \omega C_1$$

and

$$\tan \delta_2 = r_2 \omega C_2$$

Substituting the relationship

$$\frac{C_1}{C_2} = \frac{R_2}{R_1}$$

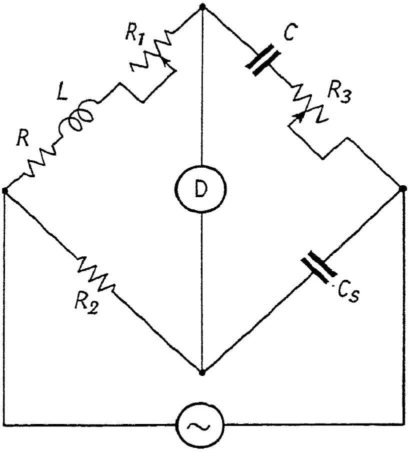


FIG. 129. OWEN'S BRIDGE

in the first of the balance conditions, we have

$$r_2 - \frac{R_2}{R_1} r_1 = \frac{L_2}{R_1 C_1} - \frac{L_1}{C_2 R_1}$$

or

$$r_2 - \frac{C_1}{C_2} r_1 = \frac{L_2}{R_1 C_1} - \frac{L_1}{C_2 R_1}$$

whence

$$\omega C_2 r_2 - \omega C_1 r_1 = \omega \left(\frac{L_2}{R_2} - \frac{L_1}{R_1} \right)$$

or

$$\tan \delta_1 - \tan \delta_2 = \omega \left(\frac{L_1}{R_1} - \frac{L_2}{R_2} \right)$$

$$= \tan \theta_1 - \tan \theta_2 \quad . \quad . \quad . \quad (163)$$

where θ_1 and θ_2 are the phase angles of the inductance arms I and II.

Owen's Bridge. This bridge, the connections for which are given in Fig. 129 is, in fact, a modification of Grover's method just described and provides a method of measuring self-inductance in terms of a standard capacitance.

The inductance under test, $R + j\omega L$, is connected in series with a variable non-inductive resistor R_1. The standard capacitance is C_s. At balance

$$\frac{R + R_1 + j\omega L}{R_2} = \frac{R_3 - \dfrac{j}{\omega C}}{-\dfrac{j}{\omega C_s}}$$

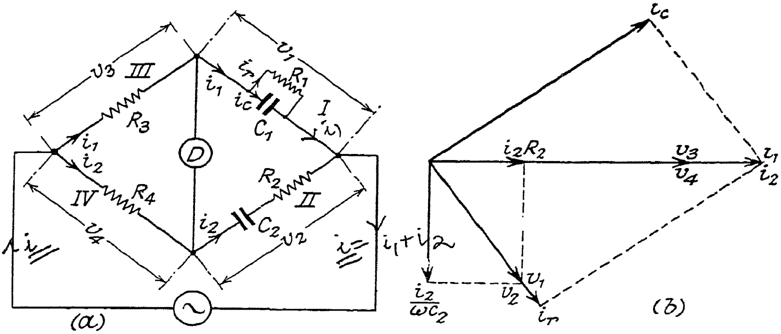


Fig. 130. Wien Bridge

from which $$R = R_2\frac{C_s}{C} -$$

and $$L = R_2R_3C_s$$

This bridge, which can cover a wide range of inductance measurements with limited apparatus, is convenient to use since the balance, which is independent on frequency and wave-form, is obtained by successive adjustment of the two resistors R_1 and R_3.

Method. This method is a convenient one when an imperfect capacitor is shunted by a resistance as is the case in cable testing. In Fig. 130 (*a*), in which the connections of the bridge are shown, C_1 is the equivalent shunt capacitance of the capacitor, and R_1 the shunt resistance. The capacitor C_2 is a standard air capacitor and R_2, R_3, and R_4 are non-inductive resistances. If the unknown capacitor is not already shunted by a resistance, the resistance R_1 is placed in parallel with it. Balance is obtained by variation of the resistances R_2, R_3, and R_4.

At balance

$$i_1\left(\frac{R_1}{1 + j\omega C_1R_1}\right) = i_2\left(R_2 - \frac{j}{\omega C_2}\right)$$

and $$i_1R_3 = i_2R_4$$

Hence, it follows that

$$C_1 = \frac{\frac{R_4}{R_3} \times C_2}{1 + \omega^2 R_2^2 C_2^2} \quad . \quad . \quad . \quad (164)$$

and

$$R_1 = \frac{R_3(1 + \omega^2 R_2^2 C_2^2)}{\omega^2 R_2 R_4 C_2^2} \quad . \quad . \quad (165)$$

The vector diagram for balance conditions is shown in Fig. 130 (*b*).

OTHER BRIDGE METHODS OF MEASURING CAPACITANCE. The *Schering Bridge* method of measuring the capacitance and power factor of capacitors has already been described in Chapter IV.

Some of the methods already described earlier in this chapter for the measurement of self- or mutual-inductance in terms of capacitance form convenient methods of measuring the capacitance of a capacitor in terms of self- or mutual-inductance, if suitable inductance standards are available. Obviously, such methods may be used either way about without modification, the theory of the method remaining the same. Two such methods are *Anderson's Bridge* and the *Carey-Foster Bridge*.*

BRIDGE METHODS FOR SPECIAL PURPOSES. *Measurement of the Self-inductance of Alternating-current Resistance Standards.* It is often important that the self-inductance of heavy-current, low-resistance standards for use in alternating-current measurements should be known. The inductance of such standards is usually very small, and its measurement necessitates special methods. Wattmeter methods, using a reflecting-type wattmeter, have been devised but do not compare favourably with alternating current bridge methods, and the latter are therefore in more general use.

Campbell's Method. The connections of this method (Ref. (13)) are given in Fig. 131. The standard resistor R_s has four terminals, c_1c_2 being its "current terminals," and p_1p_2 its "potential terminals"—hence the name "four-terminal resistors" given to such standards. The self-inductance of R_s is l. M_1, M_2, and M are three variable mutual-inductances, the former two not necessarily being of known values, and the latter being a low-reading standard inductometer. R and L are the resistance and inductance of the circuit containing the secondary winding of M_1 and the primary of M_2. D is the detector. Balance is obtained by variation of the three mutual inductances as required. These should be well spaced in order to avoid mutual inductance effects between them.

Theory. At balance the sum of the voltages in the detector circuit is zero.

Thus,

$$(R_s + j\omega l)i + j\omega Mi + j\omega M_2 i_1 = 0$$

or

$$i[R_s + j\omega(l + M)] + j\omega M_2 i_1 = 0$$

---

* Other methods of measuring both inductance and capacitance are given in Hague's *Alternating Current Bridge Methods* and in the *Dictionary of Applied Physics*, Vol. II, to which works the reader is referred for further information on the subject.

Also in the circuit containing L and R,

$$j\omega M_1 i + i_1(R + j\omega L) = 0$$

Hence

$$\frac{R_s + j\omega(l + M)}{j\omega M_1} = \frac{j\omega M_2}{R + j\omega L}$$

$$RR_s - \omega^2 L(l + M) + j\omega LR_s + j\omega R(l + M) = -\omega^2 M_1 M_2$$

Equating real and imaginary quantities, we have

$$RR_s - \omega^2 L(l + M) = -\omega^2 M_1 M_2$$

$$RR_s = \omega^2[L(l + M) - M_1 M_2]$$

and

$$R(l + M) + LR_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (166)$$

Thus the self-inductance l can be found if L, R_s, R, and M are known, the values of M_1 and M_2 being unnecessary.

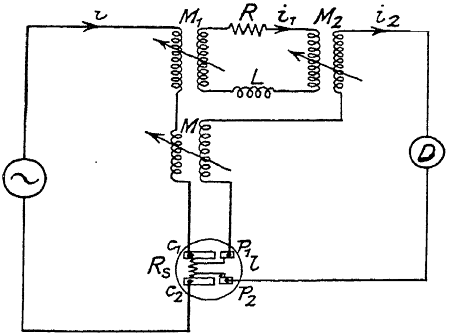


FIG. 131. CAMPBELL'S METHOD

If M_1 and M are reversed the expression for l becomes

$$\frac{M - l}{R_s} = \frac{L}{R}$$

or

$$l = M - L\frac{R_s}{R}$$

It is necessary for balance that M should be greater than l and $M_1 M_2 > \dfrac{L^2 R_s}{R}$

In the same paper Campbell describes the application of this method to the measurement of the capacitance and power factor of capacitors. For this purpose the connections of the network are the same, except that the capacitor replaces the standard resistance R_s. Balance is obtained in the same way, and it can be shown by the method used above that the balance conditions are

$$\frac{1}{C\omega^2} = M + \frac{rL}{R} \quad . \quad . \quad . \quad . \quad . \quad (167)$$

and
$$Rr = \left[M_1M_2 - \frac{L^2r}{R}\right]\omega^2 \quad . \quad . \quad . \quad (168)$$

where C is the capacitance and r the series resistance representing the loss component, of the capacitor under test. The power factor $r\omega C$ can be found from the two equations,

$$r = \frac{RM_1M_2\omega^2}{R^2 + L^2\omega^2} \quad . \quad . \quad . \quad . \quad (169)$$

and
$$\frac{1}{C\omega^2} = M + \frac{LM_1M_2\omega^2}{R^2+L^2\omega^2} \quad . \quad . \quad . \quad (170)$$

which follow from the balance conditions. If the time constant $\frac{L}{R}$ is small

$$C = \frac{1}{\omega^2 M} \quad . \quad . \quad . \quad . \quad . \quad (171)$$

to a very close approximation.

Hartshorn's Method for the Measurement of the Self-inductance of Low-resistance Standards. This method, described by L. Hartshorn (Ref. (18)), is essentially a modification of the Kelvin double bridge (see Chapter VII), and compares the phase angles of two low-resistance standards. To obtain the requisite sensitivity the supply is of telephonic frequency, and telephones are used as the detector. This is justifiable, since the frequency has little effect upon the inductance of such standards. The connections are given in Fig. 132.

S and X are the two low-resistance standards whose phase angles are to be compared, that of S being known and that of X unknown. Let their resistances be R_s and R_x and their self-inductances L_s and L_x. P, Q, p, and q are shielded non-reactive resistors, and r_1 and r_2 are low resistances—most suitably slide-wires—for fine adjustment of the resistances of the arms. C_1 and C_2 are variable air capacitors, shunting Q and q. The leads from the potential terminals of S and X are run close together to avoid inductive loops. R and R' are variable resistances with their connecting point earthed as shown. If the ratio $\frac{R}{R'}$ is made equal to $\frac{S}{X}\left(=\frac{P}{Q}=\frac{p}{q}\right)$ the potential of the detector D is that of earth, although it is not actually earthed. By this means earth-capacitance effects between the telephone detector D and the operator's head are eliminated. This point will be discussed further in connection with the Wagner earth device (see page 231).

Operation. In operation the following procedure is adopted—

1. Adjust r_1 and C_1, with connections as shown, until balance of the bridge—i.e. silence in the telephones—is obtained.

2. Remove the link M, connecting S and X, and obtain balance again by adjusting r_2 and C_2.

3. Replace the link and obtain balance again by adjusting r_1 and C_1.

4. Repeat the above procedure until balance is obtained with the link either in or out. Let the reading of C_1 for this condition be C_1'.

5. Remove the link, transfer the supply to the points T_1T_2, and adjust C_1 until balance is again obtained. Let the setting of C_1 now be C_1''.

Then the expression for the time constant of X is

$$\frac{L_x}{R_x} = \frac{L_s}{R_s} - \frac{1}{2}Q(C' + C'') - \frac{L_P}{P} + \frac{L_Q}{Q} \quad . \quad . \quad (172)$$

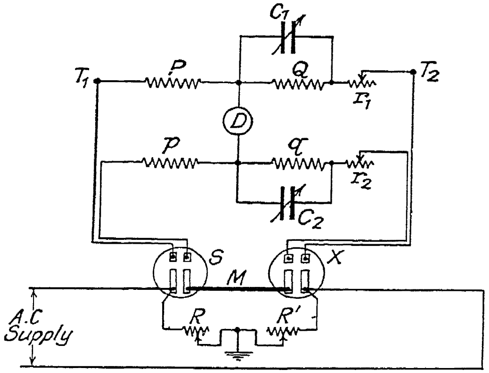


FIG. 132. HARTSHORN'S METHOD FOR THE MEASUREMENT OF SELF-INDUCTANCE OF LOW-RESISTANCE

where L_P and L_Q are the self-inductances, and P and Q the resistances of P and Q.

The original paper should be consulted for a fuller consideration of the method.

Double-ratio Bridges with Inductively-coupled Ratio Arms. The Wagner earth device (see p. 231), which is used to bring the terminals of the detector branch of a bridge to earth potential, has the disadvantage that its branches must be adjusted with every change of earth capacitance, i.e. with every bridge measurement. The two balances—that of the bridge network itself and that of the Wagner earth—are mutually dependent so that a series of alternate balances must be made to arrive at a final simultaneous balance.

As an alternative method of reducing earth capacitance effects A. D. Blumlein (Brit. Pat. No. 323037) proposed replacement of the resistive arms of the bridge network by a pair of tightly-coupled inductors. By this means it is possible to bring the three terminals of these two arms to almost the same potential so that, by connecting the centre terminal to earth, earth capacitance effects at all four corners of the bridge can be eliminated.

A development of this proposal is the use of double-ratio bridges with two sets of inductively-coupled ratio arms. A full description of the construction and application of such bridges has been given by H. A. M. Clark and P. B. Vanderlyn (Ref. (42)).

An interesting application of the double-ratio bridge is in a height indicator for aircraft. W. L. Watton and M. E. Pemberton (Ref. (43)) describe a direct-capacitance altimeter which uses the change in capacitance, with height, between the aircraft and the earth as a measure of low altitudes (up to about 200 feet). The sensitivity of the bridge for this purpose can be made sufficient to detect capacitance changes of 1 μμμF

SOURCES OF ERROR IN BRIDGE MEASUREMENTS, AND PRECAUTI

Although it is best, in considering the sources of error in a.c. bridge measurements, to treat each particular method separately, the space available here does not permit such consideration. The possible sources of error are considered in general below. For more detailed treatment the reader is referred to Dr. B. Hague's excellent book on *Alternating Current Bridge Methods,* to which work the author has referred for some of the information contained in this chapter.

Stray Field Effects. Errors may be caused by the fact that the various arms of the bridge network may be—unintentionally—either magnetically or electrostatically coupled, due to the "stray" magnetic or electrostatic fields existing round apparatus included in the network. When such effects are present the simple theory of the network—considering each arm as being entirely separate from the other arms except where intentionally coupled together—is no longer quite true. Under these conditions the detector may indicate balance—or zero deflection—when balance conditions have not really been obtained.

In networks containing two or more self- or mutual-inductances there may be mutual-inductance between two of them in different bridge arms. Usually, stray magnetic fields will be more important than electrostatic stray fields when inductances and resistances only are present. If the bridge contains capacitors the opposite is the case, errors then being caused by inter-capacitance between the various arms. Loops formed by the leads connecting a piece of apparatus to the bridge may also introduce errors owing to their inductance. In inductance measurements the leads should be twisted together to avoid such loops, while in capacitance measurements the leads should be separated from one another to avoid capacitance between them. As already pointed out, it is possible in some cases to eliminate the effects of the leads by making two measurements on the bridge—one with the apparatus under test in circuit, and one with the piece of apparatus short circuited—or by substituting a variable standard for the unknown and adjusting it to give balance with the same bridge settings as when the unknown was in circuit.

To avoid errors due to magnetic coupling between arms the inductance coils used should be wound astatically—i.e. having no appreciable stray magnetic field—or magnetic screening may be adopted. For such screening a thin sheet of high permeability material is placed so as to prevent the stray magnetic field from reaching the apparatus in the other arms. The inductance coils should, also, be arranged at some distance from one another and from the bridge. Mutual inductance between two pieces of apparatus can often be detected by altering their relative positions and observing if the bridge settings for balance are altered thereby.

In some cases errors are caused by direct induction effects between the supply to the bridge, and the detector circuit, which may cause the detector to indicate the passage of a current through it when the bridge is, in reality, balanced. Such effects may be eliminated by placing the supply alternator at some distance from the bridge and by supplying the bridge through an "inter-bridge transformer." The latter is a transformer having windings which are very well insulated from one another and which have an earthed metallic screen between them. Such transformers should have a closed magnetic circuit to avoid magnetic leakage, the core often consisting of sheet steel ring punchings. Usually there are several windings on the transformer to afford a choice in the working voltage.

When telephones are being used, direct induction between apparatus in the bridge arms and the telephone circuit may produce sound in the telephones when the bridge is balanced. The presence of such direct induction may be detected by moving the head, when the bridge is almost balanced, so as to alter the plane of the telephones relative to the bridge. If no difference in sound is detected when the telephones are thus moved about it may be assumed that such induction effects are negligible. To eliminate such effects, when present, the telephones must be disconnected from the bridge and moved so that silence is obtained when the bridge is supplied with power from the alternator. Adjustment of the bridge to obtain balance must be carried out with the telephones in this position.

Errors due to electrostatic coupling between the various arms of the bridge and to earth capacitances of the various pieces of apparatus may be guarded against by electrostatic screening. The use of such screens renders these capacitance effects definite in magnitude and independent of the distribution of the apparatus forming the bridge network. By this means the effect of such inter- and earth-capacitances upon the accuracy of the bridge may be made very small.

A similar purpose is served by the various earthing devices of which that due to Wagner (Ref. (21)) is commonly used.

Wagner Earthing Device. This device is used in conjunction with most of the bridge networks previously described, and is usually very necessary if accurate results are to be obtained. Fig. 133 shows the connections of the device for use in conjunction with the general form of bridge network. Z_1, Z_2, Z_3, and Z_4 are the impedances of the bridge arms. X and Y are the two variable impedances of the Wagner earth branch, the centre point of which is earthed as shown. These impedances may consist of variable resistances and self-inductances similar to those used in the arms of the bridge proper, but not necessarily of known value. D is the telephone detector.

If the switch S is on contact b, balance of the bridge may be obtained by adjustment of the impedances Z_1, Z_2, etc. At balance, the points a and b are at the same potential, and no current should flow in the detector branch. These two points are not, however, necessarily at earth potential, and it is found that a capacitance current flows from the detector branch to the observer's head through the telephones, thus rendering complete silence in the telephones unattainable, although a point of minimum sound can be obtained.

After adjusting the bridge to give minimum sound, the switch S is thrown on to contact c, so that the telephones are then connected between point a and earth. X and Y are next adjusted until silence is obtained. Under these conditions point a must be at the same potential as earth although it is not permanently earthed.

The telephones are next connected, by the switch, back on to ab, and Z_3 and Z_4 are adjusted to give minimum sound again. The process is continued until silence is obtained with the switch on either of the positions b and c, without further adjustment of the impedances being required when the switch is thrown over from one to the other. Then all three points a, b, and c must be at earth potential. Thus the telephones are at earth potential and the capacitance effect with the observer's head is eliminated. In the double-ratio a.c. bridges mentioned on p. 230 some of the disadvantages of the use of the Wagner earth arrangement are overcome.

Leakage Errors. If the insulation between the various pieces of apparatus forming a bridge network is not good, trouble may arise through leakage currents from one arm to another. This is especially true in the case of high impedance bridges. To avoid this the apparatus used may be mounted on insulating stands.

Eddy Current Errors. Standard resistances and inductances used in bridge networks should be so constructed as to avoid variation of their values due to eddy currents when the frequency is varied. The effective resistance and inductance of a piece of apparatus under test may vary with frequency due to this cause. Large masses of metal in the vicinity of the bridge should be avoided, as the flux produced by eddy currents induced in them may set up troublesome e.m.f.'s in parts of the network. In the case of mutual inductances eddy current effects, if present, cause the induced voltage to lag by some angle less than 90° behind the inducing current, in which case the simple theory of the network does not hold.

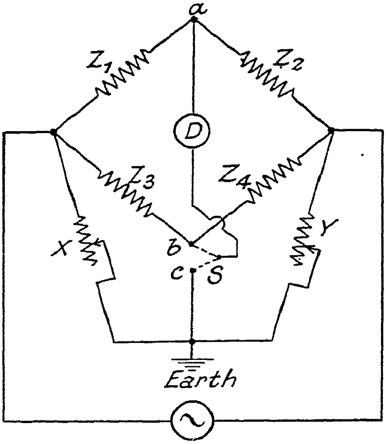


Fig. 133. Connections of Wagner Earth Device

Residual Errors. In speaking of the resistors used in the various bridge networks the description "non-inductive" or "non-reactive" has been applied to them, indicating that their inductance and capacitance are both zero. Although resistances for such purposes are constructed so that these quantities are very small, it cannot always be assumed that they are zero. The term "residual" is used to indicate the small inherent inductance or capacitance of a resistance coil. In precise work it is sometimes necessary to take these residuals into account—for which purpose they must either be measured or calculated—in order that errors due to them shall be avoided. The self-capacitance of coils is usually only important when the coil has many turns and the supply frequency is high. The resistance and inductance of such coils are increased and reduced, respectively, due to this cause by amounts which are proportional to the square of the frequency (Refs. (19), (20)).

Frequency and Wave-form Errors. Some of the bridges previously described are independent of the frequency of the supply in the sense that the balance conditions do not involve the frequency. In such cases, therefore, the frequency of the supply is only important in its effect upon the effective resistance and inductance of the apparatus under test; and the fact that the supply wave-form contains harmonics is only important for the same reason.

In the case of networks in which the balance conditions do involve the frequency, the latter is important and must be carefully measured. The wave-form of the supply is also obviously of importance, since the bridge cannot be balanced both for the fundamental and the harmonics in the wave-form (if any) simultaneously. If telephones are employed in such bridges it will be found impossible to obtain complete silence, only a point of minimum sound being obtainable.

There are two means of circumventing this difficulty. The first is by using some form of "wave filter" such as those described by Campbell (Ref. (14)), and the second is by using a tuned detector, such as a vibration galvanometer, instead of telephones. Such detectors will not respond appreciably to frequencies other than that of the fundamental of the supply.

It is advantageous, if possible, to use in the bridge network such values of the impedances that the frequency term in the expression for the quantity to be measured is reduced to zero.

Used in Conjunction with A.C. Bridge Networks.

1. SOURCES OF CURRENT. These may be conveniently divided into three classes: (*a*) microphone hummers, (*b*) alternators, (*c*) oscillators.

Oscillators are nowadays almost universally used but, for completeness, the other two methods will be described briefly.

(*a*) *Microphone Hummers* provide a means of obtaining a supply of constant frequency, and of reasonably pure sine wave form, by the use of comparatively simple apparatus. The power obtainable from such sources is, however, usually small. Fig. 134 illustrates the principle and connections of such a piece of apparatus due to Campbell.

A steel bar, 2·5 cm. diameter, and of length depending upon the desired frequency $\left(\text{given by length} \doteqdot \frac{1{,}075}{\sqrt{\text{frequency}}}\right)$ is supported at two nodal points by knife edges. This bar carries a microphone M at one end, connected to one terminal of a battery as shown. T is a transformer having three windings, an earthed screen being placed between the winding supplying the bridge and the other two windings. C_1 and C_2 are capacitors. An electromagnet, energized by two coils as shown, is placed under the centre of the bar. Coil A is for the purpose of polarizing this magnet and carries the microphone current, while coil B maintains the bar in vibration when once such vibration is started.

The frequency obtained by this means is very constant, and the wave-form is good, the disadvantage being the small amount of power available. H. W. Sullivan Ltd. make a reed hummer requiring an input of 100 mA at 6 to 8 volts.

Details of a number of other microphone hummers and of interrupters, etc., are given by Campbell (Ref. (3)) and by Hague (Ref. (1)).

(*b*) *Alternators* of the inductor type have the advantage of a comparatively large output, although the frequency obtainable is not so constant, nor, in general, is the wave-form so good as that obtained by other methods of supply. For measurements at commercial frequencies a motor-alternator set of the ordinary type may be used. For higher frequencies (of the order of 500 to 2,000 cycles per second) an alternator such as that due to Duddell (Ref. (24)) may be used. Different frequencies may be obtained by varying the speed of the driving motor, but constancy of frequency may require the use of an automatic speed regulator.

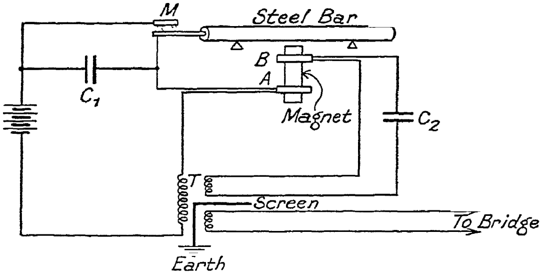


FIG. 134. CAMPBELL MICROPHONE HUMMER

The most suitable type of alternator, giving a comparatively large output and good wave-form (slight 3rd harmonic), is the Duddell alternator referred to above. This machine has a rotating field system and a stator which is in the form of a smooth ring—unslotted—carrying a Gramme ring winding to avoid tooth ripples. The rotor is a steel disc of 20 cm. diameter, with 30 projecting pole pieces each carrying a magnetizing coil, these coils being held in position by wedges in the slots separating the poles, and being supplied with direct current from a battery. The alternator is driven by a motor and a link belt. Frequencies up to 2,000 cycles per second are obtainable by variation of the motor speed, the alternator speed for this frequency being 8,000 r.p.m. A motor running at 2,000 r.p.m. may be used, the required alternator speed being obtained by gearing by means of the pulleys upon which the link belt runs.

The speed of the motor may be controlled by connecting a low fixed resistance in the armature circuit, for starting purposes, and a diverter resistance in parallel with the armature itself for speed control purposes. This method, especially at the lower speeds, gives much greater stability of speed with variation of load than the more usual variable armature-series-resistance method.

As stability of speed (and therefore frequency) is of great importance, it is often a useful precaution to include in series with the bridge network, a variable sistance and a low reading ammeter, so that the load resistance may be maintained constant by adjustment of this variable resistance whenever the impedance of the bridge network is altered. By this means the load on the driving motor is maintained constant and the speed remains steady.

The Duddell alternator as described above has an output varying from a few watts at 100 cycles to about half a kilowatt at 2,000 cycles. In a test upon a machine of this type, the following results were obtained—

Alternator field d.c. volts .	100
Alternator armature voltage	from 3 volts at 100 cycles up to 55 volts at 1,000 cycles
Alternator output . .	from 5 watts at 100 cycles to 400 watts at 1,200 cycles

Bridge Current. In connection with the supply of current to a bridge network it must be pointed out that the current-carrying capacity of the impedances forming the arms of the network should be carefully considered. It may happen that the total impedance of the two arms forming one "side" of the network (such as arms Z_1 and Z_2 in Fig. 133) is low compared with the impedances of the other two "sides." In such a case the low-impedance side may take an excessive current, with consequent damage to the apparatus in these arms. A rough rule in connection with current-carrying capacity in resistance boxes is to allow from $\frac{1}{2}$ to 1 watt per coil in circuit in the box. For example, in the case of a 10 ohm coil, allowing $\frac{1}{2}$ watt per coil, the current-carrying capacity is $\sqrt{\frac{0\cdot5}{10}} = 0\cdot22$ amp., since $I^2R = 0\cdot5$.

From this point of view it may be best to make up any required resistance by using as many coils as possible to obtain a greater radiation surface; thus 10 ohms may be made up of 5 ohms, 2 ohms, 2 ohms, and 1 ohm, instead of using one 10 ohm coil.

(c) *Oscillators.* The advantages of this method of supply are that the frequency is absolutely constant and determinable with great accuracy. The power available is sufficient for most bridge measurements, and the wave form is very close to a pure sine wave. For these reasons this method of supply has very largely displaced other methods.

Such oscillators depend upon the fact that a circuit containing inductance L and capacitance C has a natural frequency of oscillation given by $f = \frac{1}{2\pi\sqrt{LC}}$. If such a circuit has an e.m.f. induced in it and is then left to oscillate, the frequency of oscillation of the current in the circuit will be given by the above expression. The natural dying away of these oscillations, due to resistance in the circuit, is prevented by supplying energy from another circuit containing (say) a three electrode valve and hence oscillations of this frequency are continuously produced.

Fig. 135 gives the connections of a triode valve oscillator, as commonly used for purposes of bridge supply. B_1 is a battery of about 200 volts to obtain the required anode potential in the valve

V. B_2 is a 6 volt battery, with regulating resistance R, for the purpose of heating the filament of the valve. L_1 and L_2 are coils inductively coupled with coil L_3, the latter being part of a tuned circuit of which the other part is a variable capacitor C. The tuned circuit is connected in the valve anode circuit and coil L_1 in the grid circuit, as shown, while coil L_2 supplies the bridge network. The tuning is carried out by variation of the capacitance C by means of which any desired frequency within the range of the apparatus may be obtained.

In operation, when the circuit is first closed a current is produced in the anode circuit, and oscillations are set up in the tuned circuit.

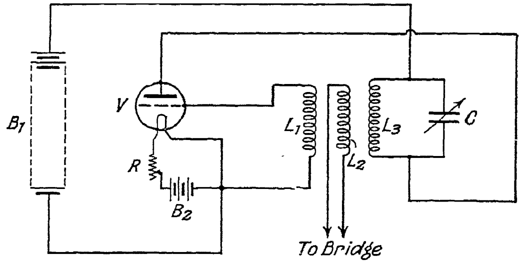


FIG. 135. CONNECTIONS OF TRIODE VALVE OSCILLATOR

Owing to the inductive coupling between coils L_3 and L_1 these oscillations cause variations of grid potential which produce currents in the anode circuit of the valve and so the oscillations are maintained, the energy required for the supply of the losses being obtained from the batteries.

Any frequency up to the extreme limit of audibility can be obtained by suitably choosing the values of the inductance L_3 and the capacitor C, e.g. if L_3 is $\frac{1}{36}$ henry and C is 1 microfarad,

$$f = \frac{1}{2\pi\sqrt{\frac{1}{36} \times \frac{1}{10^6}}} = \frac{3{,}000}{\pi}$$

$$= 955 \text{ cycles per second}$$

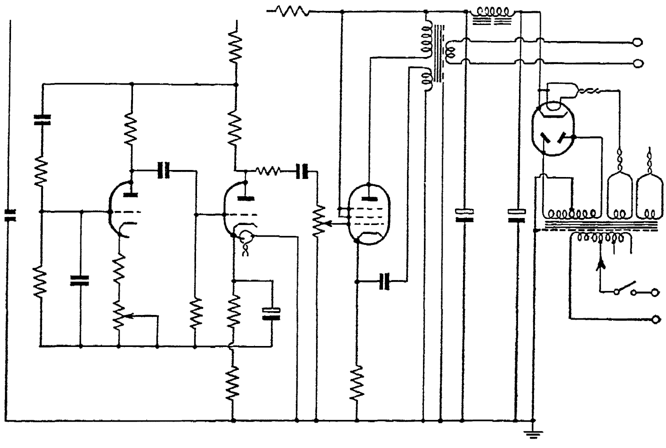
Fig. 136 gives the connections of a valve oscillator which has been developed by the Cambridge Instrument Company. In this generator the oscillation frequency is determined by the characteristics of a Wien bridge network. The output from a 2-stage amplifier, having negative feed-back, is connected back to the input circuit through the Wien bridge so that a suitable amount of positive feed-back is obtained at the characteristic frequency of the bridge.

The amplifier output is also taken to the output valve via a suitable network and variable attenuator. In order to maintain a good wave-form produced by the oscillator, considerable negative feed-back is also applied over the output stage.

The oscillator can be constructed for any frequency from a few cycles per second up to 100 kc/sec. or more. Outputs of up to 2 watts with practical sinusoidal wave-form can be obtained.

A multi-frequency bridge oscillator made by H. W. Sullivan, Ltd. gives a frequency range of 300 to 10,000 c/s or 1,000 to 30,000 c/s each to within ± 1 per cent.

An amplitude control valve ensures constant output and purity of wave-form, the output being up to 4 or 5 watts.



(*Cambridge Instrument Co. Ltd.*)

FIG. 136. MAINS FIXED FREQUENCY OSCILLATOR CIRCUIT DIAGRAM

An audio signal generator made by Advance Components, Ltd. covers the range 15 c/s to 50,000 c/s. The bridge-type resistance-capacitance oscillator used incorporates a stabilizing circuit and is followed by two stages of amplification with heavy negative feed-backs so that there is stability of frequency and negligible distortion. Either sine-wave or square-wave output can be used, the former being variable from 200 μV to 20 V and the latter from 400 μV to 40 V.

Some of the H. W. Sullivan valve oscillators covering frequency ranges of 50–170,000 c/s are precise enough to be used as frequency standards of 0·1 per cent accuracy. They have a frequency stability of a few parts in a million during a period of measurement. Although for ordinary bridge measurements of inductance and capacitance a fixed-frequency simple valve oscillator of, say, 1,000 c/s and output of about 1 watt is adequate, for more specialized bridge measurements, oscillators of the heterodyne or resistance-capacitance type, continuously variable from "supply" frequencies through the telephonic range and even including the "carrier" frequency range are preferable. These have output powers up to 5 watts. Although on occasions such high power is necessary for supply to a bridge, it is usually better practice, wherever possible, to limit the power supplied to a bridge and employ a valve detector-amplifier of an aperiodic type in which balance detection is sensed both aurally by telephones and visually by a pointer galvanometer having a logarithmic deflection (to avoid damage which may be caused by a greatly unbalanced bridge).

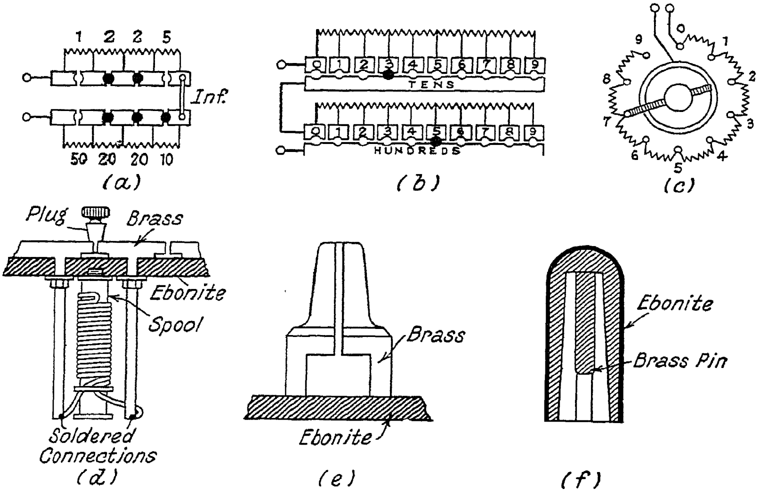


FIG. 137. CONSTRUCTION OF VARIOUS TYPES OF VARIABLE RESISTANCES

2. VARIABLE RESISTANCES, INDUCTANCES, AND CAPACITORS. Fixed standards of resistance, inductance, and capacitance have been described in Chapter II.

Variable Resistances. These are usually in the form of resistance boxes. Fig. 137 illustrates the various methods of arranging and mounting the coils in such boxes. Fig. 137 (*a*) shows the simplest arrangement, in which the removal of a plug puts a coil in circuit. The reading, as shown, is 56 ohms. An open-circuiting link or plug is usually provided. Fig. 137 (*b*) shows the "straight-decade" arrangement. One plug only is necessary for each decade, the reading with plugs as shown being 530 ohms. The dial arrangement, the value of the resistance in which is varied by the rotation of a laminated copper brush or arm, is shown in Fig. 137 (*c*). Fig. 137 (*d*) illustrates a common method of mounting the resistance coils the coil being non-inductively wound on a brass spool. These brass tubes have usually a double layer of silk ribbon wound on them, and are coated with shellac varnish and baked to remove moisture. Manganin wire is used in all high-grade resistance coils, the coils being impregnated with shellac after winding, and being annealed by baking for 10 hours at 140° C. After annealing, the coils are usually boiled in paraffin wax to prevent the absorption of moisture. The ends of the coil are soldered to the terminal blocks with silver solder. Two brass rods, connected to the brass contacts on the top of the box, serve as terminals for the coil. The Leeds and Northrup Co. use an improved method, which consists of attaching the coil spool to the contact blocks and soldering one end of the coil to the spool, the other end being soldered to the spool of an adjacent coil.

It is of great importance in accurate work that the contact resistance in such resistance boxes shall be small and constant in value. The various manufacturing firms have developed different methods of attaining this, one such method being shown in Figs. 137(*e*) and 137(*f*). These show the special type of plug developed by Messrs. Gambrell Bros. This takes the form of a hollow brass cylinder with brass centre pin, the outer surface being coated with ebonite, milled so that a good grip on the plug may be obtained. This plug fits on to two contact blocks, of the shape shown on the left in the figure. They are slightly conical and have a centre hole to take the plug pin. When the plug is inserted the conical shape of the contact blocks serves to clamp it, and a very good contact is obtained. This type of plug has the advantage that it is independent of the other contacts on the lid of the box, which is not the case with the form of contact shown in Fig. 137 (*a*), where the insertion of a plug in a hole adjacent to a plug already inserted tends to tighten the original plug in its hole.

The sliding "Dual" contact shown in Fig. 138 is used by Messrs. H. Tinsley & Co. for dial pattern resistance boxes and bridges. The contact blocks are cut away as shown, a multiple-leaf brush being fitted to make contact on both the top and bottom surfaces of the slots cut in these blocks. This method gives a very good contact on all studs, and is capable of giving a degree of precision in measurement which would normally be expected only with a plug type of contact.

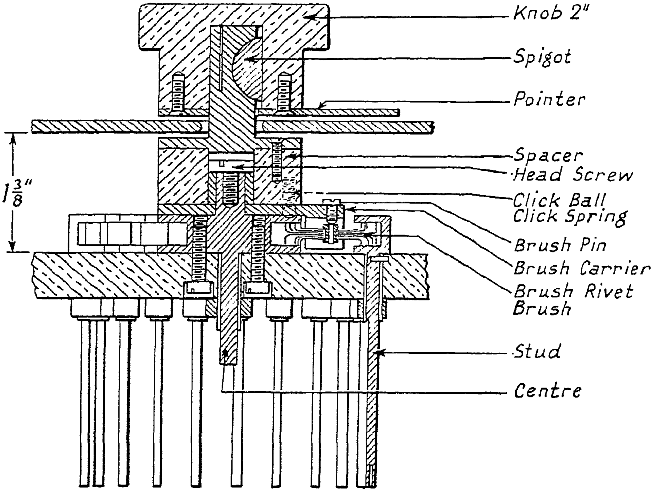
Fig. 139 shows the construction of a dial-pattern resistance box by the Cambridge Instrument Co. for use in a.c. Bridge measurements.

H. W. Sullivan a.c. Decade Resistance Standards have 0·01 per cent direct-reading accuracy and permanence of frequencies up to

50 kc/s. For bridge work, the lower decades are of constant inductance to 0·01 μH, and the time constants of the various decades are—

100 ohms	2.10^{-9}
1,000 ohms	2.10^{-9}
10,000 ohms	5.10^{-9}

A patented pre-set switching device due to W. H. F. Griffiths is employed to minimize the residual inductance and the high frequency resistance error; the screen capacitance is compensated automatically and so the effective residual inductance is rendered sensibly independent of screen connection.



(*H. Tinsley & Co. Ltd.*)

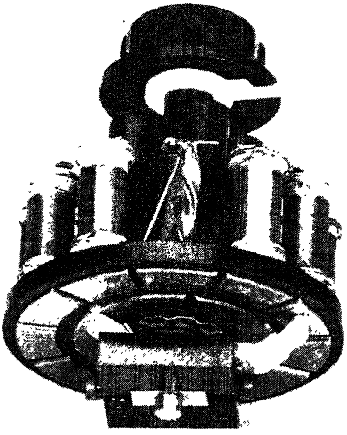
FIG. 138

When plugs are used, as in the first two methods shown in Fig. 137, care should be taken to ensure that these are firmly pressed home before measurements are made, as appreciable contact resistance may result if any of the plugs are loosely inserted. Such plugs should be inserted and withdrawn with a rotational, or screwing, motion, and not simply pushed in, or pulled out, directly. The former method of insertion ensures good contact and prevents the plug from fastening tight in the contact hole, while, if the plugs are pushed directly in, considerable difficulty may be experienced in removing them and the contact blocks may thus become loosened.

Obviously the fewer the number of plugs used, the better, and for this reason the arrangement shown in Fig. 137 (*b*) is an improvement upon that of Fig. 137 (*a*).

Resistance boxes are made up in a large variety of ranges, varying from a total of 11 ohms variable in steps of 0·1 ohm up to a total of 1 megohm. The arrangement of the coil resistance values shown in Fig. 137 (*a*)—5, 2, 2, 1; 50, 20, 20, 10, etc.—is most usual.

The adjustment of the resistance values of the coils varies according to the type of box considered. For ordinary purposes, these values may be adjusted to 1 part in 1,000, while in the case of boxes for precision purposes, the adjustment is usually to 1 part in 10,000.



(*Cambridge Instrument Co. Ltd.*)
FIG. 139

It is essential that resistance coils for use in a.c. bridge measurements shall have very small residual inductance and capacitance. Special methods of winding are necessary to fulfil this condition. To obtain a very small inductance the coil is wound so that adjacent parts of it carry currents in opposite directions. In this way the magnetic field of the coil is kept very small. The self capacitance of the coil is kept small by subdividing it so that adjacent parts have a very small capacitance and also have only a small potential difference between them.

Since the self-capacitance of coils wound on bobbins is usually quite small, the reduction of their inductance is often the most important question. The winding of alternate turns of the coil in reverse directions is one method used for this purpose. This method is illustrated in Fig. 140 (*a*), the arrangement shown being due to Grover and Curtis (Ref. (17)). The wire is wound on a cylindrical former having an axial slit along the greater part of its length. The wire passes through this slit once in every turn, so as to give reversal of winding direction. The arrow heads show the directions of current in various parts of the winding. Obviously the magnetic effects of adjacent turns neutralize one another. This type of coil has a very small inductance, but is somewhat difficult to wind.

Fig. 140 (*b*) shows the Chaperon (Ref. (22)) method of winding. This winding is really an extension of the bifilar principle, the currents in adjacent wires neutralizing one another as regards resultant magnetic field, as shown. Both the inductance and capacitance of coils wound in this way are small.

Modifications of this method have been used in which the winding is divided into a number of sections connected in series.

Campbell Constant-inductance Rheostat. There are several forms of low-resistance rheostats designed to have a very low and calculable inductance. These are usually constructed on the bifilar principle, and are often of the slide-wire form. Fig. 141 (*a*) illustrates

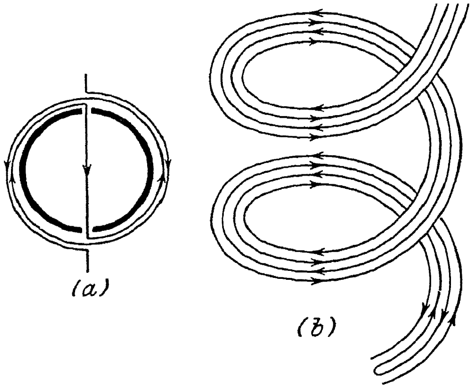


FIG. 140. NON-INDUCTIVE WINDINGS

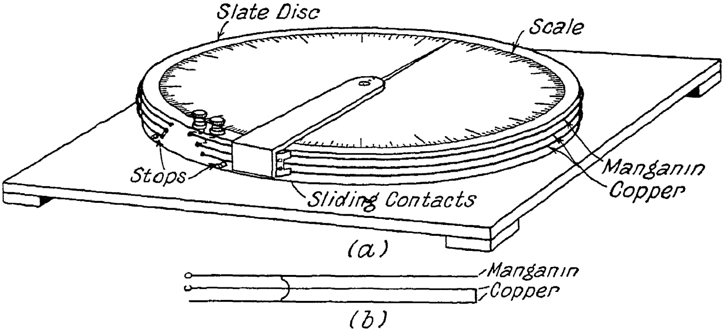


FIG. 141. CAMPBELL CONSTANT-INDUCTANCE RHEOSTAT

the construction of Campbell's constant-inductance rheostat. This is a very useful piece of apparatus for use in certain bridge measurements, since it enables a fine adjustment of the resistance of a bridge arm to be made without altering the inductance settings.

The resistance is increased by moving the sliding contact to the right (Fig. 141 *b*). Such a movement increases the length of manganin wire in circuit and reduces the length of copper wire in circuit. Whatever the position of the slider, the total length of wire in circuit is always the same, and forms a bifilar loop, thus maintaining the inductance small and constant.

A precision slide wire of this type, made by Muirhead and Co., Ltd., may have any resistance from 0·5 to 10 Ω. One made by the Cambridge Instrument Co., Ltd. has a total resistance of 0·12 Ω with a residual inductance of 0·2 μH.

A precision decade non-reactive slide resistance designed on the Kelvin-Varley slide principle (see p. 315) is made by H. W. Sullivan, Ltd. It has manganin coils. One type, having a total resistance of 1,000 ohms, has each of its five dials brought out to separate terminals so that it can also be used as a slide resistance of 200, 40, 8 or 1·6 ohms.

Variable Inductances. Such pieces of apparatus should have as high an inductance as possible compared with their resistance, i.e. their time-constant should be great. Their inductance should be continuously variable and should cover as great a range as possible between maximum and minimum settings. In addition, it is highly desirable that the variation of inductance with position of the moving part should obey a straight line law, and also that the coils shall be astatically wound. The inductance for a given position should not, of course, vary with time, and variation of frequency should not cause appreciable variation of inductance.

Most variable-inductances are so constructed that they can be used as either self- or mutual-inductances. When used as self-inductances the fixed and moving coils are connected in series and the inductance is given by

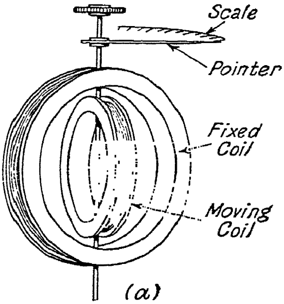


FIG. 142. AYRTON-PERRY INDUCTOMETER

$$L = L_1 + L_2 \pm 2M$$

where L_1 and L_2 are the self-inductances of the fixed and moving coils respectively, and M is the mutual-inductance (variable) between them.

In order to eliminate frequency errors the coils are usually wound with stranded wire and the use of metal parts in the construction is avoided as far as possible.

Ayrton-Perry Inductometer. Fig. 142 shows, diagrammatically, the construction of a simple form of variable inductance (self or mutual) due to Ayrton and Perry. The moving coil is mounted inside the fixed coil and is carried by a spindle which also carries a pointer and handle at the top as shown. Movements of the pointer indicate the variation in the angle between the planes of the coils, but the scale may be graduated to read the inductance directly.

When constructed for use in accurate measurements, the coils are wound on mahogany formers whose surfaces are spherical and great care is necessary in fixing the coils so as to ensure constancy of inductance with time.

This form of inductometer can be cheaply and easily constructed for use as a variable self-inductance in cases where the inductance must be variable but not necessarily known, e.g. for use in one arm of the Wagner earth device.

Its disadvantages are that the instrument produces an external magnetic field, which may be troublesome if the inductometer is placed near to the bridge network, and that the scale is not linear.

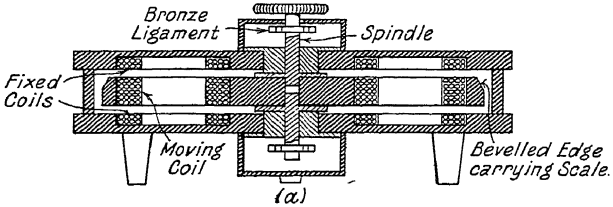


FIG. 143. CONSTRUCTION OF BROOKS AND WEAVER INDUCTOMETER

Brooks and Weaver Inductometer. This form of inductometer (Ref. (23)) is one of the best forms for general purposes. The coils are wound and connected astatically, the time-constant is high, and the scale is uniform throughout the greater part of the range. It is also fairly easily constructed, and its calibration remains reasonably constant with time even when in continuous use. The current carrying capacity, also, is high for this type of apparatus.

The construction of the inductor is shown in Fig. 143. There are, in all, six link-shaped coils—four fixed and two moving. These are wound with stranded wire. The moving coils have twice as many turns as the fixed. These coils are embedded in ebonite or Bakelite discs which are about 15 in. diameter. Bakelite has the advantage that it has less tendency to warp than ebonite. The top and bottom discs, which are fixed, are about $\frac{1}{2}$ in. thick, and are separated by ebonite or Bakelite pillars. The centre disc is thicker—to carry the larger coils—and is of slightly smaller diameter. It has a bevelled edge, upon which a scale is marked out over 180° of its circumference, this scale being used in conjunction with an index mark on the lower fixed disc. Connections to the moving coils are made through copper or phosphor-bronze ligaments soldered to the two halves of the spindle.

The dimensions of the coils are specially chosen to give a uniform scale, and also to obtain as great an inductance as possible for a given length of wire. The relationships between the various dimensions are given in Fig. 144, in terms of the mean radius R of the semicircular ends of the coils. The depth of the moving coils should be the same as their width of winding, i.e. 0·78R, and the depth of the fixed coils 0·39R.

A great advantage of this method of construction is that small variations of the length of gap between (say) the moving disc and the upper fixed disc, due to warping of the former or to wear of the bearings, have no appreciable effect upon the inductance of the instrument, since movement away from the upper fixed disc means movement towards the lower one, thus maintaining the inductance the same within narrow limits.

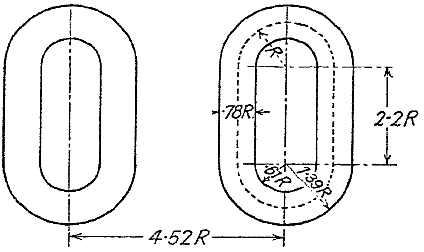


FIG. 144. RELATIVE DIMENSIONS OF BROOKS AND WEAVER INDUCTOMETER

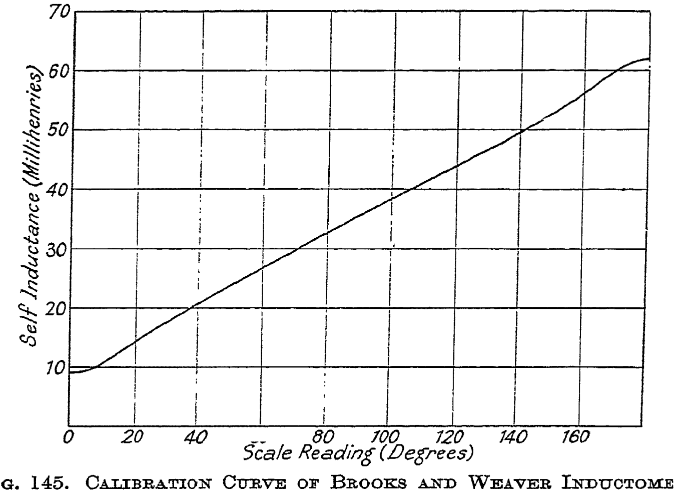


FIG. 145. CALIBRATION CURVE OF BROOKS AND WEAVER INDUCTOMETER

Fig. 145 shows the calibration curve for an inductometer of this type when used as a self-inductance (all six coils in series). The instrument had 144 turns on each of its fixed coils and 288 turns on each of its moving coils, the total resistance being 17·5 ohms at 20° C.

Campbell and Butterworth-Tinsley Mutual Inductometers. Both of these instruments are, essentially, variable mutual inductances, such instruments having the advantage that their inductance can be reduced to zero or given negative values, while a variable self-inductance can only be reduced to some minimum value depending upon the self-inductance of the coils and by the mutual-inductance between them.

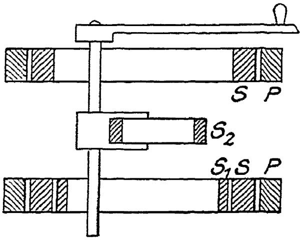


FIG. 146. CONSTRUCTION OF CAMPBELL MUTUAL INDUCTOMETER MANUFACTURED BY THE CAMBRIDGE INSTRUMENT CO.

The Campbell instrument, devised by A. Campbell (Ref. (10)) and manufactured by the Cambridge Instrument Co., has an arrangement of coils as shown in Fig. 146. *PP* are two equal coaxial fixed coils forming the primary winding. These are connected in series. The two coils *SS*, connected in series, form together one of the fixed secondary windings. S_1 is another secondary coil, also fixed, while S_2 is a movable secondary coil. The three secondary windings are connected in series. A link is provided for the purpose of reversing the connections to the moving coil S_2. Coils *S* and S_1 are each divided into ten sections of equal mutual inductance with the primary, and connections are taken from these sections to two dial switches. The mutual inductance of each of the sections of S_1 is, in one form of the instrument, 100 microhenries, and of coil *S* 1,000 microhenries, giving a total for the two coils of 11,000 microhenries. Fine adjustment is obtained by rotation of the secondary moving coil S_2, which is mounted midway between, and parallel to, the fixed coils as shown. An index mark on the handle arm serves the purpose of a pointer, readings being observed on a scale fixed under this arm on the lid of the instrument.

The coils are wound with stranded wire, and marble is used in the best instruments for the coil bobbins, and as the framework to which the coils are attached. These instruments have the advantage of high accuracy and simplicity, but possess considerable capacitance, which introduces errors at the higher frequencies.

Butterworth's mutual inductometer (Ref. (7)), manufactured by Messrs. H. Tinsley & Co., is designed so as to eliminate the defect of the Campbell and similar instruments, due to inter-capacitance between the windings. The makers claim that with this type of instrument a correction of only 0·07 per cent is necessary in the case of an instrument calibrated at 50 cycles and used at 1,000 cycles.

This instrument has a fixed primary coil and two sets of three secondary coils, also fixed. There is also a moving secondary coil for fine adjustment. Each set of fixed secondary coils consists of three coils having mutual inductances with the primary in the ratio of 6 : 3 : 1. Connections are made from each set to a commutator which is manipulated as a dial switch. The two dials are marked 1 to 10, the various inductances being obtained by connecting various combinations of the three coils, through the commutator, in series. In some cases one of the coils is reversed to give the required inductance. For example, 8 is obtained by the commutator connecting the 6 and 3 coils in series so that their magnetic effects are cumulative, and the 1 coil is reversed, thus giving the value $8 = 6 + 3 - 1$.

In a common form of the instrument one dial gives 10 millihenries in steps of 1, while the other dial has a total to 1 millihenry in steps of 0·1, the moving coil giving from $-$ 0·01 to $+$ 0·11 millihenry. The readings in the latter case are observed on a scale placed under the handle arm, as in the Campbell instrument. In this case the total range of the instrument is 11·11 millihenries.

In the Sullivan-Griffiths variable standard of self or mutual inductance the formers of both rotor and stator are constructed to have temperature compensation and so to give a high degree of stability to the windings. There are two ranges of self inductance and one range of mutual inductance, all three direct-reading in inductance, from a single calibration in terms of self inductance. Accuracies as high as 0·02 per cent are possible up to frequencies of a few kc/s but the instrument may be used at frequencies up to 50 kc/s by applying corrections reaching a maximum of 0·3 per cent at this frequency.

The Sullivan-Griffiths decade standards have a number of temperature-compensated standards, each having ten tappings taken to a rotary switch and thus providing a decade of inductance. The coils are all arranged geometrically at mutually zero magnetic coupling. In the standard having a maximum inductance of 1 henry, the finest subdivision on a sixth (continuously variable) dial is 0·05 μH. The decades are adjusted to be direct-reading to an accuracy of 0·03 per cent throughout their entire range and are provided with a calibration of 0·01 per cent.

Variable Capacitors. Variable capacitors may take the form of a subdivided fixed capacitor, various fractions of which can be obtained by movement of either a dial switch or by plugs. If continuous variation of the capacitance is required, as is often the case in a.c. bridge work, a variable air capacitor of the parallel plate type is used. In some cases a combination of both of the above types may be most useful.

Continuously variable capacitors, having air as dielectric, consist of two sets of plates, usually semicircular—one set fixed and the other moving—arranged so that the moving plates can be rotated in the air gap between the fixed plates as shown in Figs. 147 and 148. The capacitance is varied by varying the area of the moving plates interleaving with the fixed plates. The plates, which are usually of aluminium or brass, are proportioned so that the capacitance varies in almost exact proportion to the angle turned through. The plates should be made fairly thick, so as to avoid bending, which would alter the calibration, and all corners should be carefully rounded. The bearings must be well fitted, so that the axial distance between the plates shall be definite and constant. The variable air capacitor shown in Fig. 148 is of the precision type for use as a laboratory standard or for a.c. bridge measurements. It has a slow-motion, worm-gear device permitting high accuracy in setting and reading. Its temperature coefficient is 30–40 parts in 10^6 per degree centigrade, the power factor being less than 0·0001 at 1,000 c/s, and residual self inductance between 0·04 and 0·07 μH, for all settings.

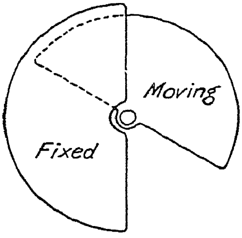


FIG. 147

Square Law Capacitors. Duddell (Ref. (26)) constructed a variable capacitor with the plates shaped so as to give a square law of capacitance variation. Such capacitors are of use in wavemeters for radio work where the wavelength is approximately proportional to the square root of the capacitance of the variable capacitor.

The plates were shaped as shown in Fig. 149. R is the inner radius of the fixed plates; r is a radius of the moving plates. The law of the curve bounding the moving plates is

$$- R^2 \quad . \quad . \quad . \quad . \quad (173)$$

K being a constant such that the interleaving area of the plates —shown shaded in the figure—is equal to $K\theta^2$. θ is the angle turned through by the moving plates from their zero position. Then, since the shaded area is proportional to θ^2 it follows that the capacitance also is very nearly proportional to θ^2. W. H. F. Griffiths* has investigated the laws of variable air capacitors with several different designs of plates. One of these, the Sullivan-Griffiths logarithmic variable air capacitor, covers a wide range of capacitance, the logarithmic scale law giving the same accuracy of scale reading throughout the range for which the law holds. This is particularly important for very low values of capacitance.

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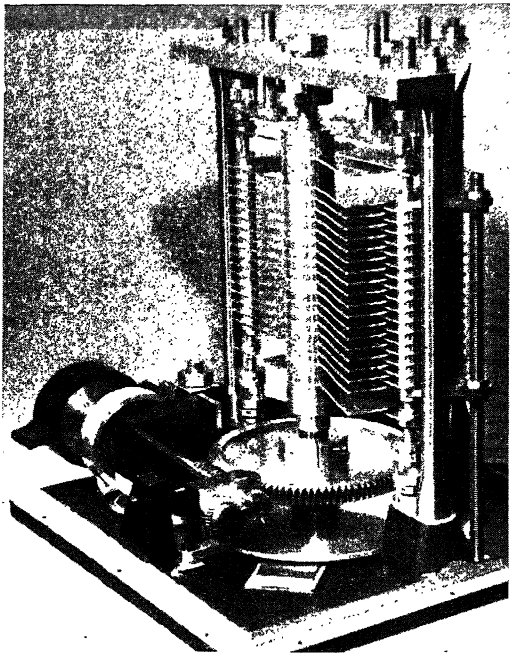
* *Experimental Wireless and The Wireless Engineer*, Vol. III, No. 28, January, 1926. and Vol. III, No. 39, December, 1926. See also Refs. (44), (49).

The Sullivan precision variable air capacitor standard has fused silica insulation ensuring a high degree of permanence in the calibration. It has very low power losses, a scale accuracy of 1 part in 20,000 (or better) and a temperature coefficient less than 10 parts in 10^6 per degree centigrade.



(*Muirhead & Co., Ltd.*)

FIG. 148. VARIABLE AIR CAPACITOR

The decade air capacitor by the same maker consists of a number of variable capacitors. The sectors are introduced in such a manner that small angular movements, due to a possible uncertainty of "click" positioning, produce only very small edge capacitance changes on one sector only.

Self-contained Bridge Networks and Meters. It is of great convenience in a.c. bridge work to have some form of permanently connected bridge. Apart from the saving of time and labour in connecting up the bridge network, such a piece of apparatus, if properly designed, minimizes errors due to inductance and capacitance in the leads, and to leakage effects. A measurement can also be repeated if necessary, with the assurance that the distribution of the bridge will be the same as on that employed in the previous measurement.

A number of self-contained a.c. bridges for the measurement of inductance and capacitance have been developed by various manufacturing firms. An interesting example is the "Mufer" capacitance bridge made by the Baldwin Instrument Company. This employs the De Sauty Circuit (see page 221) the resistance arms R_1 and R_2 being in the form of a potential divider the setting of which, for balance, indicates the value of the capacitance under test by means of an attached pointer and scale. Two standard capacitors are incorporated to give two scales (0·00005 to 0·016 microfarad and 0·015 to 4 microfarads) and the instrument also contains its own oscillator of the neon tube type, the only auxiliary apparatus required being a 120 volt dry battery and telephones. The makers claim an accuracy of within 2 per cent. While this may not be regarded as a precision instrument it has the advantages of portability, and great simplicity in use.

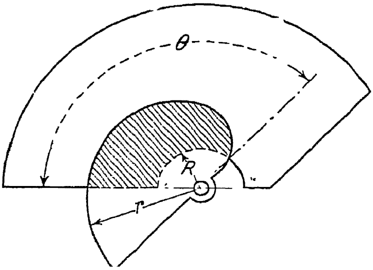


FIG. 149. DUDDELL SQUARE LAW CONDENSER

A Muirhead and Co. impedance bridge can be used for the measurement of resistance, inductance, capacitance, dissipation factor and Q-factor. It is completely self-contained, including a fixed frequency oscillator and an amplifier preceding the telephones used as detector; operation is entirely from the a.c. supply mains. Ranges covered are

Resistance, 0·001 ohm to 1 megohm (accuracy ± 1 per cent).

Inductance, 1 μH to 1,000 H (accuracy ± 1 per cent).

Capacitance, 1 $\mu\mu$F to 100 μF (accuracy ± 1 per cent).

Dissipation factor, 0 to 1·2 (accuracy ± 15 per cent or ± 0·005 whichever is the greater).

Q factor, 0 to 60 on two ranges (accuracy ± 15 per cent).

Messrs. Evershed and Vignoles make a capacitance meter based, essentially, on the well-known "Megger" Insulation Tester (see

p. 305). The capacitance meter, however, has an electro-magnet, instead of a permanent magnet, and an a.c. generator. The armature has a constant-speed centrifugal clutch and there are two independent windings displaced in phase relationship, the displacement being such that the current in the electro-magnet lags 90° behind the voltage across the measuring circuit. Thus, with capacitors present, the currents through the control and deflecting coils are 180° in phase ahead of the field and, by an appropriate arrangement of the connections, they act as if in phase. If the capacitor has imperfect insulation this gives rise to a current component in quadrature with the flux in the electro-magnet and hence this current develops no torque.

Another useful instrument is the Avo universal bridge (The Automatic Coil Winder and Electrical Equipment Co., Ltd.). This is a 50-cycle bridge, self contained, and having 20 calibrated ranges. It covers resistance measurements, in 8 ranges, from $0{\cdot}5\ \Omega$ to $50\ \mathrm{M}\Omega$, inductance from 50 mH to 500 H in four ranges and capacitance from 5 $\mu\mu$F to 50 μF in 8 ranges. The accuracy at mid scale is of the general order of ± 2 per cent.

The Cintel wide-range capacitance bridge (Cinema-Television, Ltd.) is a self-contained equipment including its own oscillator (giving a fixed frequency of 1,592 c/s) and detector. It covers capacitances from 0·002 $\mu\mu$F to 100 μF in 18 steps and the accuracy is ± 1 per cent of full scale on all ranges. It can also be used for high resistance measurements up to 30,000 megohms. The balance indication for the bridge is a dual Electron Ray tube and a fine balance is obtainable by the slow-motion dials which have a 50 : 1 step-down ratio. A mutual and self inductance bridge by the same makers covers a range from 0·001 μH to 30 mH in 12 steps.

The Sullivan-Griffiths direct-reading inductance bridge has a range of 1 μH to 100 μH with an accuracy of 0·1 per cent. It has very small frequency and temperature errors and measures also capacitance from 0·0001 μF to 1 μF, resistance, and iron-cored or air-cored inductance without or with superposed d.c. up to 2 amperes flowing through the coil being measured. This bridge is based upon the use of a standard arm comprising an inductance standard of four decades tapped by rotary switches which also maintain the resistance of the whole standard arm constant for all values of inductance.

Another Sullivan inductance bridge has been specially designed for the measurement of iron-cored inductances from 10 mH to 1,000 mH without or with superposed d.c. up to 2 amperes. It may be used also for the measurement of air- or dust-cored coils. The network is that of the "Owen" bridge (see p. 224) but several novel features have been introduced by W. H. F. Griffiths to make the bridge entirely direct-reading for both inductance and resistance.

The Sullivan-Griffiths precision decade capacitance bridge has a wide range (0·1 $\mu\mu$F to 100 μF) and a direct-reading accuracy of

0·01 $\mu\mu$F to 0·01 per cent. The accuracy of direct-reading of power-factor measurement is 0·0001 and both this and the capacitance accuracy are maintained up to high frequencies.

Detectors. Electrodynamometer instruments have been used in a modified form as detectors in a.c. bridge measurements. Sumpner (Ref. (27)) introduced an electrodynamometer having an iron core giving very high sensitivity, and Weibel (Ref. (28)) describes several similar instruments designed for the same purpose.

The detectors in most common use for a.c. bridge measurements are, however, the telephone and the vibration galvanometer.

Telephones are widely used as detectors at frequencies of 500 cycles and over, up to 2,000 or 3,000 cycles, and are the most sensitive detectors available for such frequencies. The sensitivity of a telephone varies with the frequency of the supply, since the vibrating diaphragm which produces the sound has certain natural frequencies of vibration at which frequencies resonance is obtained, giving very high sensitivity. Wien (Ref. (29)), when investigating such resonance, found that for a Bell telephone resonance was obtained—with consequent highly increased sensitivity—at frequencies of 1,100, 2,800, and 6,500 cycles per second, and in the case of a Siemens telephone at frequencies of 720, 2,100, and 5,000 cycles.

The sensitivity of the observer's ear must also be taken into account when considering the sensitivity of the telephone as a detector. This varies with frequency. For most people a frequency of 800 cycles per second is a convenient one, since a note of this frequency is easily distinguished.

In selecting a telephone it is therefore best to choose one which has maximum sensitivity at the frequency at which it is to be used. The resistance of a telephone should match that of the bridge network. The range of resistances obtainable is roughly from 50 ohms to 7,000 or 8,000 ohms, a suitable telephone resistance for bridges of medium impedance being of the order of 200 ohms.

Transformers are sometimes used in conjunction with a low resistance telephone when the bridge network is of high impedance. The telephone is connected to the transformer secondary (low voltage side), the primary (high voltage side) being connected to the branch points of the network to which the detector is usually connected. In this way the voltage applied to the telephone is stepped down and the current stepped up.

Tuned Detectors. To improve the sensitivity (1) of a detector it may be tuned so that resonance—and therefore maximum amplitude of vibration for a given current—is obtained. Such tuned detectors also have the advantage that the response to frequencies other than the fundamental frequency of the supply is very small. (2) Errors due to harmonics in the supply waveform are thus minimized.

Campbell showed that an ordinary telephone can be tuned by means of a small screw pressing against the diaphragm at an eccentric point.

Amplifiers. Thermionic amplifiers are commonly used to increase the sensitivity in bridge measurements. Several forms are described by Hague (Ref. (1)).

An amplifier-detector made by Muirhead and Co., Ltd. and operated off 200–250 V a.c. mains is especially for use, in preference to an amplifier and telephones, where measurements are being made under noisy conditions and when the frequency is above or below the audible range. In the input circuit there are two balanced and screened transformers, with different input impedances, either of which may be selected by a low-capacitance key switch; this enables the detector to be used with most a.c. bridges. The linear amplifier, covering a wide range of frequencies, is followed by a bridge-connected metal rectifier and moving-coil milliameter indicating the rectified current. The amplifier itself is two-stage, resistance-capacitance coupled and uses R.F. pentode valves. Provision is made for backing-off the standing current in the anode circuit so that the milliammeter reads zero when no input is applied to the amplifier.

A power input of 4×10^{-14} watt causes a readable deflection of the meter corresponding to about 5 or 16 microvolts across the input according to which of the two transformers is used.

Vibration Galvanometers are the most widely-used tuned detectors. They are manufactured for various frequencies from 5 cycles per second up to 1,000 cycles, but are most commonly used below 200 cycles per second over which range they are considerably more sensitive than the telephone.

Vibration galvanometers are of two types—

(*a*) Moving-magnet. (*b*) Moving

The latter type is the more generally used, the moving-magnet type having the disadvantage of being seriously affected by magnetic fields of the resonant frequency, unless adequately screened. The moving-coil galvanometers are not appreciably affected by such fields.

Moving-magnet Type. The galvanometers of this type consist of a suspended system which carries one or more small, permanent magnets, and a light mirror about 2 or 3 mm diameter. The magnets are suspended between the poles of a magnet which is, in some forms, a permanent one, and in others is an electromagnet energized by coils carrying the current to be measured or detected. In the former the current is passed through coils whose magnetic field causes the suspended magnets to oscillate, the permanent magnet acting as the control. The control in other forms is supplied by torsion of the suspension. Air friction is the chief source of damping.

The moving system is tuned to the supply frequency either by altering the tension and length of the suspension or by varying the strength of the permanent magnet field, if such a magnet is included in the instrument.

A beam of light is thrown upon the mirror and, when current is passing through the instrument, the moving system oscillates, producing a band of light on the scale. In adjusting a bridge network to give zero deflection of the galvanometer, this band of light must, of course, be reduced until it again becomes a single spot, of the same diameter as when the supply is switched off. Some practice is necessary in observing when this condition has been attained. It is usually best to switch the galvanometer in and out of circuit and to note if there is any observable difference in the size of the spot in the two cases.

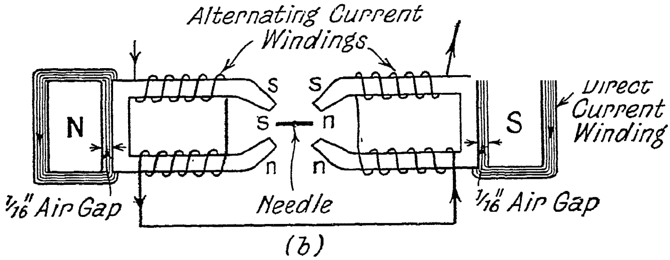


FIG. 150. CONSTRUCTION OF SCHERING AND SCHMIDT GALVANOMETER

Tuning.—To tune the galvanometer, a small current of the supply frequency is passed through it, and the tuning adjustments (variation of the tension and length of suspension or otherwise) are continued until the reflected band of light reaches its maximum length.

It is often helpful, in tuning, to adopt some method such as the following: First vary the frequency of the supply until the instrument shows maximum deflection and note this frequency. Next adjust the galvanometer and again vary the supply frequency to give maximum deflection. Note this frequency and proceed thus by successive steps until maximum deflection is produced by a supply frequency equal to that at which the measurements are to be made.

When used in the bridge network, the galvanometer should be shunted by a variable resistance to protect it against excessive currents when the bridge is out of balance. The shunting can be removed in steps until balance is almost obtained, when the shunt may be entirely removed, so that maximum sensitivity is obtained.

Drysdale Galvanometer. The Drysdale instrument can be tuned to resonance over a range of frequency of from about 20 cycles per second to 200 cycles per second. The resonance curve is so steep that at 50 cycles per second a variation in frequency of 2 per cent up or down reduces the deflection for a given current by 80 per cent. The necessity of maintaining the supply frequency constant within very narrow limits, when such instruments are being used, is thus obvious.

In this instrument the coil carrying the current to be detected is situated behind the moving system. Control is by means of a permanent magnet. Tuning is effected by varying the magnetic field in the gap of this permanent magnet by means of a magnetic shunt, the position of which is altered by means of a screw.

The sensibility of this instrument at 50 cycles per second, when a 40 ohm coil is used, is 4 millimetres per microampere with the scale distant one metre from the instrument.

Advantages of the instrument are the ease with which it can be tuned, and the fact that coils of different impedances can be inserted, as required by the bridge network used.

Schering and Schmidt Galvanometer. In this instrument the suspension is a phosphor-bronze strip and carries a light piece of iron and a mirror. This moving system is enclosed in an ebonite tube which can be slipped in between the four poles of two U-shaped magnets as shown (Fig. 150). These magnets carry four magnetizing coils, connected in series, through which the alternating current is passed. The two U-shaped magnets themselves fit in between the two poles of another magnet excited by a winding which carries direct current. The resistance of this latter winding is about 20 ohms and it can be supplied from a 10 volt battery. It is for the purpose of polarizing the iron needle of the suspended system. Oscillation of the needle is produced by the distortion of the d.c. magnet field by the superposed alternating field.

The instrument is tuned by variation of the controlling magnetic field by adjustment of the current in the d.c. exciting winding. In vibration galvanometers generally, the smaller the damping, the sharper the resonance curve. If the supply frequency is not absolutely constant it may be convenient to make the tuning curve less sharp by increasing the damping. Provision for this is made, in this instrument, by supplying a small piece of copper, adjacent to the moving needle, the position of which can be adjusted by a screw in the suspension piece. Damping is effected due to eddy currents induced in the copper by the moving needle.

Various resistances of the coils carrying the alternating current can be used, a common value being 500 ohms. With a single moving system the frequency range of an instrument of this type is about 25 to 100 cycles per second. The sensitivity, as given by the makers, when a 500 ohm coil is used, varies from 90 mm per microampere at 25 cycles, to 25 mm per microampere at 70 cycles, the scale being distant 1 metre from the instrument.

The Schering instrument is largely used in capacitance bridges at high voltages, its advantages for such work being as follows—

1. High insulation between the alternating current system and the d.c. windings, owing to the fact that a $\frac{1}{16}$ in. air gap is left between the a.c. magnets and the control magnet.

It can be tuned from a distance by variation of the d.c. magnet exciting current.

3. The instrument has a very small self-capacitance.

Messrs. H. Tinsley and Co. make an instrument of this type.

Moving-coil Vibration Galvanometers. These galvanometers are of the d'Arsonval type, having a moving-coil suspended between the poles of a strong, permanent magnet. The moving system is designed to have a very short, natural period of vibration; and the damping is very small, in order that the resonance curve shall be sharp—i.e. the deflection, for a given current passing through the instrument, is very much reduced by a small departure from the frequency to which the galvanometer is tuned. The alternating current to be detected is passed through the suspended coil, which consists of a few turns—or often of only a single loop—of wire. The moving system carries a small mirror, upon which a beam of light is cast. The system vibrates when an alternating current is passed through the coil, the reflected beam of light from the mirror thus throwing a band of light upon the scale.

These galvanometers are tuned by adjusting the length and tension of the suspended system.

*Duddell Moving-coil Vibration Galvanometer.** In this instrument the moving coil consists of a single loop of fine bronze or platinum-silver wire, this wire passing over a small pulley at the top and being pulled tight by a spring attached to the pulley (Fig. 151 (*a*)). The tension of this spring can be adjusted for tuning purposes by turning a milled head to which it is attached. The loop of wire is stretched over two ivory bridge pieces, the distance apart of these being adjustable in tuning the instrument. Variation of this distance apart obviously varies the length of the loop which is free to vibrate, and thus varies the natural period of the galvanometer. The galvanometer is roughly tuned by adjustment of the bridge pieces, fine adjustment of the tuning being obtained by varying the tension on the loop.

When a current passes through the loop a couple, tending to turn the loop about its vertical axis, is produced. When the current reverses this couple also reverses, thus causing oscillation of the loop when alternating current is passed through it.

This galvanometer can be used for frequencies between 100 and 1,800 cycles per second, the current sensitivity being about 50 mm per microampere, with a scale distance of 1 metre. The effective resistance is about 250 ohms. The sensitivity, if the loop is not too short, is almost inversely proportional to the frequency. In common with moving-coil vibration galvanometers generally, the instrument is not greatly affected by external magnetic fields. It has the disadvantage that the tuning can only be carried out by actually

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* See Ref. (25).

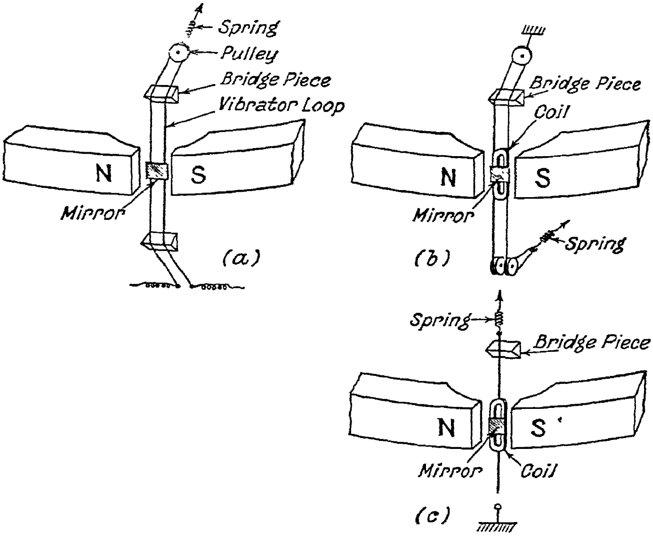
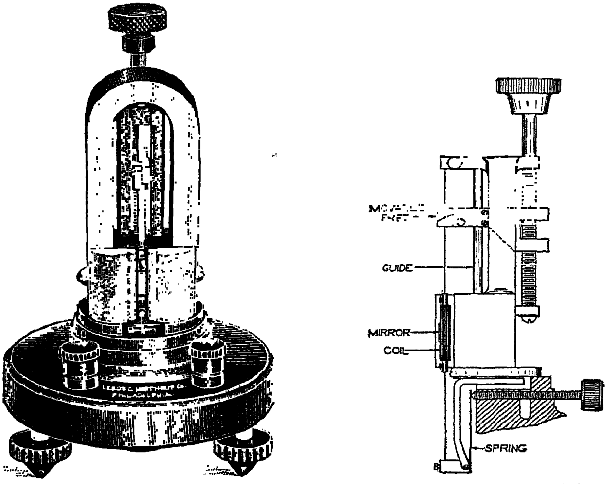


FIG. 151. CONSTRUCTION OF MOVING-COIL VIBRATION GALVANOMETERS



(*Leeds & Northrup Co.*)

FIG. 152. LEEDS AND NORTHRUP MOVING-COIL VIBRATION GALVANOMETER

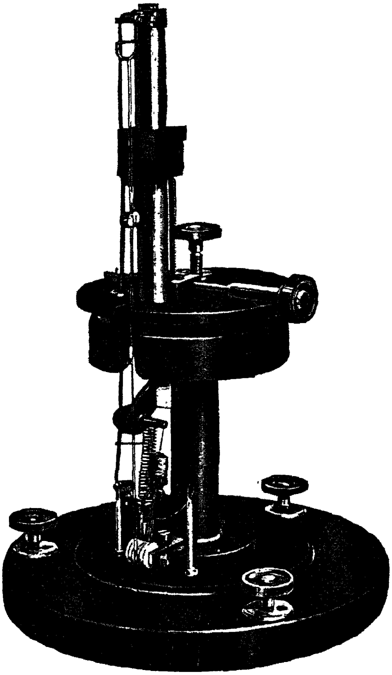
handling the instrument, and is, therefore, not very convenient for use in high-voltage work.

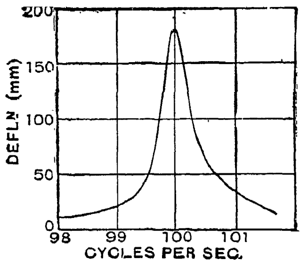
Other Moving-coil Vibration Galvanometers. A. Campbell (Refs. (15), (31), (32), (33)) has developed other moving-coil vibration galvanometers. Fig. 151 (*b*) shows the construction of his long-range instrument, and Fig. 151 (*c*) his short-range pattern. The former instrument has a bifilar suspension carrying a very light coil and a small mirror. The length of the suspension is varied, for tuning purposes, by the movement of a bridge above the coil, and the tension by means of a spiral spring at the bottom of the suspension. The range of frequency covered by such an instrument is from 50 to 1,000 cycles per second, and the sensitivity at 50 cycles is of the order of 60 mm per microampere at 1 metre scale distance, with an effective resistance of about 500 ohms, this sensitivity falling off at the higher frequencies to less than 1 mm per microampere.

The short-range instrument has a single strip suspension. Tuning is carried out in a similar way to that of the long-range instrument. The frequency range is from 10 cycles to 400 cycles per second, and has very high sensitivity at the lower frequencies (of the order of 400 mm per microampere at a scale distance of 1 metre when the frequency is 10 cycles per second).

Fig. 153 shows the construction and also a resonance curve of a Campbell moving-coil vibration galvanometer, manufactured by the Cambridge Instrument Co.

Fig. 152 gives details of the





(*Cambridge Instrument Co., Ltd.*)

FIG. 153. CAMPBELL MOVING-COIL VIBRATION GALVANOMETER AND RESONANCE CURVE

construction of a vibration galvanometer of a similar type, manufactured by the Leeds & Northrup Co. The screw adjustments for variation of the length of and tension on the suspension are clearly shown. The frequency range of this particular instrument is from 50 to 80 cycles per second, the sensitivity being stated by the makers as 40 mm per microampere at a scale distance of 1 metre and a frequency of 60 cycles per second, the resistance being 700 ohms.

To avoid the falling off in sensitivity with increase of frequency, several suspensions are usually provided for use with the same instrument at different frequencies.

Vibration galvanometers, generally, are susceptible to mechanical vibrations whose frequency is of the same order as that to which they are tuned. For this reason it is often necessary to provide some form of support which acts as a protection from such vibrations. One method of support which has been found to be satisfactory is to stand the instrument on rubber feet which rest on a heavy block of slate suspended from the ceiling by springs. Underneath the slate may be fitted damping vanes dipping into an oil dash-pot.

Theory of the Vibration Galvanometer with One Degree of Freedom. All the vibration galvanometers described above have only one degree of freedom—i.e. their suspended system only rotates about the axis of suspension. The theory of galvanometers with one degree of freedom was first given by Wenner (Ref. 30), and the following is based upon his work on the subject.

Considering, first of all, the constants of the galvanometer considered—called by Wenner the "intrinsic constants"—we have

(*a*) The "*displacement constant.*" If the suspended coil is of length l cm (measured along the axis of the suspension), has a width r cm, and has N turns, then the couple displacing the coil, when it carries i e.m. units of current, and is situated in a magnetic field of strength H, is $NHilr \cos\theta$ dyne-cm (see Fig. 154), where θ is the angle (in radians) between the plane of the coil and the direction of the magnetic field. If θ is small, $\cos\theta \fallingdotseq 1$, and the deflecting couple is $NHilr$ dyne-cm. Assuming the coil to be rectangular, lr is the area of its plane. Let $lr = A$, then the expression for the couple may be written $NHAi = Gi$. The constant G is called the "displacement constant" of the galvanometer, and is equal to NHA.

(*b*) The "*constant of inertia.*" Of the three couples retarding the motion, one is dependent upon the moment of inertia of the suspended system and upon the angular acceleration of this system. This couple may be written $a\frac{d^2\theta}{dt^2}$ where a is the "constant of inertia" or moment of inertia of the system.

(*c*) The "*damping constant.*" Another couple, retarding motion, is that due to the damping effect of air friction and elastic hysteresis in the suspension. This is usually assumed proportional to the angular velocity, and may be written

$$b\frac{d\theta}{dt}$$

where b is the "damping constant."

(*d*) The "*control or restoration constant.*" The couple due to the elasticity of the suspension is proportional to the displacement, and may be written $c\theta$, where c is the "*restoring constant.*"

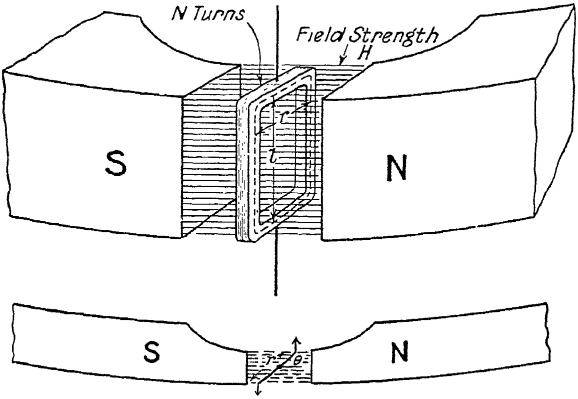


FIG. 154

We have, therefore, as the equation of motion of the system,

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = Gi \quad . \quad . \quad . \quad . \quad (174)$$

Now, if the current i is alternating, and is given by the expression

$$i = I_{max} \cos \omega t$$

we have
$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = GI_{max} \cos \omega t \quad . \quad . \quad (175)$$

The solution of this differential equation will be in two parts. The expression for θ will be the sum of a Particular Integral and the Complementary Function. The complementary function—obtained by solving the equation

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = 0$$

—will, in this case, give the expression for the angle θ when the current in the coil is zero—i.e. it will represent the natural free vibration of the coil. As will be shown later, this expression contains a factor of the form ε^{-at} so that it represents a vibration of the coil which rapidly dies away when the current is switched on. This is, therefore, the transient part of the solution of the equation of motion.

The particular integral will give an expression for θ, which represents the steady vibration of the coil after the current has been switched on for some appreciable time. Proceeding, then, to obtain the complementary function, we have

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = 0$$

The "auxiliary equation" is $am^2 + bm + c = 0$ and the roots of this equation are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and $$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Thus $$\theta = A\varepsilon^{m_1t} + B\varepsilon^{m_2t} \quad . \quad . \quad . \quad . \quad (176)$$

where A and B are constants to be determined from the initial conditions.

In vibration galvanometers the damping is small, and b^2 is less than $4ac$. Thus m_1 and m_2 are imaginary, and may be written

$$m_1 = -k_1 + jk_2$$
$$m_2 = -k_1 - jk_2$$

where $j = \sqrt{-1}$, $k_1 = \frac{b}{2a}$, and $k_2 = \frac{\sqrt{4ac - b^2}}{2a}$

$$\therefore \theta = A\varepsilon^{(-k_1 + jk_2)t} + B\varepsilon^{(-k_1 - jk_2)t}$$
$$= \varepsilon^{-k_1t}[A\varepsilon^{jk_2t} + B\varepsilon^{-jk_2t}] \quad . \quad . \quad . \quad . \quad (177)$$

Since, from trigonometry,

$$\varepsilon^{jpx} = \cos px + j\sin px$$

and $$\varepsilon^{-jpx} = \cos px - j\sin px$$

we have

$$\theta = \varepsilon^{-k_1t}[A(\cos k_2t + j\sin k_2t) + B(\cos k_2t - j\sin k_2t)]$$
$$= \varepsilon^{-k_1t}[(A + B)\cos k_2t + j(A - B)\sin k_2t]$$
$$= \varepsilon^{-k_1t}[P\cos k_2t + Q\sin k_2t]$$

where $P = A + B$ and $Q = j(A - B)$

i.e. $\theta = \varepsilon^{-k_1t}[F\sin(k_2t + \alpha)]$

where $F = \sqrt{P^2 + Q^2}$ and $\alpha = \tan^{-1}\frac{P}{Q}$

Thus $\theta = \varepsilon^{\frac{b}{2a}t}\left[F \sin\left(\frac{\sqrt{4ac - b^2}}{2a}\,t + \alpha\right)\right]$ . . . (178)

this being the transient portion of the solution which rapidly decreases in value as t is increased. The constants F and α must be determined from the initial conditions—i.e. they depend upon the position of the coil at the instant corresponding to zero time.

In this expression $\frac{\sqrt{4ac - b^2}}{2a}$ is the angular velocity ω, and is equal to 2π × the frequency of the vibratory motion.

Thus
$$f = \frac{\sqrt{4ac - b^2}}{4\pi a}$$

where f is the "natural frequency" of the vibrating system.

If the damping is negligibly small, $b = 0$, and the "*undamped natural frequency*" is given by

$$f = \frac{\sqrt{4ac}}{4\pi a} = \frac{1}{2\pi}\sqrt{\frac{c}{a}}$$

and obviously depends upon the moment of inertia of the system and upon the controlling forces.

The instrument is critically damped—i.e. it will not vibrate freely —when $f = 0$. This condition is fulfilled when $4ac = b^2$, or when the damping constant $b = 2\sqrt{ac}$.

Proceeding to find the particular integral, we have

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = GI_{max}\cos\omega t$$

or
$$\frac{d^2\theta}{dt^2} + \frac{b}{a}\frac{d\theta}{dt} + \frac{c}{a}\theta = \frac{GI_{max}}{a}\cos\omega t$$

Employing the operator D, we have

$$(D^2 + hD + g)\theta = \frac{GI_{max}}{a}\cos\omega t$$

where $h = \frac{b}{a}$ and $g = \frac{c}{a}$

$$\therefore\ \theta = \frac{\frac{GI_{max}}{a}\cos\omega t}{(D^2 + hD + g)}$$

Multiplying numerator and denominator by $(D^2 - hD + g)$ gives

$$\theta = \frac{\frac{GI_{max}}{a}(D^2 - hD + g)\cos\omega t}{(D^2 + g)^2 - h^2D^2}$$

$$= \frac{\dfrac{GI_{max}}{a}}{(D^2+g)^2 - h^2D^2}[-\omega^2 \cos \omega t + h\omega \sin \omega t + g \cos \omega t]$$

$$= \frac{GI_{max}}{a}(g-\omega^2)\frac{1}{(D^2+g)^2-h^2D^2}\cos \omega t + \frac{GI_{max}}{a}h\omega \frac{1}{(D^2+g)^2-h^2D^2}\sin \omega t$$

$$= \frac{\dfrac{GI_{max}}{a}(g-\omega^2)\cos \omega t}{(g-\omega^2)^2+h^2\omega^2} + \frac{\dfrac{GI_{max}}{a}h\omega \sin \omega t}{(g-\omega^2)^2+h^2\omega^2}$$

Substituting for h and g, we have

$$\theta = \frac{\dfrac{GI_{max}}{a}\left[\left(\dfrac{c}{a}-\omega^2\right)\cos \omega t + \dfrac{b}{a}\,\omega \sin \omega t\right]}{\left(\dfrac{c}{a}-\omega^2\right)^2 + \dfrac{b^2}{a^2}\omega^2}$$

$$= \frac{GI_{max}[(c-\omega^2 a)\cos \omega t + b\omega \sin \omega t]}{(c-a\omega^2)^2+b^2\omega^2}$$

$$= \frac{GI_{max}\sqrt{(c-a\omega^2)^2+b^2\omega^2}\left[\dfrac{c-\omega^2 a}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}\cos \omega t + \dfrac{b\omega}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}\sin \omega t\right]}{(c-a\omega^2)^2+b^2\omega^2}$$

or

$$\theta = \frac{GI_{max}}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}[\cos(\omega t - \beta)] \quad . \quad . \quad . \quad (179)$$

where $\beta = \tan^{-1}\dfrac{b\omega}{c-a\omega^2}$

This obviously represents a steady vibratory motion of amplitude $\dfrac{GI_{max}}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}$ and of frequency $\dfrac{\omega}{2\pi}$.

The complete solution for θ, being the sum of this expression and the expression derived previously as the complementary function, is thus

$$\theta = \varepsilon^{-\frac{b}{2a}\cdot t}\left[F \sin\left(\frac{\sqrt{4ac-b^2}}{2a}t + \alpha\right)\right] + \frac{GI_{max}}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}\cos(\omega t - \beta)] \quad . \quad (180)$$

Since the first expression is a transient which usually affects only the first few vibrations after switching on, we may neglect it and take as the law of the displacement simply

$$\theta = \frac{GI_{max}}{\sqrt{(c - a\omega^2)^2 + b^2\omega^2}} \cos(\omega t - \beta) \quad . \quad . \quad . \quad (181)$$

Example. Fig. 155 shows how the transient term, at the beginning of the vibration period, produces an unsteady state which gradually disappears, giving, finally, a steady vibration.

The curves shown are based on data given by Campbell* for a vibration galvanometer of the moving-coil type developed by him.

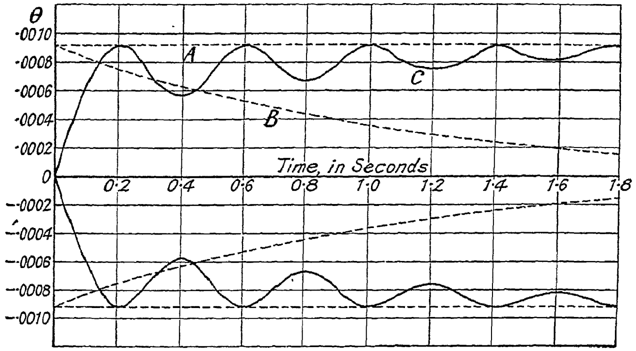


Fig. 155

The data given are as follows—

Number of turns	40
Mean area of turn. . . .	0·07 sq. cm.
Strength of magnetic field . .	2,700 lines per sq. cm.
Effective resistance . . .	1,540 ohms
Resonance frequency . . .	100 cycles per sec.
Inertia constant a. . . .	26×10^{-6}
Damping constant b . . .	49×10^{-6}
Restoring constant c . . .	10·4
G	8,600

Thus, in the expression for the deflection θ as derived above,

$$\frac{b}{2a} = 0{\cdot}942$$

$$\frac{\sqrt{4ac - b^2}}{2a} = 632$$

(This expression is equal to $\sqrt{\frac{c}{a}}$ to a very close approximation, and should therefore equal 628 when the instrument is tuned (see page 265), slight errors in the values of the constants probably being responsible for the discrepancy.)

$$\beta = 0° \ 11'$$
$$\omega = 628 \ (= 2\pi \times 100)$$

---

* *Dictionary of Applied Physics*, Vol. II, p. 974.

Assuming that I_{max} has the value 10 microamps, the expression

$$\frac{GI_{max}}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}$$

for the amplitude of the steady deflection has the value 0·000917.

The value of $\frac{b}{2a}$ being somewhat small, the transient term persists for an appreciable time, taking 2 sec. to fall to 0·153 of its initial amplitude. For this reason it is impossible to show, in the figure, the full curves for the two terms in the expression for the deflection θ. Lines passing through successive maximum points of these curves are shown instead. The dotted lines A are the lines passing through the maximum points of the steady deflection curve given by $\theta_1 = 0{\cdot}000917\cos(628t - 0°\ 11')$, while the dotted curves B pass through the maximum points on the curve $\theta_2 = \varepsilon^{-0{\cdot}942t}\,[F\sin(628t+\alpha)]$. The full line curves C pass through the maximum points of the total deflection curve (given by the summation of curves θ_1 and θ_2).

The effect of the transient term is shown by the "beat" effect which gradually dies away as the transient terms disappear. In the figure these beats have been drawn approximately owing to the difficulty of showing the full curves with a scale which is, necessarily, very cramped.

Tuning. In tuning the galvanometer, the object is to make the amplitude $\frac{GI_{max}}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}$ as great as possible for a given current I_{max}, which means that $\frac{G}{\sqrt{(c-a\omega^2)^2+b^2\omega^2}}$ must be made as great as possible. This expression can be increased by increasing the numerator G and by reducing the denominator $\sqrt{(c-a\omega^2)^2+b^2\omega^2}$.

Since $G = NHA$, it may be made large by using a coil of large area A and with a large number of turns N. G is obviously increased also by increasing the strength H of the magnetic field in which the coil lies. This latter method is the more important, since increasing the area and number of turns on the coil will increase its moment of inertia so that a will be increased and, thereby, the denominator may be increased.

Considering the denominator: of the three constants a, b, and c contained in it, c is the only one which can usually be varied. The constant c is the control constant and is varied by adjusting the length and tension of the suspension of the moving system, or by variation of the polarizing field of the galvanometer, in the case of moving-magnet instruments.

If the supply frequency is fixed—as it usually is in bridge measurements—the tuning process consists of varying c until $c - a\omega^2$ is zero, thus making the denominator of the amplitude expression a minimum. Since $\omega = 2\pi \times$ the supply frequency, we have the condition $c - a(2\pi f)^2 = 0$ (f = supply frequency), which must be satisfied in tuning the instrument.

Thus, c must equal $a(2\pi f)^2$ or $f = \frac{1}{2\pi}\sqrt{\frac{c}{a}}$

It should be noted that this expression for the supply frequency f, in terms of c and a, is the same as the expression for the frequency of the undamped vibration of the galvanometer (see page 262). This means that resonance occurs when the supply frequency is equal to the undamped natural frequency of the galvanometer. The amplitude under resonance conditions is obviously $\frac{GI_{max}}{b\omega}$.

Consider the case of the vibration galvanometer whose constants have already been given.

The constants are—

$$a = 26 \times 10^{-6}$$

$$b = 49 \times 10^{-6}$$

$$c = 10{\cdot}4$$

and the resonance frequency is given as 100 cycles per second. More exactly the frequency for resonance is $\frac{1}{2\pi}\sqrt{\frac{10{\cdot}4 \times 10^6}{26}}$ or 100·7 cycles per second. The deflection at resonance is

$$\frac{GI_{max}}{b\omega} = \frac{GI_{max}}{\frac{49}{10^6} \times 2\pi \times 100{\cdot}7} = GI_{max} \times 32{\cdot}27$$

The deflection at other frequencies is calculated from the expression $\frac{GI_{max}}{\sqrt{(c - a\omega^2)^2 + b^2\omega^2}}$ since $(c - a\omega^2)$ is zero only at the resonance frequency. A table showing the deflection for a range of frequency from 98·5 to 102 cycles per second is given below, and these values are plotted in Fig. 156. The sharpness of the resonance curve of a vibration galvanometer is well illustrated in the curve obtained.

TABLE IX

Frequency	ω	Deflection
98·5	618·8	GI_{max} × 2·27
99	622	,, × 2·93
99·5	625·1	,, × 4·21
100	628·4	,, × 7·63
100·5	631·4	,, × 22·3
100·7	632·6	,, × 32·3
101	634·6	,, × 13
101·5	637·7	,, × 5·78
102	640·9	,, × 3·57

To compare the response of the galvanometer to harmonics in the supply waveform consider a third harmonic when the supply frequency is that to which the galvanometer is tuned—namely, 100·7 cycles per second. The frequency of the third harmonic is 302·1 cycles per second. At this frequency the value of the expression $\sqrt{(c - a\omega^2)^2 + b^2\omega^2}$ is 83·32, so that, even if the amplitude of the third harmonic were equal to that of the fundamental, the amplitude of the deflection would be only GI_m

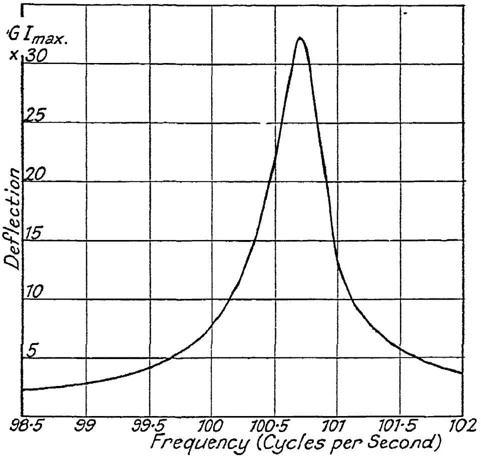


FIG. 156. RESONANCE CURVE OF VIBRATION GALVANOMETER

Thus the sensitivity to the fundamental compared with the sensitivity to the third harmonic is $\frac{32{\cdot}3}{0{\cdot}012} = 2{,}690$, showing that an entirely negligible error is introduced by the fact that the supply waveform contains harmonics.

The above theory assumes a current, $i = I_{max} \cos \omega t$, flowing through the galvanometer, and therefore refers to the "*current sensitivity*" of such instruments.

The "*voltage sensitivity*" may be determined by considering the case of a given voltage applied to the instrument terminals. In this consideration the voltage *induced* in the coil owing to the fact that, while vibrating, it is cutting through the magnetic field of strength H must be taken into account. The reader is referred to Hague's *Alternating Current Bridge Methods*, 4th Edition, p. 277, or to the *Dictionary of Applied Physics*, Vol. II, p. 971, for the full theory in this case.

BIBLIOGRAPHY AND REFERENCES

(1) *Alternating Current Bridge Methods,* B. Hague.

(2) *The Calculation and Measurement of Inductance and Capacity,* W. H. Nottage.

(3) *Dictionary of Applied Physics,* Vol. II.

(4) *Electrical Measurements,* F. A. Laws.

(5) *Radio Frequency Measurements,* E. B. Moullin.

(6) S. Butterworth, *Proc. Phys. Soc.,* Vol. XXXIV, p. 1 (1921).

(7) S. Butterworth, *N.P.L. Report,* 1922, p. 82.

(8) S. Butterworth, *Proc. Phys. Soc.,* Vol. XXIV, p. 75 (1912).

(9) D. W. Dye, *Exp. W. and W. Eng.,* Vol. I, p. 691 (1924).

(10) A. Campbell, *Proc. Phys. Soc.,* Vol. XXI, p. 75 (1910).

(11) A. Campbell, *Proc. Phys. Soc.,* Vol. XXII, p. 214 (1910).

(12) A. Campbell, *Electrician,* Vol. LX, p. 626 (1908).

(13) A. Campbell, *Proc. Phys. Soc.,* Vol. XXIX, p. 347 (1917).

(14) A. Campbell, *Proc. Phys. Soc.,* Vol. XXIV, pp. 107 and 158 (1912).

(15) A. Campbell, *Proc. Phys. Soc.,* Vol. XX, p. 626 (1907).

(16) F. W. Grover, *Bull. Bur. Stands.,* Vol. III, p. 375 (1907).

(17) F. W. Grover and H. L. Curtis, *Bull. Bur. Stands.,* Vol. VIII, pp. 463 and 495 (1913).

(18) L. Hartshorn, *Proc. Phys. Soc.,* Vol. XXXIX, p. 337 (1927).

(19) E. Giebe, *Ann. der Phys.,* Vol. XXIV, p. 941 (1907).

(20) E. Giebe, *Zeits. fur Inst.,* Vol. XXXI, pp. 6 and 33 (1911).

(21) K. W. Wagner, *Elekt. Zeits.,* Vol. XXXII, p. 1001 (1911).

(22) G. Chaperon, *Comptes Rendus,* Vol. CVIII, p. 799 (1889).

(23) H. B. Brooks and F. C. Weaver *Bull. Bur. Stands.,* Vol. XIII, p. 569, (1917).

(24) W. Duddell, *Phys. Soc. Proc.,* XXIV, p. 172 (1912).

(25) W. Duddell, *Phys. Soc. Proc.,* XXI, p. 774 (1909).

(26) W. Duddell, *I.E.E. Journal,* Vol. LII, p. 275 (1914).

(27) W. E. Sumpner, *Jour. Soc. Tel. Engrs.,* XVI, p. 344 (1887).

(28) E. Weibel, *Bull. Bur. Stands.,* XIV, p. 23 (1918).

(29) M. Wien, *Ann. der Phys.,* Vol. IV, p. 450 (1901).

(30) F. Wenner, *Bull. Bur. Stands.,* Vol. VI, p. 347 (1910).

(31) A. Campbell, *Proc. Phys. Soc.,* Vol. XXV, p. 203 (1913).

(32) A. Campbell, *Proc. Phys.* Soc., Vol. XXVI, p. 120 (1914).

(33) A. Campbell, *Proc. Phys. Soc.,* Vol. XXXI, p. 85 (1919).

(34) W. H. F. Griffiths, *Experimental Wireless and the Wireless Engineer,* January and February, 1929.

(35) "The Design, Construction and Use of Resistors of Calculable Reactance," N. F. Astbury, *I.E.E. Journal,* Vol. LXXVI, p. 389.

(36) *The Measurement of Inductance, Capitance and Frequency,* A. Campbell and E. C. Childs.

(37) *Basic Electrical Measurements,* M. B. Stout.

(38) *Radio-frequency Measurements by Bridge and Resonance Methods,* L. Hartshorn.

(39) *Electrical Measurements and the Calculation of the Errors Involved,* Part II, D. Karo.

(40) "The Practical Construction of Vibration-free Mounting with Auxiliary Mass," J. A. Haringx, *Philips Tech. Rev.,* Vol. 9, No. 3, p. 85.

(41) "Vibration-free Mountings with Auxiliary Mass," J. A. Haringx, *Philips Tech. Rev.,* Vol. 9, No. 16.

(42) "Double-ratio a.c. Bridges with Inductively-coupled Ratio Arms," H. A. M. Clark and P. B. Vanderlyn, *Jour. I.E.E.,* Vol. 96, Pt. II, p. 365.

(43) "A Direct Capacitance Aircraft Altimeter," W. L. Walton and M. E. Pemberton, *Jour. I.E.E.,* Vol. 96, Pt. II, p. 379.

(44) "Law Linearity of Semi-circular Plate Variable Condensers," W. H. F. Griffiths, *The Wireless Engineer,* Vol. XXII, No. 258, pp. 107–18.

(45) "The Temperature Compensation of Condensers," W. H. F. Griffiths, *The Wireless Engineer*, March, 1942.

(46) "Recent Improvements in Air Cored Inductances," *The Wireless Engineer*, Vol. XIX, No. 220, pp. 8–19, and No. 221, pp. 56–63.

(47) "The Temperature Compensation of Condensers," *The Wireless Engineer*, Vol. XIX, No. 222, pp. 101–11, and No. 223, pp. 148–57.

(48) "The Direct Reading of the Frequency of Resonant Circuits," *The Wireless Engineer*, Vol. XX, No. 242, pp. 524–38.

(49) "Wide Range Variable Condenser for Special Laws," *The Wireless Engineer*, Vol. XI, No. 131, pp. 415–18.

(50) "Standards and Standardization," *The Wireless Engineer*, Vol. XX, No. 234, pp. 109–26.

(51) "The Losses in Variable Air Condensers," *The Wireless Engineer*, Vol. VIII, No. 90, pp. 124–6.

(52) "The Accuracy of Calibration Permanence of Variable Air Condensers for Precision Wavemeters," *The Wireless Engineer*, Vol. V, No. 52, pp. 17–24, and No. 53, pp. 63–74.

(53) "Further Notes on the Calibration Permanence and Overall Accuracy of the Series-Gap Precision Variable Air Condenser," *The Wireless Engineer*, Vol. VI, No. 64, pp. 23–30, and No. 65, pp. 77–80.

(54) "Electrical Standards for Research and Industry," a catalogue published by Messrs. H. W. Sullivan, Ltd. in 1954.

CHAPTER VII

MEASUREMENT OF RESISTANCE

FROM the point of view of measurement, resistances can be classified generally as follows—

(*a*) LOW RESISTANCES. All resistances of the order of 1 ohm and under may be classified thus. In practice such resistances may be met with in the armatures and series windings of large machines, in ammeter shunts, cable lengths, contacts, etc.

(*b*) MEDIUM RESISTANCES. This class includes resistances from about 1 ohm upwards to about 100,000 ohms. In practice the majority of the pieces of electrical apparatus used have resistances which lie between these limits.

(*c*) HIGH RESISTANCES. Resistances of 100,000 ohms and upwards must be so classified.

A classification such as the above is not rigid, but forms a guide as to the method of measurement to be adopted in any particular case.

Measurement of Low Resistance. Methods of measurement which are suitable for medium resistances are in most cases unsuitable for low resistance measurements, chiefly because contact resistances cause serious errors. It is clear that contact resistances of the order (say) of 0·001 ohm—negligible though they may be when a resistance of 100 or more ohms is to be measured—are of great importance when the resistance to be measured is of the order of 0·01 ohm.

Again, it is usually essential, with low resistances, that the two points between which the resistance is to be measured shall be very definitely defined. Thus the methods which are specially adapted to low resistance measurement employ *potential connections* —i.e. connecting leads which form no part of the circuit whose resistance is to be measured, but which connect two points, in this circuit, to the measuring circuit. These two points are spoken of as the *potential terminals,* and serve to fix, definitely, the length of the circuit under test. In the methods used for the precise measurement of low resistance, the "unknown" resistance is compared with a low-resistance standard of the same order as the unknown, and with which it is connected in series. Both resistances are fitted with four terminals—two "current terminals," to be connected to the supply circuit, and two "potential terminals" to be connected to the measuring circuit. This arrangement is shown in Fig. 157.

AMMETER AND VOLTMETER METHOD. This method, which is the simplest of all, is in very common use for the measurement of low resistances when an accuracy of the order of 1 per cent is sufficient.

It must be realized, however, that it is, essentially, a comparatively rough method, the accuracy being limited by those of the ammeter and voltmeter used, even if corrections are made for the "shunting" effect of the voltmeter. In Fig. 158, R is the resistance to be measured and V is a high-resistance voltmeter of resistance R_v. A current from a steady direct-current supply is passed through R in series with a suitable ammeter.

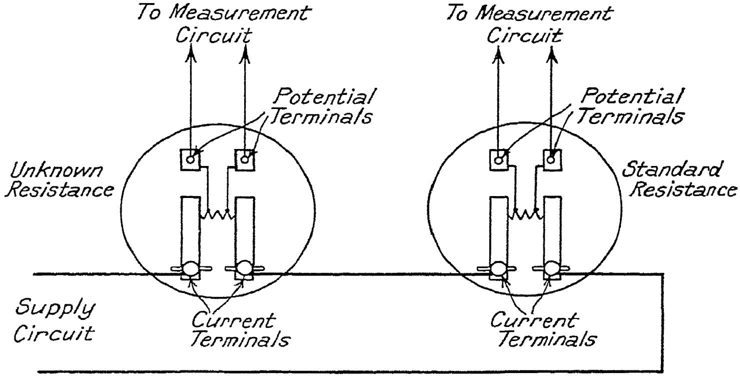


FIG. 157. MEASUREMENT OF LOW RESISTANCE

Then, assuming the current through the unknown resistance to be the same as that measured by the ammeter A, the former is given by

$$R = \frac{\text{voltmeter reading}}{\text{ammeter reading}}$$

If the voltmeter resistance is not very large compared with the resistance to be measured, the voltmeter current will be an appreciable fraction of the current I, measured by the ammeter, and a serious error may be introduced on this account.

Example. A resistance whose actual value is 1 ohm, is to be measured by the ammeter and voltmeter method. The carrying capacity of the resistance is 100 milliamperes, which is the current used in making the measurement. The voltmeter used has a resistance of 5 ohms, and reads up to 100 millivolts. What is the measured value of the 1 ohm resistance?

Let resistance of 1 ohm resistance and voltmeter in parallel = r.

Then
$$\frac{1}{r} = \frac{1}{1} + \frac{1}{5} = 1{\cdot}2$$

$$r = \frac{1}{1{\cdot}2} = 0{\cdot}833 \text{ ohm}$$

Volt drop across the resistance to be measured

$$= 0{\cdot}833 \times 0{\cdot}1$$
$$= 0{\cdot}0833 \text{ volt}$$
$$= \text{voltmeter reading}$$

$$\text{Ammeter reading} = 0{\cdot}1 \text{ amp.}$$

$$\text{Thus the measured value of the resistance} = \frac{0{\cdot}0833}{0{\cdot}1}$$
$$= 0{\cdot}833 \text{ ohm}$$

This means that, even if the ammeter and voltmeter give readings which are exactly correct, an error of 0·166 ohm, or 16·6 per cent, is introduced by the fact that the voltmeter takes an appreciable fraction of the total current.

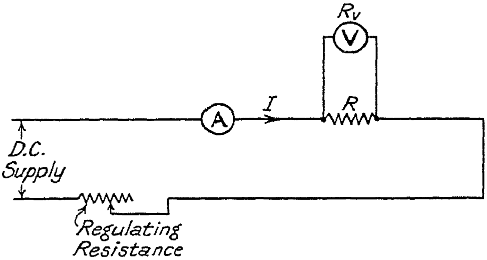


FIG. 158. AMMETER AND VOLTMETER METHOD OF RESISTANCE MEASUREMENT

If, on the other hand, the current-carrying capacity of the 1 ohm resistance had been such that a much greater current could have been passed through it, so that a voltmeter of resistance (say) 500 ohms, reading up to 10 volts, could have been used, then the error introduced would have been only 0·2 per cent, as can be seen from a similar calculation to the above.

Correction for Shunting Effect of the Voltmeter. In general terms, if the actual value of the unknown resistance is R, and its measured value R_m, the voltmeter resistance being R_v, and the ammeter current I, we have—

$$\text{Resistance of voltmeter and } R \text{ in parallel} = \frac{RR_v}{R + R_v}$$

$$\text{Voltage drop across } R = \frac{RR_v\, I}{R + R_v}$$
$$= \text{voltmeter reading}$$

(assuming the voltmeter to read correctly).

Thus, upon the assumption that the ammeter reading also is exactly correct,

$$R_m = \frac{RR_v I}{(R + R_v)I} = \frac{RR_v}{R + R_v}$$

or

$$R = \frac{R_m R_v}{R_v - R_m} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (182)$$

This method is useful in practice in the measurement of such resistances as those of armatures, and of joints and contacts when the current-carrying capacity is fairly great and when the results are only required to within the limits of accuracy of the ammeter and voltmeter used.

Potentiometer Method. In the potentiometer method of measuring a low resistance the unknown resistance is compared with a standard resistance of the same order of magnitude.

These standard low resistances are of the type described in Chapter II. The following table gives the resistances and current-carrying capacities of some of a range of standards as manufactured by Messrs. H. Tinsley and Co.

TABLE X

Resistance (Ohms)	Current-carrying capacity (Amp.)
10	1
5	1·4
2	2·2
1	3
0·5	4·5
0·1	22
0·01	150
0·001	700
0·0005	1000
0·0001	2250

The standards are adjusted to within 0·03 per cent of their nominal resistance (0·05 per cent for the last three).

The complete range of these resistance standards is from 10,000 Ω down to 0·0001 Ω. They are suitable for use at frequencies up to 1,000 c/s and their time constants for frequencies up to this value are 1×10^{-6} down to the 0·01 standard. For this and lower resistances the time constant is 3×10^{-6}. The ratings in watts are 10 W over the range 10,000 Ω to 0·2 Ω; 50 W between 0·2 Ω and 0·02 Ω; 200 W between 0·02 Ω and 0·005 Ω and 500 W between 0·001 Ω and 0·0001 Ω. Below 0·02 Ω water cooling, with a motor-driven stirrer, is used.

The unknown resistance and a standard of the same order of resistance are connected in series as in Fig. 157. A steady current is passed through them from a battery of the heavy-current type. The magnitude of this current should be chosen so that a voltage drop of the order of 1 volt is obtained, if possible, across each of the resistances.

The voltage drops across both the unknown resistance and the standard are then measured on the potentiometer (see Chapter VIII), several measurements being made, alternately, and with as small a time interval as possible between the measurements. The mean values of these are taken as the correct voltage drops across the two resistances. By carrying out the measurements in this way the error due to possible variation of the supply current is minimized.

The potential leads to the potentiometer carry no current when the potentiometer is balanced, and thus the current through the two resistances is the same. Then

$$\frac{\text{Resistance of the unknown}}{\text{Resistance of the standard}} = \frac{\text{Voltage drop across the unknown}}{\text{Voltage drop across the standard}}$$

from which the resistance of the unknown is obtained in terms of that of the standard resistance.

Precautions. When used for precise work resistance standards should be frequently checked against National Physical Laboratory standards, and the most recent calibration of the resistor should be used in calculating the resistance of the unknown. If the standard resistance is subject to appreciable variation with time, it may be necessary to estimate its probable variation from the time of the last calibration, on the assumption that its rate of variation is uniform, and the same as that between the dates of the two preceding calibrations. A standard resistor in which such variation with time is large is, of course, useless for precise work. The temperature of the two resistors, during the test, should be measured and the resistance of the standard, at the measured temperature, should be obtained from its resistance-temperature curve. The measured resistance of the unknown is that at the measured temperature, and this should be stated, as its temperature coefficient may be large, and hence its resistance may be appreciably different at other temperatures.

In precise work also, a second measurement should be made with the supply reversed (care must be taken to reverse the potential leads to the potentiometer at the same time), and thermo-electric effects should be taken into account as described in the next chapter.

When the necessary precautions are taken, and a good potentiometer and sensitive galvanometer are used, the accuracy obtainable by this method may be within a few parts in 100,000, or within rt in 10,000.

Kelvin Double Bridge. This method is one of the best available for the precise measurement of low resistances. It is a development of the Wheatstone Bridge by which the errors due to contact and leads resistances are eliminated.. The connections of the bridge are shown in Fig. 159.

In the figure, X is the low resistance to be measured, and S is a standard resistance of the same order of magnitude. These are connected in series with a low-resistance link r, connecting their adjacent current terminals. A current is passed through them from a battery supply. A regulating resistance and ammeter are connected in the circuit for convenience. Q, M, q, and m are four known, non-inductive resistances, one pair of which (M and m, or Q and q) are variable. These are connected to form two sets of ratio arms as shown, a sensitive galvanometer G connecting the dividing points of QM and qm. The ratio $\frac{Q}{M}$ is kept the same as $\frac{q}{m}$, these ratios being varied until zero deflection of the galvanometer is obtained. Then $\frac{X}{S} = \frac{Q}{M} = \frac{q}{m}$, from which X is obtained in terms of S, Q, and M.

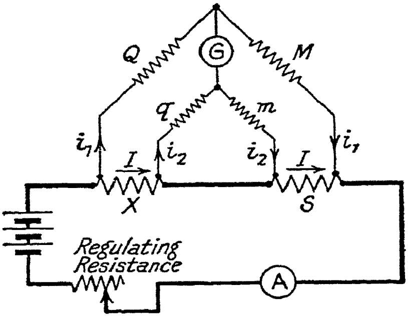


FIG. 159. KELVIN DOUBLE BRIDGE METHOD OF MEASURING LOW RESISTANCE

Theory. At balance of the bridge (i.e. zero galvanometer deflection) the current in arm Q = current in arm M. Let this current be i_1. Also, current in arm q = current in arm m. Let this current be i_2. Therefore, the current in X = current in S. Let this current be I.

Again, voltage drop across Q = voltage drop across X + voltage drop across q

i.e. $$i_1Q = IX + i_2q$$

In the same way, $$i_1M = IS + i_2m$$

Now, since q and m are in parallel with the resistance r, the current I in X divides so that $\frac{r}{r + q + m} I$ passes through q and m, i.e.

$$i_2 = \frac{r}{r + q + m} \cdot I$$

Substituting this value of i_2 in the above equations, we have

$$i_1Q = IX + \frac{rq}{r + q + m} \cdot I$$

and $$i_1M = IS + \frac{rm}{r + q + m} \cdot I$$

By division
$$\frac{Q}{M} = \frac{X + \dfrac{rq}{r + q + m}}{S + \dfrac{rm}{r + q + m}}$$

from which
$$MX = QS + \frac{Qmr}{r + q + m} - \frac{Mqr}{r + q + m}$$

$$X = \frac{QS}{M} + \frac{r}{r + q + m}\left(\frac{Qm}{M} - q\right)$$

or
$$X = \frac{QS}{M} + \frac{mr}{r + q + m}\left(\frac{Q}{M} - \frac{q}{m}\right) \quad . \quad . \quad (183)$$

The term $\frac{mr}{r + q + m}\left(\frac{Q}{M} - \frac{q}{m}\right)$ can be made very small by making the resistance of the link r very small, and also by making the ratio $\frac{Q}{M}$ as nearly as possible equal to $\frac{q}{m}$.

If this term is made negligibly small—which is not difficult to accomplish in practice—the expression for X becomes simply

$$X = \frac{Q}{M} . S$$

which gives the resistance of the unknown in terms of the resistance of the standard.

In order to take into account thermo-electric e.m.f.s (see next chapter), a measurement should also be made with the direction of the current reversed and the mean of the two readings should be taken as the correct value of X.

In a Kelvin double bridge manufactured by Messrs. H. Tinsley & Co. the range of resistance covered is 0·1 microhm to 1 ohm. Under specified conditions of test the accuracy is stated as

From 1,000 microhms to 1 ohm . . . 0·05 per cent.

From 100 microhms to 1,000 microhms . . . 0·2 per cent to 0·05 per cent.

From 10 microhms to 100 microhms . . . 0·5 per cent to 0·2 per cent, limited by thermo e.m.f.'s.

There are four internal resistance standards of 1 Ω, 0·1 Ω, 0·01 Ω and 0·001 Ω respectively.

H. W. Sullivan, Ltd. make a precision Kelvin and Wheatstone bridge covering a range of 1 microhm to 1 megohm.

Operation of Kelvin Double Bridge in Practice. The method of operation of the Kelvin double bridge in practice is often somewhat different from that described above, especially when precise measurements of low resistance are to be made.

Instead of varying the ratio arms, keeping the ratio $\frac{Q}{M}$ equal to $\frac{q}{m}$, to obtain balance of the bridge, the resistances Q, M, q, and m

are often made up of resistance coils whose resistances are fixed and are accurately known, together with their temperature coefficients. The ratios $\frac{Q}{M}$ and $\frac{q}{m}$ thus remain fixed during the test, and are made equal to one another, and roughly equal to the ratio $\frac{X}{S}$ assuming this to be known, approximately. If not known, the ratio can easily be determined approximately by measuring X, first of all, using a less accurate method such as the ammeter and voltmeter method, or, better, by the potentiometer method.

Adjustment of the bridge to obtain balance is then carried out by shunting either the unknown resistance or the standard by a variable resistance, such as a resistance box. Then, assuming balance to be obtained by shunting the unknown by a resistance x, let the resistance of X and x in parallel, at balance, be X', then

$$X' = \frac{Q}{M} \cdot S$$

and also
$$\frac{1}{X'} = \frac{1}{X} + \frac{1}{x}$$

from which the value of X can be obtained. As before, measurements are made also with the supply current reversed, and the average value of the two results taken as the final value. The sensitivity of the bridge can be determined by noting the smallest variation of the shunting resistance x, which produces an observable deflection of the galvanometer. The difference in X' for such a variation of x can then easily be calculated, thus giving the sensitivity of the bridge. This method of obtaining fine adjustment, by shunting a low resistance by a resistance box of much greater resistance, will be found a very useful one in electrical measurements generally. It has the advantage, also, that the resistances of the coils in the resistance box need not be known to within any high degree of accuracy, since slight errors in their values introduce negligible errors in the resistance of the combination.

The ratios $\frac{Q}{M}$ and $\frac{q}{m}$ may be made exactly equal by adjustment of the resistances of the leads to be used in connecting up the bridge, since these leads resistances are obviously included in the arms as well as the resistances of the coils themselves. By suitably proportioning the resistances of these copper leads the ratios $\frac{Q}{M}$ and $\frac{q}{m}$ can also be made independent of temperature—if this is necessary—to a very close approximation. The resistances of the leads need only be known with an accuracy of (say) 1 per cent, since they are usually small compared with those of the coils which they connect.

Example.

Resistance of link r = 0·0001 ohm

Resistance of coil in arm Q = $10{\cdot}0027_0$ at 20° C.
(Temperature coefficient 0·00003)

Resistance of coil in arm M = $20{\cdot}0142_5$ at 20° C.
(Temperature coefficient 0·00002)

Resistance of coil in arm q = $10{\cdot}0027_1$ at 20° C.
(Temperature coefficient 0·00003)

Resistance of coil in arm m = $20{\cdot}0067_0$ at 20° C.
(Temperature coefficient 0·000025)

Resistances of copper leads—

$$\left.\begin{array}{llll}\text{In arm } Q & = 0{\cdot}0146 & \text{at } 20^\circ & \text{C.} \\ \;\;,, \;\; ,, \;\; M & = 0{\cdot}0660 & ,, & ,, \\ \;\;,, \;\; ,, \;\; q & = 0{\cdot}0061_5 & ,, & ,, \\ \;\;,, \;\; ,, \;\; m & = 0{\cdot}0122 & ,, & ,, \end{array}\right\} \text{Temperature coefficient } 0{\cdot}0043$$

Resistance of standard = $0{\cdot}0100120_2$ at 20° C.

Resistance of "unknown" = 0·005 (about)

To balance the bridge the standard resistance is shunted by a resistance of 18·1 ohms, measurement being made at 20° C.

First, the values of the two ratios $\frac{Q}{M}$ and $\frac{q}{m}$ are calculated for a temperature of 20° C.

Then,
$$\frac{Q}{M} = \frac{10{\cdot}0027_0 + 0{\cdot}0146}{20{\cdot}0142_0 + 0{\cdot}0660} = 0{\cdot}49886_5$$
$$\frac{q}{m} = \frac{10{\cdot}0027_1 + 0{\cdot}0061_5}{20{\cdot}0067_0 + 0{\cdot}0122} = 0{\cdot}499970$$

To make these ratios more nearly equal the coil in arm q is shunted by some resistance y, the value of which is found as follows.

Let q' be the shunted value of this coil, then
$$\frac{q' + 0{\cdot}0061_5}{20{\cdot}0189_0} = 0{\cdot}49886_5$$
$$q' = 9{\cdot}9805_8$$

Now,
$$\frac{1}{q'} = \frac{1}{y} + \frac{1}{10{\cdot}0027_1}$$

from which
$$y = 3{,}532 \text{ ohms}$$

Since the two ratios have been thus made exactly equal at 20° C., if the measurement is made at 20° C. the value of the unknown resistance is given simply by
$$X = \frac{Q}{M} \cdot S'$$

where S' is the shunted value of the standard. Since the shunt for balance is 18·1 ohms,
$$\frac{1}{S'} = \frac{1}{0{\cdot}0100120_2} + \frac{1}{18{\cdot}1}$$

from which
$$S' = 0{\cdot}010006_5$$
$$\therefore X = 0{\cdot}010006_5 \times 0{\cdot}49886_5$$
$$= 0{\cdot}0049919_0 \text{ at } 20^\circ \text{ C.}$$

Consider, now, the effect upon the ratios $\frac{Q}{M}$ and $\frac{q}{m}$ of a rise in temperature of 5° C.

$$\text{Then}\quad \frac{Q}{M} = \frac{10{\cdot}0027_0[1 + 0{\cdot}00003 \times 5] + 0{\cdot}0146[1 + 0{\cdot}0043 \times 5]}{20{\cdot}0142_5[1 + 0{\cdot}00002 \times 5] + 0{\cdot}0660[1 + 0{\cdot}0043 \times 5}$$

$$= 0{\cdot}49886_3$$

$$\text{and}\qquad \frac{q}{m} = \frac{9{\cdot}9805_8[1 + 0{\cdot}00003 \times 5] + 0{\cdot}0061_5[1 + 0{\cdot}0043 \times 5]}{20{\cdot}0067_0[1 + 0{\cdot}00002_5 \times 5] + 0{\cdot}0122[1 + 0{\cdot}0043 \times 5]}$$

$$= 0{\cdot}49887_8$$

$$\text{Then}\ \frac{Q}{M} - \frac{q}{m} = -0{\cdot}000010$$

and the correction term $\frac{mr}{r + q + m}\left(\frac{Q}{M} - \frac{q}{m}\right)$ is equal to

$$-\frac{0{\cdot}0001 \times 20{\cdot}0067_0}{10{\cdot}0027_1 + 20{\cdot}0067_0 + 0{\cdot}0001} \times 0{\cdot}00001$$

which is roughly $\frac{2}{3} \times 0{\cdot}000000001$ and is therefore entirely negligible.

Sensitivity. Suppose that in the above measurement the smallest change in the value of the shunt across the standard which can be detected is 0·5 ohm. Then, giving this shunt the value 18·6 instead of 18·1 we have

$$\overline{S'} \qquad \overline{0{\cdot}0100120_2} \qquad \frac{1}{18{\cdot}6}$$

from which $S' = 0{\cdot}010006_8$ instead of $0{\cdot}010006_5$, which means a change in S' of 1 part in 100,000, and therefore a change in the value of X of 1 part in 100,000. Thus the sensitivity of the bridge under the above conditions is 1 part in 100,000.

In general, if x is the value of the shunt across S, we have

$$\frac{1}{S'} = \frac{1}{S} + \frac{1}{x}$$

$$\text{or}\qquad S' = \frac{Sx}{S + x}$$

Differentiating with respect to x,

$$\frac{dS'}{dx} = S\left[\frac{(S + x) - x}{(S + x)^2}\right] = \frac{S^2}{(S + x)^2} \doteqdot \frac{\Delta S'}{\Delta x}$$

where $\Delta S'$ is the change in S' for a given change Δx in x.

$$\text{Thus,}\qquad \Delta S' = \frac{S^2}{(S + x)^2} \cdot \Delta x$$

Smith (Ref. 4) has shown that, if X is changed to $X + \delta X$, the galvanometer current is given by

$$\frac{I\,.\,\delta X}{G + \dfrac{qm}{q + m} + \left[\dfrac{(X + Q)\,(S + M)}{X + Q + S + M}\right]}\left[\frac{S + M}{X + Q + S + M}\right]$$

where G is the resistance of the galvanometer. Thus, if Y is the sensitivity of the galvanometer used, in millimetres per micro-ampere, the deflection for a change of δX in the value of X is given in millimetres by

$$D = \frac{YI\delta X \,.\, 10^6 (S + M)}{\left[G + \frac{qm}{q + m} + \frac{(X + Q)(S + M)}{X + Q + S + M}\right](X + Q + S + M)} \qquad (184)$$

The best value for the galvanometer resistance is

$$\frac{qm}{q + m} + \frac{(X + Q)(S + M)}{X + Q + S + M}$$

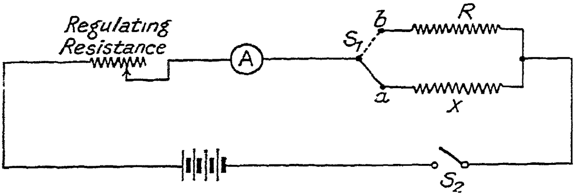


FIG. 160. MEASUREMENT OF RESISTANCE BY SUBSTITUTION

Measurement of Medium Resistance. The methods used for such measurements are—

(*a*) Ammeter and voltmeter method.
(*b*) Substitution method.
(*c*) Differential galvanometer method.
(*d*) Wheatstone bridge.

(*a*) This method has been considered in the section on low resistance measurements earlier in the chapter.

(*b*) SUBSTITUTION METHOD. The diagram of connections for this method is given in Fig. 160. X is the resistance to be measured, while R is a variable known resistance. A battery of ample capacity is used for the supply, since it is important in this method that the supply voltage shall be constant. A is an ammeter of suitable range, or a galvanometer with a shunt which can be varied as required.

With switch S_2 closed, and with switch S_1 on stud a, the deflection of the ammeter or galvanometer is observed. S_1 is then thrown on to stud b and the variable resistance is adjusted until the same deflection is obtained on the indicating instrument. Then, the value of R which produces the same deflection gives the resistance of the unknown directly.

The resistances of R and X should be large compared with that of the rest of the circuit. The method is chiefly used—somewhat modified—in the measurement of high resistance. The accuracy of the measurement obviously depends upon the constancy of the supply voltage, of the resistance of the circuit excluding X and R, and upon the sensitivity of the indicating instrument, as well as upon the accuracy with which the resistance R is known.

(c) DIFFERENTIAL GALVANOMETER METHOD. A differential galvanometer has two similar, but separate, windings, insulated from one another. The windings are of equal resistance and are wound with twinned wire so that they may be as nearly coincident as possible and thus produce almost exactly equal and coincident magnetic fields when equal voltages are applied to their terminals. If equal and *opposite* voltages were applied to the two windings the resultant magnetic field would be very nearly zero. This resultant field is made zero by the use of a small external coil connected in series with the coil having the weaker magnetic field and placed so that its magnetic effects add to that of this coil. This external coil is adjustable so that zero magnetic field may be obtained. A small magnetic needle, to which a pointer is attached, is pivoted in the field produced by the windings.

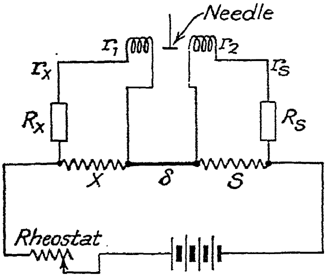


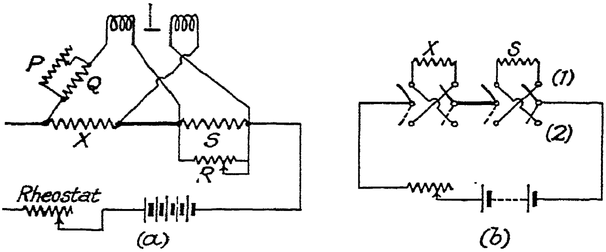
FIG. 161. MEASUREMENT OF RESISTANCE BY DIFFERENTIAL GALVANOMETER

The circuit connections when the instrument is to be used for the measurement of resistance are shown in Fig. 161. X is the unknown resistance, while S is a standard resistor with which it is to be compared. R_s and R_x are resistance boxes connected in circuit as shown, their positions in the circuit being such that the two galvanometer coils, of resistances r_1 and r_2, are very nearly at the same potential, to avoid leakage effects between them. r_s and r_x represent the resistances of the leads and δ is the resistance of the connecting link between X and S, and should be very low.

The galvanometer is first adjusted to zero by connecting its two coils in series, so that their magnetic effects are in opposition, and moving the compensating coil until zero deflection is obtained. Then, with connections as in Fig. 161, R_x is given some suitable value and the resistance R_s adjusted until zero deflection of the galvanometer is obtained, when the resistance X is given by

$$X = \left(\frac{R_x + r_x + r_1}{R_s + r_s + r_2}\right) . S \qquad . \quad . \quad . \qquad (185)$$

Theory. When zero deflection of the galvanometer is obtained, the currents in the two galvanometer coils are equal. Let i be this current, and let I be the current passing through X and S.



(a) (b)

FIG. 162. KOHLRAUSCH METHOD OF USING THE DIFFERENTIAL GALVANOMETER

Then
$$i = \frac{IX}{R_x + r_x + r_1}$$
$$= \frac{IS}{R_s + r_s + r_2}$$

From which
$$\frac{X}{R_x + r_x + r_1} = \frac{S}{R_s + r_s + r_2}$$
$$X = \left(\frac{R_x + r_x + r_1}{R_s + r_s + r_2}\right) . S$$

It follows from this expression for X that the resistances of the connecting leads and of the two galvanometer coils must be known, or must be measured separately, although if R_x and R_s are fairly large the error introduced by neglecting the resistances of the leads will probably be small. The method has been displaced, for resistance measurements, by the Wheatstone bridge but the differential galvanometer is used, in conjunction with a portable bridge, for the location of faults in telephone cables.

Kohlrausch Method. This method, using a differential galvanometer, is more suited to precision measurements of resistance than the above. It eliminates errors arising from variation of the resistances of the galvanometer circuits due to contact resistances, etc.

The connections are shown in Fig. 162 (*a*), X being the unknown resistance and S a standard resistor approximately equal in resistance to X. This standard—assuming it to be larger than X—is shunted by a variable resistance R. P and Q are resistances in parallel in one of the galvanometer circuits, as shown, P being variable. Arrangements are made for reversing the battery connections to X and S by means of a specially designed switch or commutator. This has mercury contacts and is of heavy section to give low contact resistance. This arrangement is shown in Fig. 162 (*b*) and is used in conjunction with the circuit of Fig. 162 (*a*).

The galvanometer is first adjusted to give zero deflection. The resistances P and R are adjusted so that zero deflection is obtained with the switch in *both* of the positions (1) and (2).

Then $X = S'$ where S' is the shunted value of S and is given by $\frac{RS}{R+S}$. If X is shunted instead of S, then, at balance, $S =$ the shunted value of X.

If X and S are nearly equal the resistance R will be large compared with S and X, and its value need not be known with any high degree of precision.

The same relationship between X and S exists if, instead of zero deflection being obtained in both positions of the switch, the same deflection is obtained in both positions, the direction of the deflection being the same as well as its magnitude. This condition is obtained if the galvanometer is not adjusted exactly to zero initially, so that such exact adjustment is not necessary in this method.

(*d*) WHEATSTONE BRIDGE. This is the best and commonest method of measuring medium resistances. The general arrangement is shown in Fig. 163. P and Q are two known fixed resistances, S being a known variable resistance and R the unknown resistance. G is a sensitive D'Arsonval galvanometer shunted by a variable resistance N to avoid excessive deflection of the galvanometer when the bridge is out of balance. This shunt is increased as the bridge approaches balance, so that the shunting is zero—giving full sensitivity of the galvanometer—when balance is almost obtained. B is a battery of two or three cells and M is a reversing switch so that the battery connections to the bridge may be reversed and two separate measurements of the unknown resistance made in order to eliminate thermo-electric errors. K_B and K_G are keys fitted with insulating press-buttons, so that the hand does not come in contact with metal parts of the circuit, thus introducing thermo-electric e.m.f.s. The battery key, K_B, should be closed first, followed by the closing of K_G after a short interval. This avoids a sudden (possibly excessive) galvanometer deflection, due to self-induced e.m.f.s when the unknown resistance R has appreciable self-inductance.

At balance—obtained by adjustment of S—the same current i_1 flows in both of the arms P and Q, since the galvanometer takes no current, and the same current i_2 flows also in arms R and S.

Also, voltage drop across arm P = voltage drop across arm Q and voltage drop across arm R = voltage drop across arm S

Thus,

$$i_1 P = i_2 R$$

$$i_1 Q = i_2 S$$

By division

$$\frac{P}{Q} = \frac{R}{S}$$

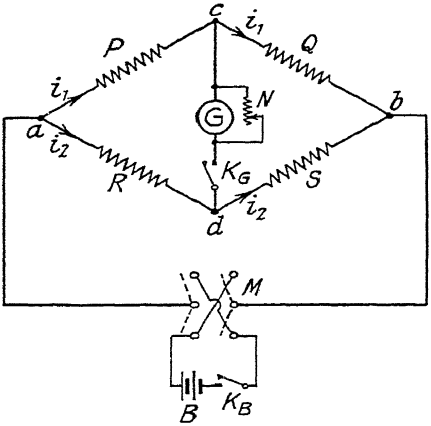


FIG. 163. CONNECTIONS OF WHEATSTONE BRIDGE

or

$$R = \frac{P}{Q} \cdot S \quad . \quad . \quad . \quad . \quad (186)$$

from which R is found in terms of P, Q, and S.

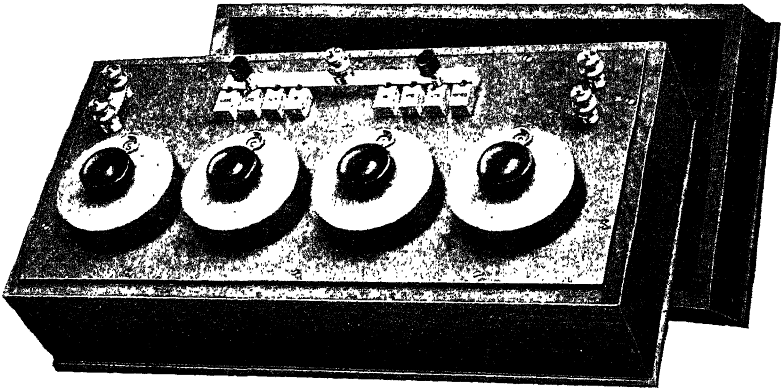
The arms P and Q are the "ratio arms" of the bridge and the ratio $\frac{P}{Q}$ may be varied as required to increase the range of the bridge.

In the elementary forms of the Wheatstone bridge the arms P and Q are replaced by a slide-wire of uniform cross-section whose resistance per unit length is constant. A scale is fitted under this slide-wire and a sliding contact—corresponding to the point c of Fig. 163—connects one terminal of the galvanometer to the wire.

S is a resistance of the same order of magnitude as the unknown. The sliding contact c is moved until zero deflection of the galvanometer is obtained. Then, since the slide-wire is of uniform cross-section, the ratio $\frac{P}{Q}$ is given by $\frac{\text{length of slide-wire } ac}{\text{length of slide-wire } cb}$, these lengths being obtained from the scale.

As before, $X = \frac{P}{Q} . S$

A precision form of Wheatstone bridge as manufactured by Messrs. Muirhead & Co. is shown in Fig. 164. Such forms of bridge contain either four or five pairs of ratio coils—tens, hundreds, thousands, and ten-thousands, in the bridge containing four pairs—and either four or five decades of resistance coils which constitute the variable arm S. These may be either of the plug pattern or sliding contact pattern. The internal connections of a Tinsley bridge are given, together with the external connections when the instrument is used in conjunction with a slide wire, in Fig. 165.



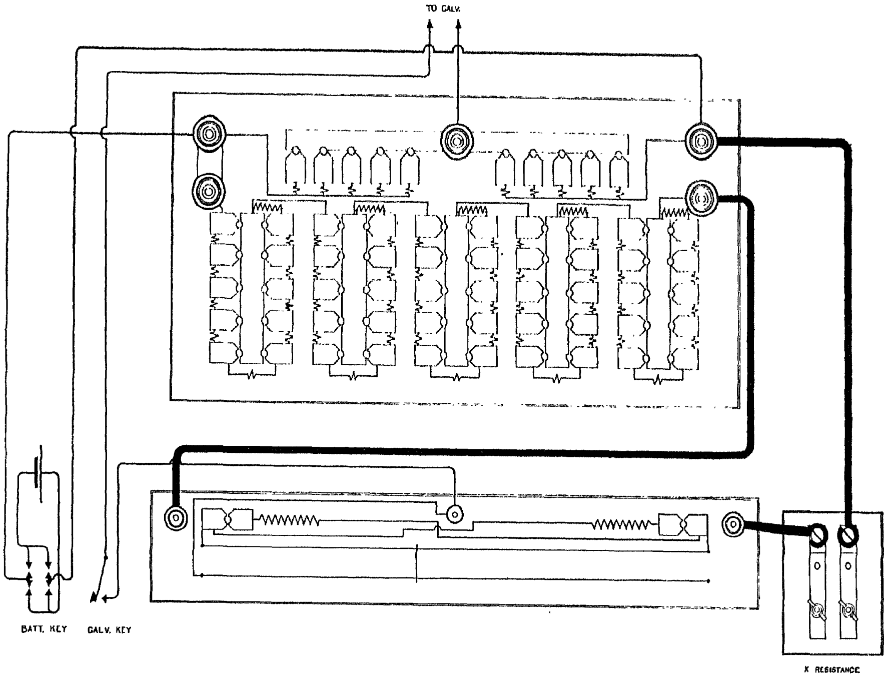
(*Muirhead & Co. Ltd.*)

FIG. 164. WHEATSTONE BRIDGE, SLIDING CONTACT PATTERN

Operation of the Bridge. The method of operation can best be illustrated by an example.

Suppose that the actual value of the resistance to be measured is 57·63 ohms and that a four-dial Wheatstone bridge—the dials containing units, tens, hundreds, and thousands—is to be used in conjunction with a galvanometer of ample sensitivity.

The bridge is connected up according to the arrangement shown in Fig. 163, care being taken to ensure that all the connections are firmly made and that all the plugs in the bridge blocks are firmly pressed home so that contact resistances may be small and definite. The galvanometer is at first heavily shunted and the ratio arms are made equal (each 10 ohms, say). The battery supply switch is closed. Assuming the magnitude of the resistance to be measured to be entirely unknown, first set the variable resistance arm, S, to some small value—say 1 ohm. Depress the key K_B and then lightly press the galvanometer key K_G (immediately raising it again if the galvanometer deflection is excessive). Note the direction of the galvanometer deflection—right or left. Next set the arm S to some high resistance—say 10,000 ohms—and again note the direction of the deflection obtained. If this direction is opposite to the previous one, the unknown resistance has some value between 1 and 10,000 ohms, being nearest in value to the setting of S which gives the



(*H. Tinsley & Co., Ltd.*)

FIG. 165. CONNECTIONS FOR SETTING UP 5-DIAL BRIDGE AND SLIDE WIRE

smaller deflection. Adjust the resistance S until approximate balance is obtained, and then remove the shunt from the galvanometer, in steps, adjusting S as required, until balance is obtained with the full galvanometer sensitivity. In the case under consideration ($R = 57{\cdot}63$), it will be found that the galvanometer deflects to the left (say) with S set at 57 and to the right with S set at 58, meaning that R lies between the two. This is the best that can be done with equal ratio arms.

To obtain greater accuracy, make Q 100 ohms, keeping P 10 ohms, at the same time altering S to 570. Then adjust S until approximate balance is obtained again. It will now be found that the balance point is between $S = 576$ and 577. Q should then be made 1,000 ohms (keeping P 10 ohms) and the process repeated. Final balance will be obtained when $S = 5{,}763$ ohms. Then

$$R = \frac{P}{Q} \,.\, S = \frac{10}{1000} \,.\, 5763 = 57{\cdot}63$$

The battery connections are then reversed by the switch M, and the measurement repeated. If it is found that an alteration of 1 ohm in S disturbs the balance, then the sensitivity is at least 1 part in 5,763. As a check on the balance it should always be ascertained that a slightly smaller value of S causes a galvanometer deflection to the left (say) and that a slight increase in S above the balance point causes a galvanometer deflection in the opposite direction.

If, with equal ratio arms at the beginning of the measurement, the unknown resistance does not lie within 1 and 10,000, the ratio arms must be adjusted until an approximate balance is obtained between these limits of S—e.g. if R is greater than 10,000, arm P must be made greater than Q, and if smaller than 1 ohm, Q must be made greater than P.

An accuracy of a few parts in 10,000 is usually obtainable with a good bridge of the type described above.

Best Galvanometer Resistance. The current through the galvanometer for a given change δR in the unknown resistance is given by—

$$i_g = \frac{i_2 \,.\, \delta R}{G + \left[\dfrac{(R+P)(Q+S)}{R+P+Q+S}\right]} \frac{Q+S}{R+P+Q+S}$$

where G is the galvanometer resistance.

Let $i_2\, \delta R \dfrac{Q+S}{R+P+Q+S} = K$, and $\dfrac{(R+P)(Q+S)}{R+P+Q+S} = A$

Then $$i_g = \frac{K}{G+A}$$

For given dimensions of the galvanometer coil

Number of turns on coil, N, $\propto \sqrt{G}$

$\therefore$ Deflecting torque $\propto Ni_g \propto i_g \sqrt{G}$

Hence, deflection for a given change δR, and current i_2 is

$$\theta \propto i_g \sqrt{G} \propto \frac{K\sqrt{G}}{G+A}$$

Differentiating we have

$$\frac{d\theta}{dG} \propto \frac{A-G}{2\sqrt{G}(G+A)^2} \text{ which is zero when } G = A$$

Thus, maximum deflection for a given change in the resistance R, and a given current i_2, is obtained when the galvanometer resistance is given by

$$G = \frac{(R + P)(Q + S)}{R + P + Q + S}$$

Obviously, the sensitivity may be increased, also, by increasing the current i_2. The galvanometer may be of the D'Arsonval type and should be as nearly critically damped as possible.

Precision Modifications of the Wheatstone Bridge. There are three principal types of such bridges—

(*a*) The slide-wire type.
(*b*) The shunt type.
(*c*) The Reichsanstalt type.

Space is available here for the description of only one of these types. Full descriptions of the other types may be found in the *Dictionary of Applied Physics*, Vol. II, p. 714.

Carey Foster Slide-wire Bridge. The connections of this bridge are shown in Fig. 166A, a slide-wire of length L being included between R and S as shown. This bridge is specially suited to the comparison of two nearly equal resistances.

Resistances P and Q are first adjusted so that the ratio $\frac{P}{Q}$ is approximately equal to the ratio $\frac{R}{S}$. Exact balance is obtained by adjustment of the sliding contact on the slide-wire. Let l_1 be the distance of the sliding contact from the left-hand end of the slide-wire. The resistances R and S are then interchanged and balance again obtained. Let the distance now be l_2.

Then, for the first balance, $$\frac{P}{Q} = \frac{R + l_1 r}{S + (L - l_1)r}$$

where r is the resistance per unit length of the slide-wire.

For the second balance,

$$\frac{P}{Q} = \frac{S + l_2 r}{R + (L - l_2)r}$$

Now, $$\frac{P}{Q} + 1 = \frac{R + l_1 r + S + (L - l_1)r}{S + (L - l_1)r} = \frac{R + S + Lr}{S + (L - l_1)r}$$

also $$\frac{P}{Q} + 1 = \frac{S + l_2 r + R + (L - l_2)r}{R + (L - l_2)r} = \frac{S + R + Lr}{R + (L - l_2)r}$$

Hence $$S + (L - l_1)r = R + (L - l_2)r$$

or $$S - R = (l_1 - l_2)r \quad . \quad . \quad . \quad . \quad . \quad (187)$$

Thus the difference between S and R is obtained from the resistance per unit length of the slide-wire together with the difference $(l_1 - l_2)$ between the two slide-wire lengths at balance.

The slide-wire is calibrated—i.e. r is obtained—by shunting either S or R by a known resistance and again determining the difference in length $(l_1' - l_2')$.

Suppose that S is known and that S' is its resistance when shunted by a known resistance, then

$$S - R = (l_1 - l_2)r$$

and

$$S' - R = (l_1' - l_2')r$$

$$\frac{S - R}{l_1 - l_2} = \frac{S' - R}{l_1' - l_2'}$$

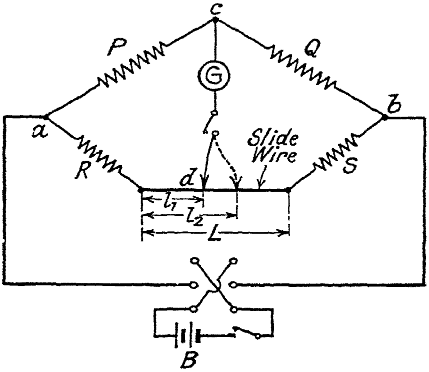


FIG. 166A. CAREY FOSTER SLIDE-WIRE BRIDGE

from which

$$R = \frac{S(l_1' - l_2') - S'(l_1 - l_2)}{(l_1' - l_2' - l_1 + l_2)} \quad . \quad . \quad . \quad (188)$$

From this expression it can be seen that this method gives a direct comparison between S and R in terms of lengths only, the resistances of P and Q, contact resistances, and the resistances of connecting leads being eliminated.

As it is important that the two resistances R and S shall not be handled or disturbed during the measurement, a special switch is used to effect the interchanging of these two resistances during the test.

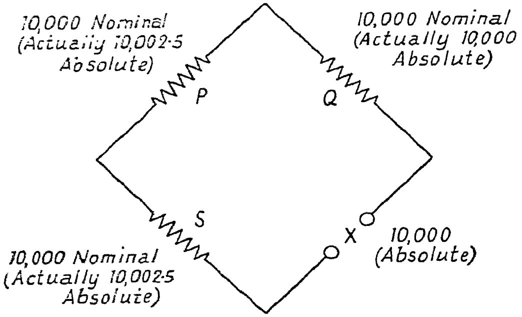
The Sullivan Wheatstone Bridge, which has an accuracy of 0·01 per cent, is designed (on a patented principle due to W. H. F. Griffiths) to measure with equal accuracy in either the new *Absolute* units or the older *International* units.

The principle is as follows—

Let P and Q be ratio arms each of four values, say, 10, 100, 1,000, 10,000 ohms, and let S be the standard (or rheostat) arms of four or five decades and X be the unknown resistance to be measured.

Q ratios are all adjusted to be correct to their nominal values in Absolute ohms.

P ratios are all adjusted to be $2{\cdot}5 \times 10^{-4}$ higher than nominal Absolute ohms, i.e. midway between Absolute and International ohms.


10,000 Nominal (Actually 10,002·5 Absolute)
P
10,000 Nominal (Actually 10,000 Absolute)
Q
S
10,000 Nominal (Actually 10,002·5 Absolute)
X
10,000 (Absolute)

166B

S is adjusted so that all its nominal values are midway between Absolute and International ohms, e.g. a nominal 1 ohm is actually 1·00025 ohm Absolute and is, therefore, 0·99975 ohm International.

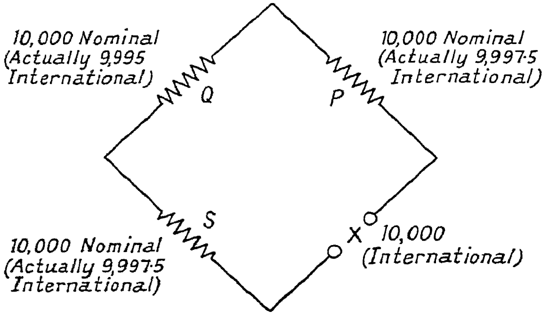

10,000 Nominal (Actually 9,995 International)
Q
10,000 Nominal (Actually 9,997·5 International)
P
S
10,000 Nominal (Actually 9,997·5 International)
X
10,000 (International)

FIG. 166C

When measuring in Absolute units, arms P and S are arranged to be adjacent as shown in Fig. 166B. Then, as an example, for the measurement of an X of 10,000 ohms (Absolute) in Absolute units with even ratios, we have the resistance values in the other three arms as shown.

It should be noted that the nominal values of all three "known" arms, P, Q and S, will be read as 10,000 notwithstanding their actual values (which are given in brackets) so that X will be measured simply as 10,000 ohms (Absolute). The balance equation in Absolute units is actually—

$$X = \frac{QS}{P} = \frac{10{,}000 \times 10{,}002{\cdot}5}{10{,}002{\cdot}5} = 10{,}000$$

When using the same bridge for measuring in International units the ratio arms P and Q are transposed so that Q and S are now adjacent as shown in Fig. 166c. This transposition is effected automatically upon connecting the resistance to be measured to the pair of "X" terminals appropriate to the system of units required. Then, for the measurement of 10,000 ohms (International) in International units we have the resistance values shown.

The same three settings of the known arms P, Q and S are used and their nominal values are all read as 10,000 ohms as before, so that X will be read as 10,000 ohms (International), although the balance equation, in International ohms, is in fact—

$$X = \frac{PS}{Q} = \frac{9997{\cdot}5 \times 9997{\cdot}5}{9995}$$

The value of X from this expression is 10,000 to an accuracy of 1 part in 10^7.

Applications of Resistance Measurements by Wheatstone Bridge. Several quite different quantities can be measured through the change which they bring about in the resistance of some form of element adapted to the particular measurement concerned and used in conjunction with a Wheatstone bridge to indicate this change.

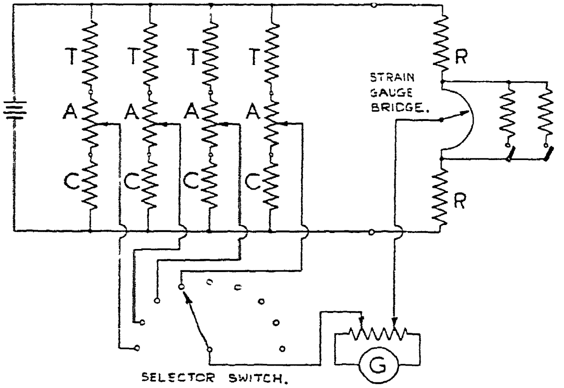
Examples in other chapters are the bismuth spiral (see p. 371) the resistance of which increases when it is placed in a magnetic field, the resistance thermometer (see p. 502) and the Ionic wind voltmeter (see p. 457).

Some important applications of resistance measurements which do not fall conveniently into the groundwork of other chapters are described below.

Strain Gauge Measurements. For the measurement of stresses and strains in structures and machines the technique of using electrical resistance strain gauges has been built up, particularly during World War II, in the aircraft industry. Thin copper-nickel alloy or nickel-chrome wires are used, these having a linear relationship—over the limited range required—between strain and electrical resistance. Nickel-chrome has the higher resistance but is less commonly used because it has some strain hysteresis and also a high temperature coefficient.

These gauges enable accurate measurements of strain to be made in any surface to which they can be cemented so that no slip occurs. They consist of grids of fine wires cemented to a thin paper membrane which is cemented, with nitro-cellulose cement, to the surface under test. The gauge is usually covered with a layer of felt to protect it and to reduce the effect of draughts. Particularly to eliminate electrolytic effects in the gauge, and also to avoid moisture absorption by the matrix, it is coated—when completely dry—with a protective wax such as Di Jell 171.

Care is necessary in preparing the surface to receive the strain gauge and in fixing the gauge itself and it is important to follow the manufacturer's instructions in this.



(*H. Tinsley & Co., Ltd.*)

FIG. 167. STRAIN GAUGE BRIDGE NETWORK

The gauge current may be as high as 25 mA for short periods but 10 mA is recommended for longer tests.

The ratio $\dfrac{\text{Percentage change of resistance}}{\text{Percentage strain}}$ is called the "gauge factor." The resistance change is greater than may be accounted for by the change in the wire dimensions and this factor is commonly about 2·2.

Sample gauges are made up and gauges are rejected if test results show a variation of more than ± 1 per cent from the mean.

In the measurements of strain, temperature effects are compensated for by using a dummy gauge which is connected in the appropriate arm of a Wheatstone bridge network (see Fig. 167). The dummy must be fixed to a piece of the same material as that under test but which is not, of course, under stress. The two gauges must be under the same conditions of ambient temperature.

The arrangement in Fig. 167, which is for static measurements, has a number of gauges *T*, each with its compensating gauge *C*. Zero balance is obtained by adjustment of the apex resistor *A* and calibration is by shunting one of the bridge arms *R* by a high resistance.

Measurement may be either by direct deflection, using a calibrated galvanometer, to measure the out-of-balance current, or by a null method through variation of the slide-wire setting. The selector switch shown must be free from thermo-electric effects. The slide wire is necessitated by the sensitivity required in such measurements, the change in gauge resistance being very small.

In some measurements all four arms of the bridge can be gauges mounted, for example, on the same specimen in tension and compression. Then, the zero adjusting and calibrating resistors can be located at a distance from the bridge.

For dynamic measurements, which are usually deflectional and use an oscilloscope or oscillograph, the bridge may be supplied at a carrier frequency from an ordinary a.c. amplifier having a flat response over the frequency range to be covered. Fig. 168 shows a bridge arrangement for carrier frequency operation. A capacitance balance is necessary in addition to a resistance balance and calibration and zero adjustment are effected by shunting the arms.

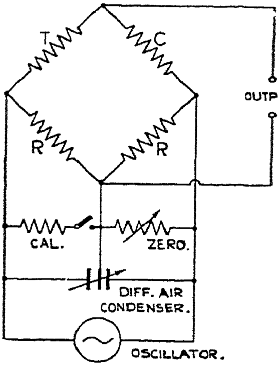


FIG. 168

Elliott Brothers (London), Ltd. have introduced a "ten-channel strain gauge equipment" to display, on one cathode-ray tube, the strain at ten different points in a test piece.

Fig. 169 shows the schematic layout and also the form in which the strain amplitudes and waveforms are displayed. Elliott publication No. G.121A gives a full description of the equipment.

The above is but a brief outline of the strain gauge technique. References (18), (19) and (20) deal with the subject comprehensively.

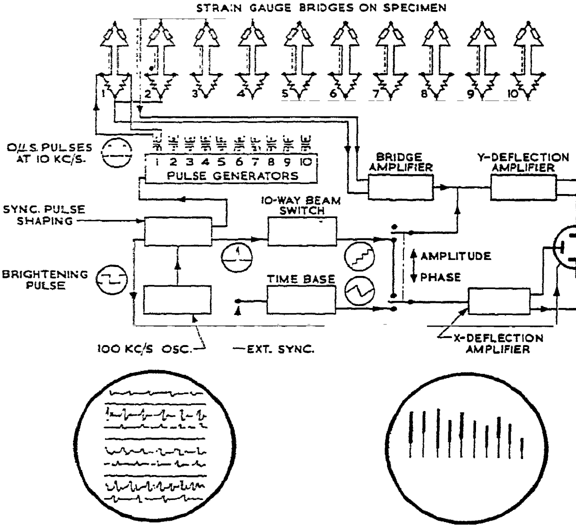
Electrical Weighing. H. I. Andrews (Ref. (21)) has described the application of the strain gauge principle to the measurement of weight or force in the testing of locomotives or trains under working conditions. For this purpose the apparatus must have reliability, rigidity, compactness and simplicity and must give reasonable accuracy during accelerations of several g with temperature changes up to 30° F.

In its development the aims were to obtain an increased output voltage and to enable robust measuring instruments to be used. The solution adopted was to employ, as the active element, a coil of enamel-insulated constantan wire wound directly on a round bar of nickel-chrome-molybdenum steel, or of mild steel or of duralumin. The coil is included in one arm of a Wheatstone bridge network and, as an example, with an applied voltage of 230 V and a coil on a mild steel bar stressed to 14 tons/in.^{2} an output voltage of 0·034 V could be obtained. Constantan wire gives a good strain sensitivity, maintains its calibration indefinitely and has a low temperature coefficient.

When the active coil, of resistance R, forms one arm of a bridge of which the other three arms are each of resistance R also, the output voltage is given by

$$v = \frac{V}{4} \cdot \frac{P}{A} \cdot K \cdot \frac{m}{E}$$

where V is the voltage applied to the bridge, P is the applied load, A is the cross sectional area of the bar, m and E are Poisson's ratio and Young's modulus, respectively, for the material of the bar and K is the sensitivity ratio which is the ratio between the proportional change of resistance of the wire and the circumferential strain in the bar.



(*Elliott Bros.* (*London*) *Ltd.*)

FIG. 169. TEN-CHANNEL STRAIN GAUGE EQUIPMENT

The Measurement of Pressure. Another application of similar principles is to the measurement of pressure. Thus, for example, N. Gross and P. H. R. Lane (Ref. (22)) have described an accurate wire-resistance method of measuring pulsating pressures up to 6,000 lb./in.^{2} As the pressure-sensitive elements, to replace the normal Bourdon tube types of gauge which are not sufficiently accurate for rapidly fluctuating pressures, cylindrical steel tubes with a wire winding are used. Two coils of 48 s.w.g. manganin wire are wound side-by-side on the tube. These are dried and sealed against moisture. The coils are connected in two diagonally opposite arms of a Wheatstone bridge network of which the detector is a mirror galvanometer having a sensitivity of 100 mm/μA at 1 metre, and a periodic time of 0·1 sec. The bridge is supplied from a battery.

The proportionate change $\frac{\Delta R}{R}$ in the resistance of the pressure cell coils is related to the pressure by the expression

$$\frac{\Delta R}{R} = K(D - t)^2(1 - \frac{1}{2}m) \cdot \frac{p}{2EDt}$$

where K is the constant relating electrical strain to mechanical strain, m and E are Poisson's ratio and Young's modulus for the material of the pressure tube, p is the pressure and D and t are, respectively, the diameter of the middle layer and the thickness of the coils.

For recording the pressure pulsations the light reflected from the galvanometer mirror is allowed to fall on to 35-mm recording paper in an oscilloscope camera, the lens of the camera being replaced by a metal disc with a narrow slit parallel to the direction of movement of the light.

The accuracy is stated as within 0·5 per cent of the full scale deflection.

Hot-wire Anemometer. In this instrument air velocity is measured by its cooling effect upon a fine wire which is heated by the passage of a small current through it. L. F. G. Simmons and A. Bailey (Ref. (26)) described on early anemometer of this type developed at the National Physical Laboratory, and Simmons and J. A. Beavan (Ref. (27)) gave an account of its use for the measurement and recording of gusts. For this purpose advantage was taken of the quick response of the hot wire to the cooling effect. Two elements, each included in its own Wheatstone bridge network, were used to record both the velocity and direction of gusts the detectors in the networks being oscillograph elements.

King's formula (Ref. (28)) for the cooling of fine wires was

$$H = (a + b\sqrt{V})(T - T_a)$$

where H is the rate of heat loss per unit length of a circular wire which is maintained at a temperature T in a steady wind of velocity V, and T_a is the air temperature.

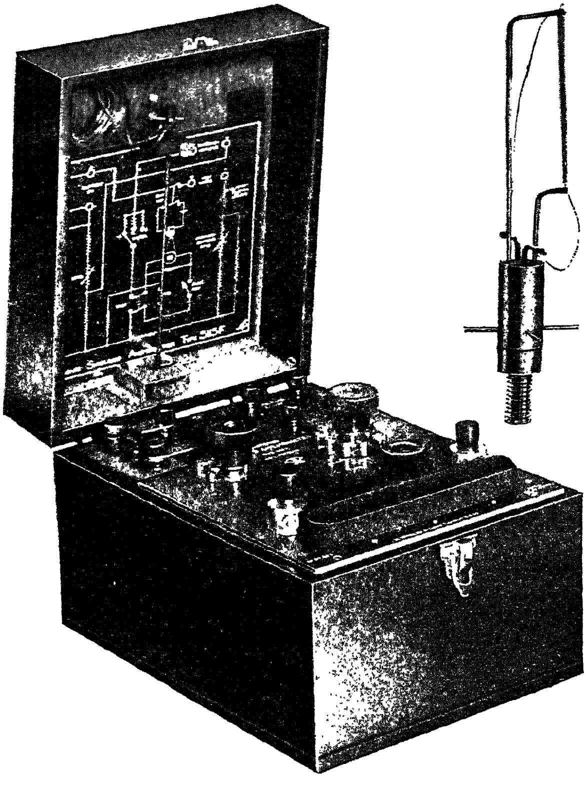
When current i flows in the wire the formula can be written

$$0{\cdot}24\, i^2r = (a + b\sqrt{V})(T - T_a)$$

The left-hand side represents the heat loss in calories per second; r is the resistance per unit length of wire: a and b do not remain constant over an appreciable range of temperature and each is expressible in the form $p + qT$ (Ref. (27)).

Later, Simmons described (Ref. (29)) a shielded form of hot-wire anemometer for low speeds, and an instrument of this type is now manufactured by H. Tinsley & Co.

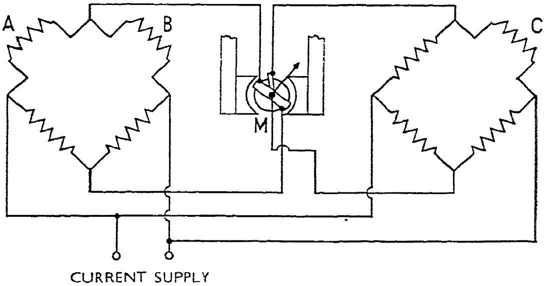
The hot-wire element, shown in Fig. 170, has a fine wire, about 3 cm long housed in one bore of a small twin-bore silica tube. The other bore contains a thermo-couple for the measurement of the hot-wire temperature. A standard heating current of 0·6 A is used and standardization is carried out by covering the element with a box and then adjusting the controlling resistors until the galvanometer used with the instrument gives a specified deflection. The standing e.m.f. of the thermo-couple is backed off by the zero setting control.



(*H. Tinsley & Co., Ltd.*)

Fig. 170. Simmons Hot-wire Anemometer

During the air-speed measurements the output from the thermo-couple is read on the galvanometer scale. Calibration of the scale is necessary because the deflection/air-speed relationship is non-linear. The Tinsley instrument has 3 ranges, for 0 to 0·4, 0 to 2·0 and 0 to 5·0 feet per second. L. L. Fox, P. L. Palmer and D. Whittaker (Ref. (30)) describe a compensating circuit for this shielded hot-wire anemometer. This overcomes the difficulty that small changes in the wire-heating current cause large errors in air flow measurement. A small e.m.f. is injected from the heater-wire circuit into the thermo-couple circuit so that the measured e.m.f. is largely independent of small current changes.



(*Elliott Bros. (London), Ltd.*)

FIG. 171. ELECTRICAL HUMIDITY METER

Electrical Humidity Meter. Yet another application of the Wheatstone bridge is the measurement of the humidity of the air by wet- and dry-bulb thermometers, the latter being of the electrical resistance type. Fig. 171 shows the circuit diagram of the Elliott Humidity meter. There are two, interconnected, Wheatstone bridges; one contains dry-bulb *A* and wet-bulb *B* thermometers and the other a second dry-bulb thermometer *C*. In the first, the galvanometer current is proportional to the temperature difference $T - T_1$ between the dry and wet bulbs while, in the second bridge, the galvanometer current is proportional to the dry-bulb temperature T. Instead of actual galvanometers two coils of a "cross-coil" indicator are used and the pointer reads the psychrometric difference $T - T_1$ corrected by the dry-bulb temperature T, i.e. the instrument reads the *relative humidity*.

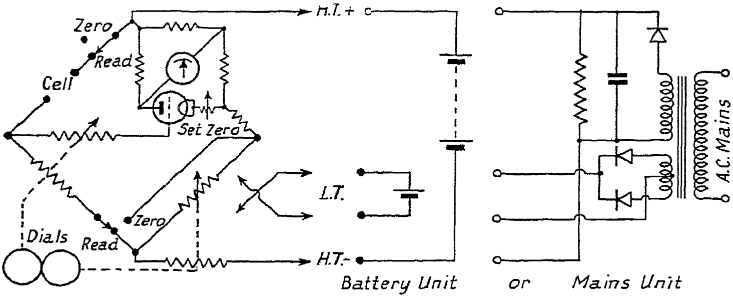
The three thermometers are mounted in a "humidity transmitter" through which the air under test is drawn, first over the two dry bulbs and then over the wet bulb.

The meter is independent of voltage variations in the supply since both bridges are supplied from the same source. A.C. mains can be used with a transformer and rectifier.

Moisture Meter. The principle of the Marconi moisture meter, designed for the measurement of the moisture content of grain and other hygroscopic materials, is shown in Fig. 172.

The circuit used (Brit. Patent No. 635, 674) is an extension of the Wheatstone bridge principle, there being a subsidiary bridge network in the arm opposite to that containing the sample under test. Out-of-balance voltage reduces or increases the resistance of the triode which, in turn, alters the effective resistance of the subsidiary bridge network. This self-balancing feature is included to render the reading of the meter independent of valve characteristics and supply voltage variations.

The sample is contained in a test cell in which a great mechanical pressure is applied to the material to bring it to a uniform state and eliminate packing errors.



(*Marconi Instruments, Ltd.*)

FIG. 172. MARCONI MOISTURE METER

The range of moisture content covered corresponds to equilibrium with atmospheres having relative humidities from about 20 per cent to 90 per cent and the accuracy is ± 1 per cent m.c.

Marconi Instruments, Ltd. also make a moisture-in-timber meter though the construction and principle of operation are different from those of the meter just described.

Measurement of High Resistance. When the resistance to be measured is of the order of one or more megohms, the methods of measurement described in the foregoing pages are unsuitable. In such cases the resistance offered to the passage of current along the surface of the insulation is often comparable with the resistance itself, and special methods have to be adopted to take such "surface leakage" into account.

Amongst the high-resistance measurements which are required to be made in practice those of the insulation resistance of cables are very important. The absorption effects in dielectrics have already been mentioned in Chapter IV, and such effects are apt to destroy the value of insulation resistance measurements unless special precautions are taken.

A simple method of measuring insulation resistance is the direct deflection method. A very sensitive moving-coil galvanometer of high resistance (1,000 ohms or more) is connected in series with the resistance to be measured, and to a battery supply. The deflection of the galvanometer gives a measure of the insulation resistance. This method is, however, only sufficient to indicate whether the insulation is faulty or otherwise, and cannot be regarded as a precise method.

PRICE'S GUARD-WIRE METHOD. Fig. 173A gives the connections for a direct deflection method in which the errors due to surface leakage are eliminated by the use of a guard wire. In the figure the resistance to be measured is the insulation resistance of a length of cable. The cable is immersed in water, which is contained in a metal tank, 24 hours being allowed to elapse—the temperature meanwhile being maintained constant—before the test is carried out. This enables the water to soak through any defects which may exist in the insulation, and also allows the insulation to attain the same temperature as the water. The ends of the cable are trimmed as shown, the outer protective covering being removed at these ends for a length of at least 12 in. A bare wire, twisted round the insulation near the end, is connected to the negative pole of the supply battery—the positive pole of which is connected to the metal tank—so that any current which leaks across the insulation surface is taken direct to the battery instead of passing through the galvanometer, and increasing its deflection. The galvanometer

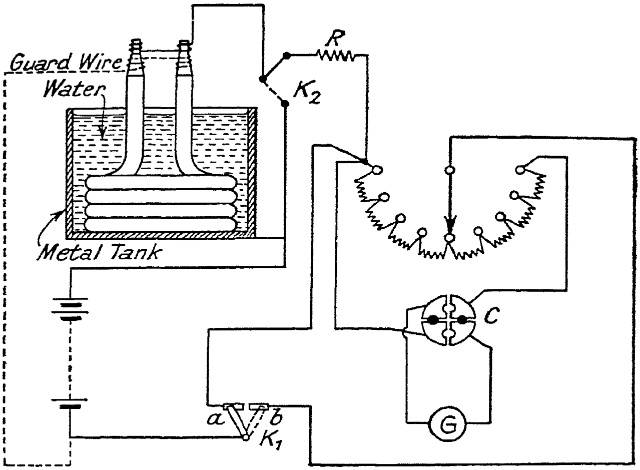


FIG. 173A. PRICE'S GUARD-WIRE METHOD OF MEASURING HIGH RESISTANCE

is shunted as shown, the shunt being of the Ayrton universal type.

The deflection of the galvanometer is observed, and its scale is afterwards calibrated by replacing the insulation resistance by a standard high resistance (usually 1 megohm), the galvanometer shunt being varied, as required, to give a deflection of the same order as before. The galvanometer, which is of the D'Arsonval type, should be very sensitive (at least 1,000 mm per microampere at a scale distance of 1 metre), should have high resistance, and, also, its deflection should be directly proportional to the current flowing through it. The resistance of the universal shunt across the galvanometer may be so chosen that the galvanometer is critically damped, thus saving time in making observations.

The battery should be of about 500 volts, and its e.m.f. should be constant. C is a four-part commutator for reversal of the galvanometer connections. R is a protective resistance of about 100,000 ohms in series with the galvanometer. K_1 is a key which is closed on contact "*a*" when the battery is first switched on. The galvanometer is thus protected from the sudden initial rush of current which charges the cable—the latter acting, of course, as a capacitor. Contact "*b*" is sufficiently close to "*a*" for the circuit to remain closed when the key is being moved over. K_2 is another key for the purpose of discharging the capacitance of the cable.

The galvanometer and its circuit, together with the keys, must be well insulated to prevent leakage currents.

Fig. 173B shows the connection diagram for insulation testing when a universal shunt on the Kelvin Varley slide principle (see p. 315) is used. The resistance of this (Sullivan) four-dial shunt is variable in steps of 1/10,000th part of the whole.

LOSS OF CHARGE METHOD. In this method the insulation resistance to be measured is connected in parallel with a capacitor and electrostatic voltmeter. The capacitor is charged, by means of a battery, to some suitable voltage, and is then allowed to discharge through the resistance, its terminal voltage being observed over a considerable period of time during discharge.

Then, assuming the capacitor to be perfect, if V is its terminal voltage at any time t, Q being the charge, in coulombs, still remaining in the capacitor, and C its capacitance, we have for the current i at time t,

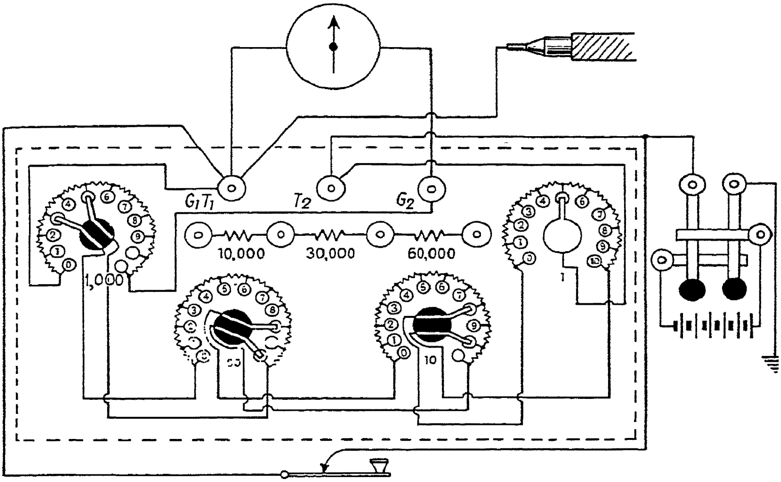
$$i = -\frac{dQ}{dt} = -C\frac{dV}{dt}$$

But $i = \frac{V}{R}$ where R is the resistance to be measured, and through which the capacitor is discharging.

$$\therefore \frac{V}{R} = -C\frac{dV}{dt}$$

or $$\frac{V}{R} + C\frac{dV}{dt} = 0$$

Solving this differential equation for V gives $V = E\varepsilon^{\frac{-t}{CR}}$ where E is the voltage when $t = 0$ (i.e. the voltage to which the capacitor was originally charged), and ε is the base of Napierian logarithms.



(*H. W. Sullivan, Ltd.*)

FIG. 173B. FOUR-DIAL SHUNT FOR USE IN INSULATION TESTING

The box contains an independent and standard resistance of 100,000 ohms (between the inner line of terminals), which is used for taking "constants" in direct deflection tests, as also in the fall of potential method of localizing faults in a cable, where the potentials are measured by direct deflection, the galvanometer being earthed through this high resistance. In addition, the high resistance is specially subdivided into three sections, 10,000 ohms, 30,000 ohms, and 60,000 ohms, for conveniently observing earth current readings in Schaefer's test for locating breaks and faults in submarine cables.

Thus $$\log_\varepsilon V = \log_\varepsilon E - \frac{t}{CR}\log_\varepsilon \varepsilon = \log_\varepsilon E - \frac{t}{CR}$$

or $$R = \frac{t}{C\log_e \frac{E}{V}} = \frac{0{\cdot}4343t}{C\log_{10}\frac{E}{V}} \quad . \quad . \quad . \quad (189)$$

R will be given in ohms if t is in seconds and C in farads.

Example. If $C = 0{\cdot}2$ microfarad, $E = 400$ volts, and the time taken for the capacitor terminal voltage to fall to 250 volts is $1\frac{1}{2}$ min.,

$$\frac{0{\cdot}4343 \times 90}{\frac{0{\cdot}2}{10^6} \quad \frac{400}{250}} = 957{\cdot}6 \times 10^6 \text{ ohms or } 957{\cdot}6 \text{ megohms.}$$

If the resistance R is very large, the time for an appreciable fall in voltage is also large. The voltage-time curve will thus be very flat, and, unless great care is taken in measuring the voltages at the beginning and end of a given time t, a serious error may be made in the value of the ratio $\frac{E}{V}$, causing a corresponding error in the measured value of R. More accurate results may thus be obtained by measuring the *change* in the voltage $(E - V)$ directly. Calling this change v, the expression for R then becomes

$$R = \frac{0{\cdot}4343t}{C \log_{10} \frac{E}{E - v}}$$

Laws (Ref. (2)) gives details of a method of measuring this change in voltage v, using a ballistic galvanometer.

If the insulation resistance of a length of cable is being measured, the cable itself may be used as the capacitor, but its capacitance must be either known or must be determined.

Several serious difficulties are encountered when this method is used. Owing to absorption effects in the dielectric under test, the current actually flowing through the resistance is not simply dependent upon its resistance, since an absorption current also flows into the insulation. For this reason observations should be continued for a long period—several hours—if the results of such measurements are to be of value. Again, the insulation resistance of the voltmeter and of the capacitor must be exceedingly high in order that leakage effects shall be negligible.

Effect of the Time of Electrification. As absorption effects are present to a greater or less degree in all insulation resistance measurements, it is important that their influence upon the measured value of the resistance should be considered. In Fig. 81 a curve showing the variation of the absorption current in a capacitor with time was given. This current falls away fairly rapidly at first, the decrease thereafter being more gradual. Evershed (Ref. (5)) found, in the case of a length of rubber-covered cable, that the absorption current was some five or six times the true leakage current—dependent upon the resistance only—after a time of application of voltage of 1 min., and was about equal in value to the true leakage current after 7 min. After the voltage had been applied for 6 or 7 hours, the absorption current still formed some 5 to 10 per cent of the total current flowing through the insulation. From these results it is obvious that the insulation resistance as defined by

$$\frac{\text{Applied voltage}}{\text{Current flowing through the insulation}}$$

depends very much upon the time of application of the voltage. In commercial testing a time of application of 1 min. is normally specified, but the resistance value so obtained will be very much less than the true value. The magnitude of the insulation resistance itself will influence the effect which the absorption has upon the measured value. If this resistance is low, the absorption current may be negligible compared with the true leakage current, but the former will be of much greater importance in the case of very high resistance insulation.

Effect of Temperature upon Insulation Resistance. The resistance of insulating materials, generally, falls with increase of temperature, the change in resistance being, in some cases, very rapid. For this reason it is important in such measurements that the result should be stated together with the temperature at which the test was carried out.

Koenigsberger and Reichenheim's formula (Ref. 6) for the variation of the resistance of hard insulating materials with temperature is as follows—

$$R_t = R_0 \varepsilon^{\frac{-kt}{(273 + t)\,273}}$$

where R_t is the insulation resistance at temperature t° C.
R_0 ,, ,, ,, ,, 0° C.
k is a constant for any given material.
ε is the base of Napierian logarithms.

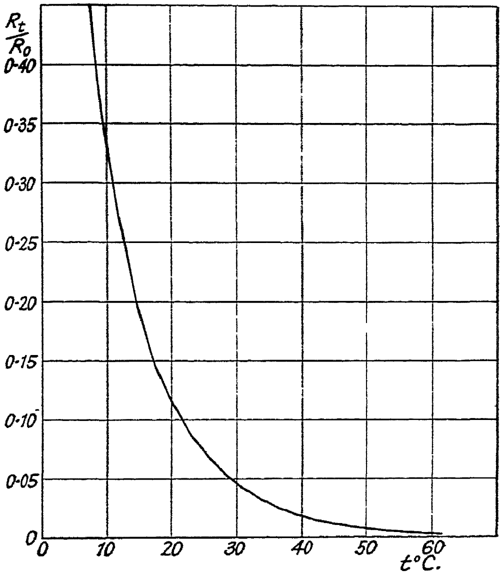


FIG. 174. RESISTANCE-TEMPERATURE CURVE FOR RED FIBRE

Fig. 174 shows the curve of R_t (expressed as a fraction of R_0) against temperature, for red fibre, for which $k = 8{,}477$. The very great reduction of resistance with increase of temperature can be observed. Curtis (Ref. (7)) gives a table of values of the resistivity at 30° C., and of the ratio $\frac{\text{resistivity at } 20^\circ\text{ C.}}{\text{resistivity at } 30^\circ\text{ C.}}$ for a number of hard insulating materials. The value of this ratio varies from 1 for India ruby mica to 16·0 for yellow beeswax.

Measurement of Resistance of Specimens of Insulating Materials. SURFACE AND VOLUME RESISTIVITY. In the above tests the resistance to be measured has been assumed to be that of the insulation of a length of cable. The direct deflection method is often applied to measurements of the resistance of insulating materials, small samples of the material, in sheet form, being used. In such cases it is necessary to distinguish between the "*surface resistivity*" and the "*volume resistivity*," or specific resistance of the material.

The "*surface resistivity*" is defined as the resistance between opposite edges of a unit square area of the surface of the material. This quantity depends upon the general condition of the surface and upon the humidity, and is not a constant.

The form of specimen used for such measurements is shown in Fig. 175. The sample rests in a pool of mercury to which the negative pole of the battery is connected, an annular groove containing mercury forming the other electrode. There is a pool of mercury also inside the specimen, and a wire from this is taken to the positive side of the battery as shown, being connected so as to form a guard wire to prevent the leakage current which actually passes through the body of the specimen from passing through the galvanometer.

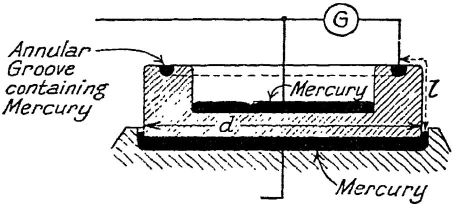


FIG. 175. FORM OF SPECIMEN FOR SURFACE RESISTIVITY MEASUREMENTS

Then, if R is the measured value of the resistance, the surface resistivity is given by $\frac{R \times \pi d}{l}$ where d is the diameter of the specimen and l is the length of the surface path from the annular groove to the outer mercury pool (see figure).

This arrangement and shape of specimen are recommended in the British Electrical and Allied Industries Research Association Report (Ref. (11)). A number of such reports, dealing with the measurement of volume and surface resistivity of insulating materials and the effect of temperature humidity, etc., upon these quantities, have been published (Refs. (8) to (13)). These reports give very full directions for the study of many different forms of insulating materials—vulcanized fibre, hard composite dielectrics, unvarnished textile fabrics, insulating oils, papers, etc., and also lay down the conditions under which the various tests should be made. Recommendations for testing procedures are also given in British Standard specifications dealing with various classes of insulating materials.

Portable Resistance Testing Sets. A portable, and reasonably accurate, form of testing set is often necessary in order that insulation tests may be made on cables and wiring systems after installation. A number of such sets have been developed and are manufactured by various instrument makers. Most of these sets are modifications of the ohm-meter originally designed by Ayrton and Perry. The principle of this instrument is illustrated by Fig. 176.

Two coils, *C* and *P*, are fixed at right angles to one another, and so that their magnetic fields—when current flows through them—both exert a turning moment upon the pivoted magnetic needle *M*, to which a pointer is attached. *SS* are the supply terminals, the supply usually being obtained from a small generator, giving about 500 volts, which is turned by hand. *P* is the pressure coil and is connected, in series with a resistance, across the supply terminals. *C* is the current coil, connected in series with the resistance *X*, to be measured. This coil carries a current which is inversely proportional to the resistance *X*.

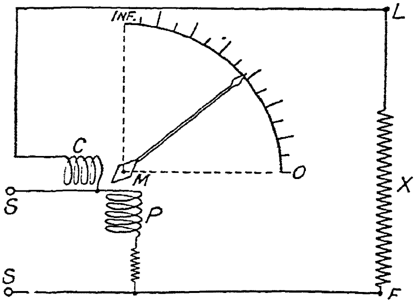
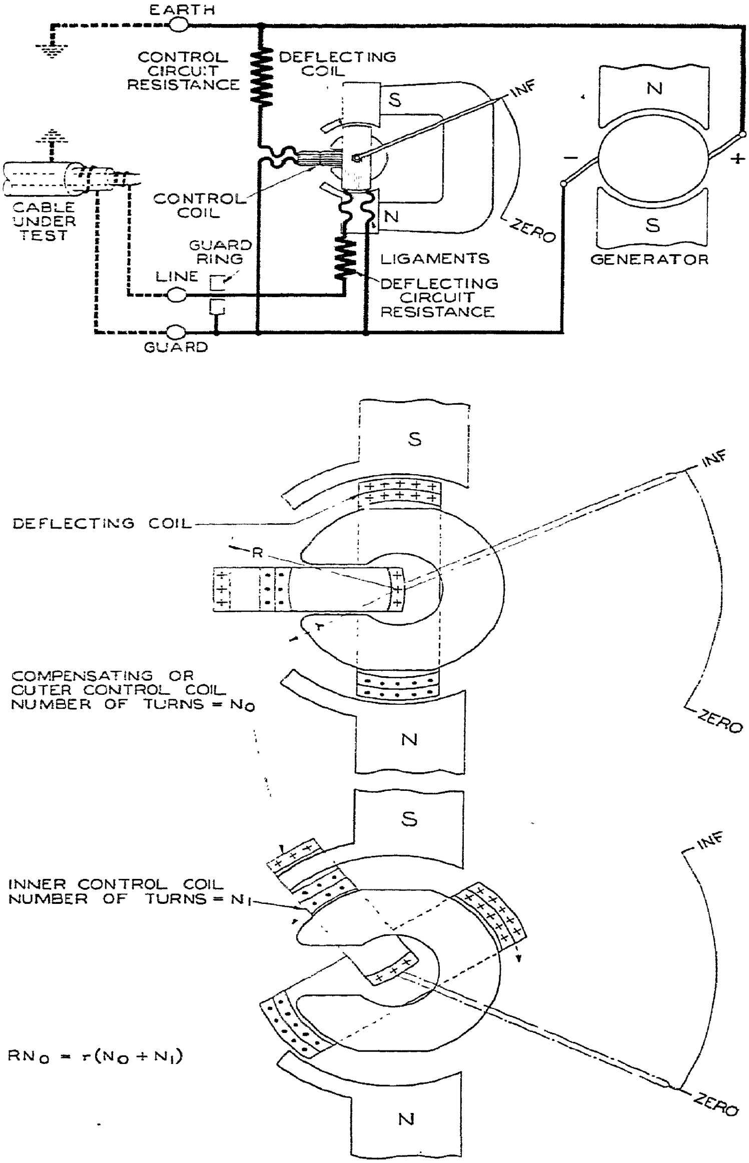


FIG. 176. CONNECTIONS OF SIMPLE AYRTON AND PERRY OHMMETER

The current in coil *P*, which is directly proportional to the supply voltage, is fixed, and is independent of the resistance to be measured. The magnetic field of this coil tends to turn the needle in an anti-clockwise direction, while the field of coil *C* tends to cause clockwise rotation.

The balance position of the needle is such that these two turning moments are equal. If *X* is infinite, there is no current in *C*, and the needle sets along the axis of coil *P*, whereas if the resistance *X* is very low, the turning moment of *C* is far greater than that of *P*, and the needle sets along the axis of *C*. The scale is graduated in resistance values (usually megohms), the intermediate points between infinity and zero being obtained by calibration.

The commonest of the more modern testing sets is the "Megger" insulation tester, manufactured by Messrs. Evershed and Vignoles, the construction and connections of which are shown in Fig. 177. The moving system consists of two coils—the "control coil" and the "deflecting coil"—rigidly mounted at an angle to one another and connected, in parallel across a small generator, with polarities such


EARTH
CONTROL CIRCUIT RESISTANCE
DEFLECTING COIL
S
INF
N
−
+
S
CABLE UNDER TEST
CONTROL COIL
N
ZERO
GUARD RING
LINE
LIGAMENTS
GENERATOR
DEFLECTING CIRCUIT RESISTANCE
GUARD
S
INF
DEFLECTING COIL
R
r
COMPENSATING OR OUTER CONTROL COIL NUMBER OF TURNS = N_0
ZERO
N
S
INF
INNER CONTROL COIL NUMBER OF TURNS = N_1
$RN_0 = r(N_0 + N_1)$
ZERO
N

(Evershed & Vignoles, Ltd.)

FIG. 177. INTERNAL CONNECTIONS OF THE MEGGER

that the torques produced by them are in opposition. The coils move in the air gap of a permanent magnet. The control coil is in series with a fixed control circuit; the deflecting coil is connected in series with a fixed deflecting circuit resistance and the resistance under test. If this last is infinitely high no current flows in the deflecting coil and the control coil sets itself perpendicular to the magnetic axis, the pointer indicating "Infinity." A lower test resistance allows current to flow in the deflecting coil and turns the movement clockwise. The control torque produces a restoring torque which progressively increases with the angular deflection, and the equilibrium position of the movement is attained when the two opposing torques balance.

The control coil is actually in two parts, in series, the outer part being a compensating coil. The two parts are arranged with numbers of turns and radii of action such that, for external magnetic fields of uniform intensity, their torques cancel one another thus giving an astatic combination.

The instrument has a small permanent magnet d.c. generator (developing 100, 250, 500, 1,000 or 2,500 V in different instruments). This may be hand-driven, through gearing and a centrifugally controlled clutch which slips at a predetermined speed so that a steady voltage can be obtained, or it may be motor-driven.

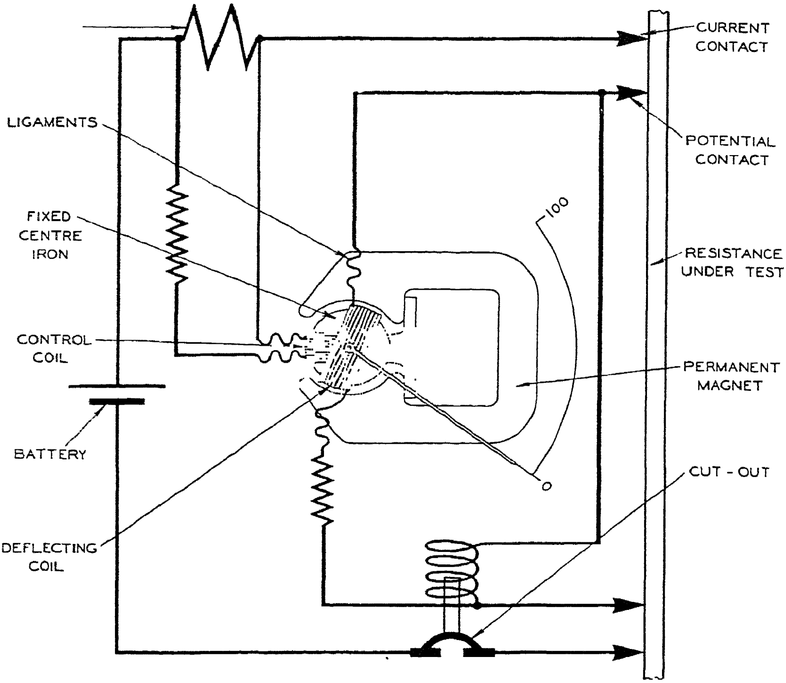
The "Bridge-Meg" and "Bridge-Megger" testing sets, also made by Evershed and Vignoles, Ltd., can be used for insulation resistance measurements, as a Wheatstone bridge for a wide range of resistance values or for fault localization by the Varley loop method (see p. 491). Variable ratio arms and a four-decade resistance are included for these uses.

Two other instruments made by the same manufacturers should be mentioned. The first is the "Ducter" ohmmeter, a low-resistance testing set for resistances from a few ohms down to one microhm, and the other is the "Megger Capacity Meter."

In the former, the connections for which are shown in Fig. 178, the control coil is connected across a shunt in the main circuit, so that the torque produced by it is proportional to the current in the resistance under test, while the deflecting coil is connected across this resistance and so carries a current proportional to the potential drop. The position taken up by the moving system is thus dependent on the ratio of potential drop and current, i.e. on the resistance being measured.

The capacity meter resembles the Megger Insulation Tester in general form but contains a hand-driven alternating current generator and an a.c. ratiometer movement calibrated in microfarads. It compares the capacitance under test with a standard capacitance and covers a range from 0 to 0·1 μF up to 0 to 10 μF.

Measurement of Insulation Resistance when the Power is On. It may be necessary, in some cases, to measure the insulation resistance to earth of a distribution system, while the power is on. Such a measurement may be made as follows. The voltage E between the two mains—positive and negative—is measured, together with the voltage V_1 from the positive main to earth, and the voltage V_2 from the negative main to earth. These measurements are made with a high resistance voltmeter whose resistance R_v should be comparable with the insulation resistances to be measured.



(*Evershed & Vignoles, Ltd.*)

FIG. 178. EVERSHED'S "DUCTER" OHMMETER

Let R_1 = resistance between + ve main and earth

Fig. 179 (*a*) shows, diagrammatically, the system when the voltmeter is connected between the positive main and earth, and Fig. 179 (*b*) the system with the voltmeter between the negative main and earth.

If I_1 is the current flowing from the positive main to the negative main, through R_2 and R_1—the latter being in parallel with R_v—in the first case we have

$$V_1 = \frac{R_1 R_v}{R_1 + R_v} \cdot I_1$$

and $$E - V_1 = R_2 I_1$$

Thus, $$\frac{E - V_1}{V_1} = \frac{R_2}{\dfrac{R_1 R_v}{R_1 + R_v}} \text{ or } \frac{E}{V_1} = \frac{R_1 R_2 + R_v(R_1 + R_2)}{R_1 R_v}$$

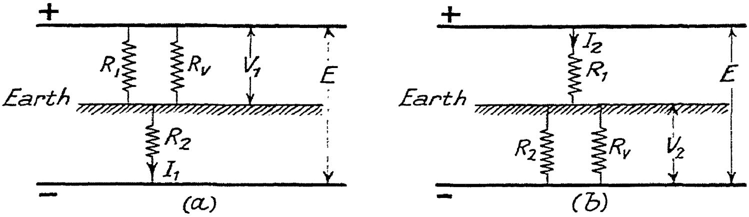


FIG. 179. MEASUREMENT OF INSULATION RESISTANCE WHEN THE POWER IS ON

By similar reasoning, in the second case we have

$$\frac{E}{V_2} = \frac{R_1 R_2 + R_v(R_1 + R_2)}{R_2 R_v}$$

Hence, $$\frac{\dfrac{E}{V_1}}{\dfrac{E}{V_2}} = \frac{\dfrac{R_1 R_2 + R_v(R_1 + R_2)}{R_1 R_v}}{\dfrac{R_1 R_2 + R_v(R_1 + R_2)}{R_2 R_v}} = \frac{R_2}{R_1} = \frac{V_2}{V_1}$$

Substituting $R_2 = R_1 \cdot \dfrac{V_2}{V_1}$ in the expression

$$\frac{E}{V_1} = \frac{R_1 R_2 + R_v(R_1 + R_2)}{R_1 R_v}$$

we have $$\frac{E}{V_1} = \frac{R_1 \dfrac{V_2}{V_1}(R_1 + R_v) + R_1 R_v}{R_1 R_v} = \left(\frac{R_1 + R_v}{R_v}\right)\frac{V_2}{V_1} + 1$$

from which $$R_1 = \left[\frac{E - (V_1 + V_2)}{V_2}\right] R_v \quad . \quad . \quad . \quad (190)$$

Similarly, $$R_2 = \left[\frac{E - (V_1 + V_2)}{V_1}\right] R_v \quad . \quad . \quad . \quad (191)$$

This method cannot be used if one of the mains is earthed, and is generally only applicable if the insulation resistances to be measured are not more than 1 or 2 megohms.

The Measurement of the Resistance of Earth Connections. The resistance between an earthing plate and the surrounding ground is often an important quantity in distribution systems. It is usually measured by a fall-of-potential method as illustrated by Fig. 180.

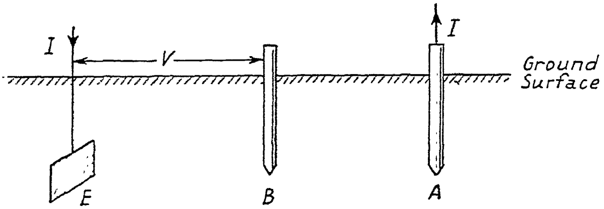


FIG. 180

A current is passed through the plate E to an auxiliary electrode A in the earth at a distance away from the plate. A second auxiliary electrode B is inserted between E and A and the potential difference V, between E and B is measured for a given current I so that the resistance of the earth connection is $\frac{V}{I}$. The placing of the auxiliary electrodes is, however, important and G. F. Tagg (Ref. (31)) has called attention to the errors which can arise from incorrect placing of these electrodes, especially when the earth resistance is low. He gives a curve, as in Fig. 181, for the resistance measured when B is at various distances from E. The correct value for the resistance of the earth connection is that measured (R_E) when B is at such a distance that the resistance lies on the horizontal part of the curve. Tagg points out that, when the earthing resistance is low the spacing

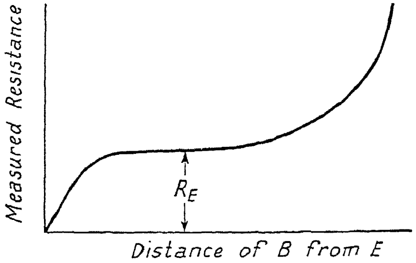
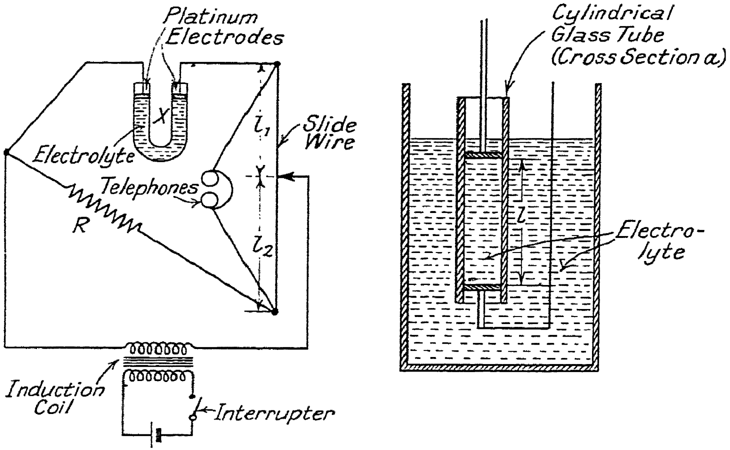


FIG. 181

between the earth plate and auxiliary electrodes may need to be hundreds of feet.

Following these principles, Messrs. Evershed and Vignoles, Ltd. make a "Megger Earth Tester" containing a direct-reading ohmmeter and a hand-driven generator.

Measurement of Resistance of Electrolytes. Owing to the fact that a polarization e.m.f. is produced whenever a current passes through an electrolyte, the usual methods of measuring resistance cannot be used to measure the resistance of electrolytes.



(a) (b)

FIG. 182. MEASUREMENT OF THE RESISTANCE OF ELECTROLYTES

Kohlrausch devised a method of measuring their resistivity. The electrolyte is contained in a glass tube having two platinum electrodes dipping into it. This cell is connected in one arm of a Wheatstone bridge network as shown in Fig. 182 (*a*). The bridge is of the slide-wire form, and is supplied from an induction coil, a telephone being used as the detector. The slide-wire may be of special form consisting of a long manganin wire wound spirally in a groove cut in a marble cylinder, the cylinder being stationary and the contact —of hard steel mounted in a manganin rod to avoid thermo-electric e.m.f.'s—sliding round the cylinder. The spindle carrying the contact has a thread of the same pitch as that of the groove in which the slide-wire lies.

R is a known resistance of the same order as that of the electrolyte. Balance is obtained by adjusting the sliding contact until no sound can be detected in the telephone.

Then, if l_1 and l_2 are the two lengths into which the slide-wire is divided by the sliding contact, the electrolyte resistance X is given

$$X = \frac{l_1}{l_2} R \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

If the resistivity of the electrolyte is to be measured it is best to use a cylindrical glass tube, of uniform cross-section, supported vertically in a vessel containing the electrolyte, and with its upper end above the surface of the liquid (Fig. 182 (*b*)). The electrodes should be of platinum and should be circular, fitting tightly inside the cylinder. The lower electrode may be pierced to allow liquid to flow through it, and the glass tube may be graduated so that the length of the column of electrolyte between the two electrodes can be accurately determined. Then, if a is the cross-sectional area of the column of electrolyte and l is its length, the resistivity is given by $\frac{Xa}{l}$ where X is the measured resistance of the column.

The temperature should be carefully observed when making such measurements, and this temperature stated when the results are given.

Measurement of Water Purity. Closely connected with the measurement of the resistance of electrolytes is that of the purity of water in terms of its electrical conductivity.

The Evershed and Vignoles "Dionic" Water Purity Meters (or Electric Salinometers) are calibrated in conductivity units taking as unity water having a resistance, at 20° C., of 1 megohm between the opposite faces of a centimetre cube. Then, water having a resistance of $\frac{1}{2}$ megohm per cm.^{3} has a conductivity of 2 and so on. These conductivity measures can be interpreted in terms of water purity from known data concerning the influence of various concentrations of salt upon conductivity.

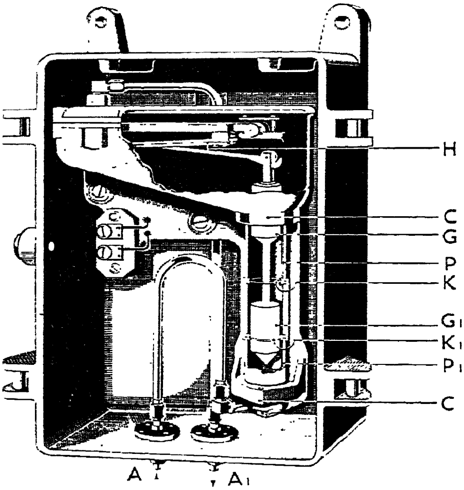
For example, at 20° C

0·823 grain of common salt per gallon gives conductivity	23 units
5 grains per gallon give conductivity	139 units
7 grains per gallon give conductivity	192 units

The apparatus consists of a water conductivity tube (see Fig. 183) which is used in conjunction with an indicating (or recording) ohmmeter having a moving system of two rigidly fixed and magnetically opposing coils moving in a permanent magnet field as in the "Megger Insulation Tester" (see p. 305). The apparatus is, however, operated from a d.c. supply.

Allowance is made in the calibration for the back e.m.f. and there is an automatic device to compensate for changes in temperature over the range 60° to 160° F. This device consists of plungers, G and G_1, which are lowered into the water tubes, P and P_1, by the bimetal levers H. This varies the effective cross sections of the water columns under test.

These water columns are contained in two insulating tubes, P and P_1, which are separated by a third insulating tube containing two platinum electrodes K and K_1. The testing current flows upwards through P and downwards through P_1 and the columns of water are of accurately known cross section. Electrically the columns are in parallel because the electrodes K, K_1 are at the same positive potential, the negative electrode being the gunmetal case C, C which carries the insulating tubes. (This case is of course insulated from the case of the instrument itself and from the water supply.) The water flows through the tube continuously, entering at A and discharging at A_1.



(*Evershed & Vignoles, Ltd.*)

FIG. 183. "DIONIC" WATER PURITY METER

Apparatus Used in Resistance Measurements. Resistance standards and resistance boxes have already been discussed and their construction described. Several other pieces of apparatus, used in connection with resistance measurements, merit description.

AYRTON UNIVERSAL SHUNT. This is an important accessory in galvanometer work. Fig. 184 (*a*) shows the connections of such a shunt in diagrammatic form.

The galvanometer G, of resistance R_G, is connected across the outer terminals ac of the shunt, whose total resistance is S; b is a moving contact and x the resistance between points ab and depends upon the position of b.

Let I be the current flowing in the main circuit (i.e. into the parallel combination of G and S).

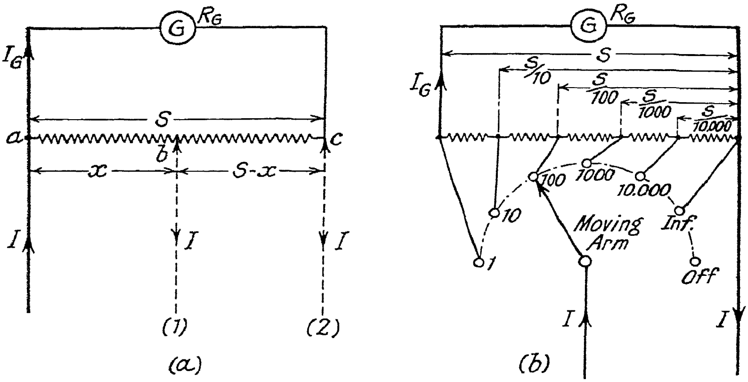


FIG. 184. AYRTON UNIVERSAL SHUNT

Then, with b in position (1) the galvanometer current

$$I_G = \frac{x}{x + S - x + R_G} \cdot I = \frac{x}{S + R_G} \cdot I$$

Again, with b in position (2)—when $x = S$—the galvanometer current is

$$I_G' = \frac{S}{S + R_G} \cdot I$$

Thus, in moving b from position (2) to position (1), the galvanometer current is reduced in the ratio $\frac{x}{S}$, this ratio being, therefore, independent of the resistance of the galvanometer.

The "multiplying power" of the shunt—i.e. the ratio

$$\frac{I}{\text{Galvanometer current}}$$

—for any given position, is $\frac{S + R_G}{x}$

By arranging the contact b to be moved in steps (by means of a moving arm and studs connected to tapping points on the shunt S), the galvanometer current for any position of b may be made a definite fraction of the current which flows through the galvanometer when b is on the point c. Thus, if b is in such a position that $x = \frac{1}{10} \cdot S$, the galvanometer current is $\frac{1}{10}$th of the current flowing when $x = S$.

These shunts are often made up so that the ratios of $\frac{x}{S}$ obtainable are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc. (see Fig. 184 (b)).

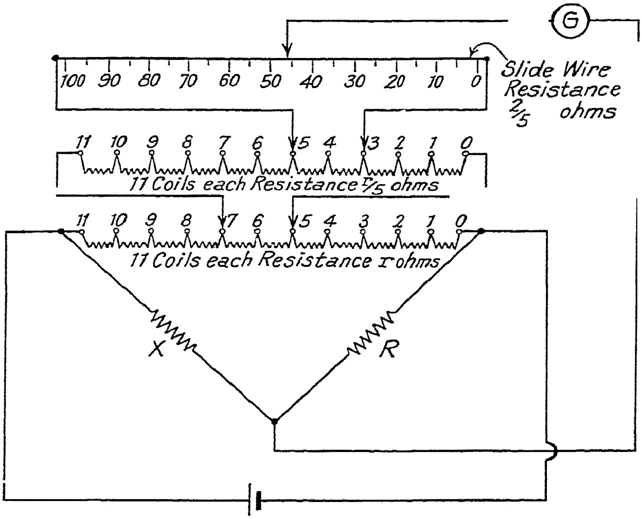
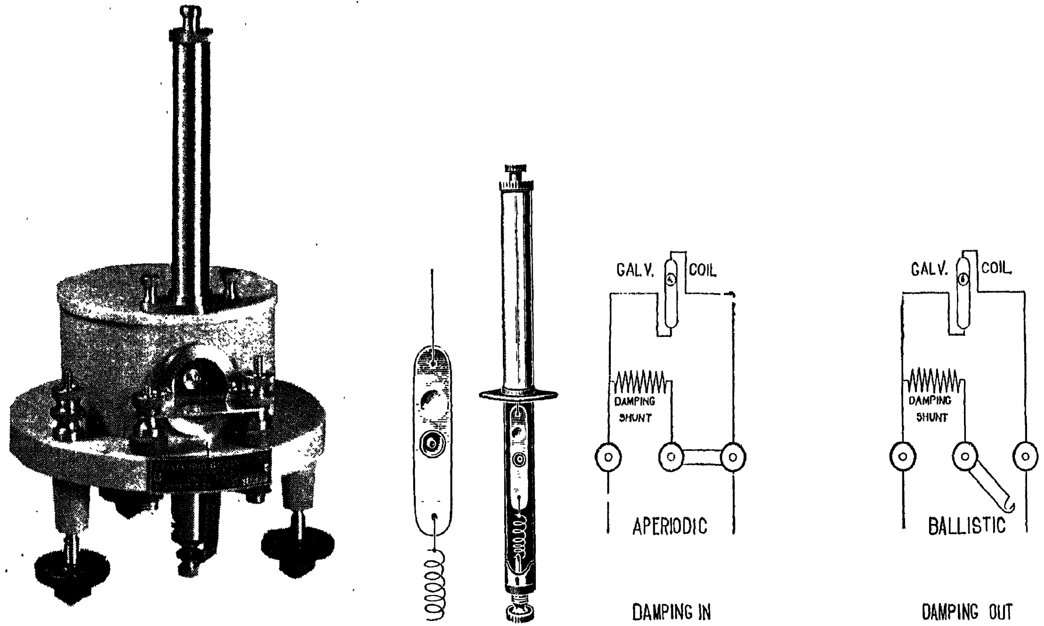

G
100
90
80
70
60
50
40
30
20
10
0
Slide Wire Resistance $\frac{2}{5}$ ohms
11 10 9 8 7 6 5 4 3 2 1 0
11 Coils each Resistance $\frac{r}{5}$ ohms
11 10 9 8 7 6 5 4 3 2 1 0
11 Coils each Resistance r ohms
X
R

FIG. 185. KELVIN AND VARLEY SLIDE

It should be noted that the resistance across the galvanometer terminals is S, no matter what the position of the contact b. The resistance S of the shunt should be chosen about 10 times that of the galvanometer, so that the latter may not be over-damped, and that its sensibility may not be appreciably reduced by being thus shunted.

KELVIN AND VARLEY SLIDE. This device may be used to replace a simple slide-wire in a Wheatstone bridge network. The principle is used, also, in the construction of potentiometers and universal shunts. It consists of a slide-wire and a number of resistance coils connected as shown in Fig. 185, where the apparatus is inserted in a Wheatstone bridge network.

The lower row of coils consists of eleven coils, each of resistance



(H. Tinsley & Co., Ltd.)

FIG. 186. D'ARSONVAL GALVANOMETER

r ohms. Shunting two of the coils is another row of eleven coils, each of resistance $\frac{r}{5}$ ohms, the shunting connections being made through two sliding contacts which always move together, so that two coils in the bottom row are shunted by the row above, throughout. Again, two of the coils in this second row are shunted in the same way by a slide-wire whose resistance is equal to that of the two coils which it shunts, namely $\frac{2}{5}r$. Obviously, the number of rows of coils can be increased as far as is justifiable, taking into consideration contact resistance errors, etc.

Since in each case the two coils are shunted by a resistance equal to their own, the resistance of the combination is the same as the resistance of one coil, so that the total effective resistance of each of the rows of coils is in each case that of ten coils only.

The reading of the slide in the figure is 5,346. Thus $\frac{R}{X} = \frac{5,346}{4,654}$

The] Galvanometer. This instrument, which is largely used in the various methods of measurement of resistance, and also for potentiometer work, consists essentially of a circular or rectangular coil of many turns of fine wire suspended between the poles of a permanent magnet. There is often a fixed cylindrical iron core inside the coil, the coil wires being situated in the two air gaps between this core and the permanent magnet. The length of the air gaps between the coil and pole faces, and between the coil and core, is usually about $\frac{1}{16}$ in., and the pole faces are shaped so as to give a radial field. The suspension is a single fine strip of phosphor-bronze, and serves as one lead to the coil, the other lead taking the form of a loosely coiled spiral of fine wire leading downwards from the bottom of the coil. The suspension carries a small mirror upon which a beam of light is cast through a glass window in the outer brass case surrounding the instrument. The beam of light is reflected on to a scale—usually at a distance of 1 metre from the mirror—upon which the deflection is measured. A torsion head is provided for adjustment of the coil position and zero setting.

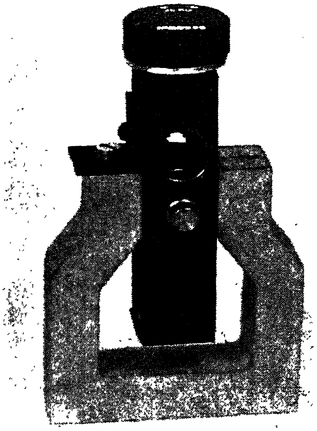
In order to save time in using the galvanometer, damping is provided by winding the coil on a light metal former. The damping is produced by the torque—opposing motion—due to the permanent magnet field in conjunction with currents which are induced in the metal former when it rotates in this magnetic field.

Damping may also be obtained by connecting a fairly low resistance across the galvanometer terminals. The damping being then dependent upon the magnitude of this resistance, it is possible, by suitably adjusting this resistance, to make the damping critical.

Fig. 186 shows a D'Arsonval galvanometer with a freely-suspended coil. The sensitivities and periodic times for three resistances are given below. This type of instrument is very suitable for ballistic measurements (see Chapter IX).

Resistance (Ohms)	Sensitivity at 1 metre		Period Time (sec.)
	mm/microamp	mm/microcoulomb	
500	250	310	5
1,000	1,100	860	8
4,000	3,500	2,000	11

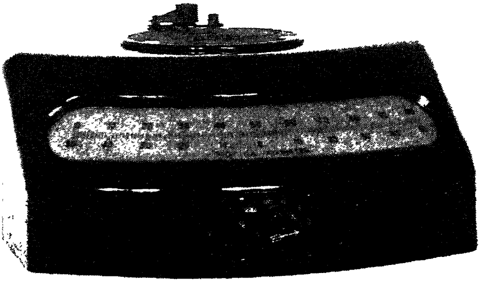
Fig. 187 shows a portable galvanometer of high sensitivity (up to 1,500 mm/microamp at 1 metre) with a moving coil which is tautly suspended at both ends by silver-gilt suspension strip. This form of suspension is independent of level.



(*H. Tinsley & Co., Ltd.*)
G. 187. SUSPENSION TYPE GALVANOMETER

The Cambridge Instrument Co. make a very robust form of galvanometer which requires no levelling or clamping. This is the Cambridge "Pot" Galvanometer. It is fitted with a pointer as well as a mirror for use with lamp and scale and gives a deflection per micro-ampere of 12 millimetres at a scale distance of 1 metre. Its resistance is 50 ohms and its period 1·3 seconds.

K. Copeland, A. C. Downing and A. V. Hill (Ref. (25)) describe a moving-coil galvanometer of extreme sensitivity. Used with photelectric amplification, the instrument is capable, in a few seconds, of reading to a few thousandths of a microvolt in a 50-Ω circuit. It is stable enough to allow 20-fold to 50-fold magnification under laboratory conditions and the magnified deflection is read on a microammeter. Two of these authors have also described (*Jour. Sci. Insts.*, Vol. 25, pp. 225 and 230) a rapid galvanometer which can record to within 0·02 μV in 0·01 sec.



(*Cambridge Instrument Co., Ltd.*)

FIG. 188. SPOT GALVANOMETER

H. W. Sullivan, Ltd. have introduced a very compact, sensitive, yet robustly suspended galvanometer fitted with shock-absorbing stops which permit overloads of up to 100 times full-scale current without risk of damage to the movement. Sensitivities up to 2,300 mm/μA can be obtained with a periodic time not greater than 2 seconds.

Theory. Let i be the current (assumed constant) flowing through the galvanometer coil. Then the equation of motion of the galvanometer is, from Equation (174) (Chapter VI),

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = Gi$$

where θ = the deflection in radians
t = time in seconds
a = moment of inertia of the moving system
b = the damping constant
c = the restoring constant
G = the displacement constant

(see theory of the vibration galvanometer, Chapter VI).

Referring to Equations (176) and (179) (*loc. cit.*), we have for the solution of the above equation

$$\theta = A\varepsilon^{m_1t} + B\varepsilon^{m_2t} + \frac{Gi}{c} \quad . \quad . \quad . \quad . \quad . \quad (193)$$

since the current is now constant and equal to i, ω in Equation (179) being zero. A and B are constants to be determined from the initial conditions.

Let θ_D be the final steady deflection of the galvanometer. Then $\theta_D = \frac{Gi}{c}$ the expression $A\varepsilon^{m_1t} + B\varepsilon^{m_1t}$ representing a motion which may, or may not, be oscillatory, according to the relative values of the constants a, b, and c.

To determine A and B, suppose that when t is zero, the deflection θ is zero and also $\frac{d\theta}{dt} = 0$ (i.e. the galvanometer moving system is stationary in its zero position).

Then, since when $t = 0$, $\theta = 0$,

$$0 = A + B + \frac{Gi}{c} = A + B + \theta_D$$

Also, since when $t = 0$, $\frac{d\theta}{dt} = 0$

$$0 = Am_1 + Bm_2$$

Hence,

$$A = \frac{m_2\theta_D}{m_1 - m_2} \text{ and } B = \frac{-m_1\theta_D}{m_1 - m_2}$$

Hence, the equation for θ becomes

$$\theta = \theta_D - \theta_D\left[\frac{m_1}{m_1 - m_2}\varepsilon^{m_2t} - \frac{m_2}{m_1 - m_2}\varepsilon^{m_1t}\right] \quad . \quad . \quad . \quad (194)$$

Now, $m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Thus, if $b^2 > 4ac$, both m_1 and m_2 are real and negative. The motion of the galvanometer is thus non-oscillatory, the deflection gradually rising from zero to its maximum value θ_D. The galvanometer under these conditions is said to be "*over-damped.*" If $b^2 < 4ac$, both m_1 and m_2 are imaginary. Then referring to Equation (180), the equation of motion is

$$\theta = \theta_D + \varepsilon^{-\frac{b}{2a}t} F \sin\left(\frac{\sqrt{4ac - b^2}}{2a}t + \alpha\right) \quad . \quad . \quad . \quad . \quad (195)$$

This equation represents an oscillatory motion, the oscillations dying away gradually as the time t is increased, giving finally a steady deflection θ_D.

If f is the frequency of these oscillations, then

$$2\pi f = \frac{\sqrt{4ac - b^2}}{2a}$$

or

$$= \frac{\sqrt{4ac - b^2}}{4\pi a} \quad . \quad . \quad . \quad . \quad . \quad (196)$$

The galvanometer, under these conditions, is *under-damped.* If there is no damping $b = 0$ and $T = 2\pi\sqrt{\frac{a}{c}}$ (where T = the periodic time of the oscillations).

If $b^2 = 4ac$, then $m_1 = m_2 = -\frac{b}{2a}$

In this case of equal roots ($m_1 = m_2$), the general solution for θ takes the form

$$\theta = \theta_D + \varepsilon^{-\frac{b}{2a}t}[A + Bt] \quad . \quad . \quad . \quad (197)$$

To find A and B, let $\theta = 0$ and $\frac{d\theta}{dt} = 0$ when $t = 0$.

Then $$0 = \theta_D + A$$

or $$A = -\theta_D$$

and $$0 = B - \frac{b}{2a} \cdot A$$

or $$B = -\frac{b}{2a} \cdot \theta_D$$

Hence, $$\theta = \theta_D - \theta_D\,\varepsilon^{-\frac{bt}{2a}}\left[1 + \frac{b}{2a}t\right] \quad . \quad . \quad (198)$$

Under these conditions the motion of the galvanometer is just non-oscillatory and the damping is said to be "*critical.*"

The undamped natural frequency of the instrument is $f = \frac{1}{2\pi}\sqrt{\frac{c}{a}}$ (see p. 262) and, with critical damping, $b^2 = 4ac$ so that $f = \frac{1}{2\pi}\sqrt{\frac{b^2}{4a^2}}$ or $\frac{b}{2a} = 2\pi f$. The equation for the deflection becomes

$$\theta = \theta_D(1 - \varepsilon^{-2\pi ft} - 2\pi ft\varepsilon^{-2\pi ft})$$

Now, when $t = \frac{1}{f}$, the value of the factor in brackets is

$$1 - \varepsilon^{-2\pi} - 2\pi\varepsilon^{-2\pi}$$
$$= 1 - 0{\cdot}00185 - 0{\cdot}0116 = 0{\cdot}9865$$

Thus, in a time equal to the undamped periodic time, the instrument will, when critically damped, reach a deflection equal to 98·7 per cent of its final deflection when a current I is suddenly applied.

Fig. 189 shows the forms of the deflection-time curves in the three cases when the galvanometer is (*a*) over-damped, (*b*) under-damped, (*c*) critically damped. The curves on the left show the rise of the deflection, starting at $\theta = 0$ when $t = 0$, while those on the right show the dying away of the deflection, starting at $\theta = \theta_D$ when $t = 0$.

Influence of the Resistance of the Galvanometer Circuit upon the Damping. In the above, the damping constant b was assumed to be dependent merely upon air friction and elastic hysteresis in the suspension. If the coil is wound upon a metal former, an e.m.f. will be induced in this former when it moves through the magnetic field of the permanent magnet. A current will flow in a closed circuit in this former, and will produce damping even though the galvanometer circuit may be open. The induced e.m.f. is proportional to the angular velocity of the coil $\frac{d\theta}{dt}$ and this additional damping may be taken into account by making the original damping b, now b'.

If the galvanometer circuit is closed, R being the resistance of this circuit, the current flowing through the galvanometer—neglecting its inductance—will be given by

$$Ri = E - G \cdot \frac{d\theta}{dt}$$

where E is the voltage applied to the galvanometer circuit. The term $G\frac{d\theta}{dt}$ represents a "back e.m.f." induced in the coil due to its motion through the magnet field, G being $NHlr$, N being the number of turns on the coil, l its active length, r its breadth and H the intensity of the permanent magnet field.

The equation of motion now becomes

$$a\frac{d^2\theta}{dt^2} + b'\frac{d\theta}{dt} + c\theta = \frac{G}{R}\left(E - G\frac{d\theta}{dt}\right)$$

or

$$a\frac{d^2\theta}{dt^2} + \left(\frac{G^2}{R} + b'\right)\frac{d\theta}{dt} + c\theta = G\frac{E}{R} = GI \quad . \quad . \quad . \quad (199)$$

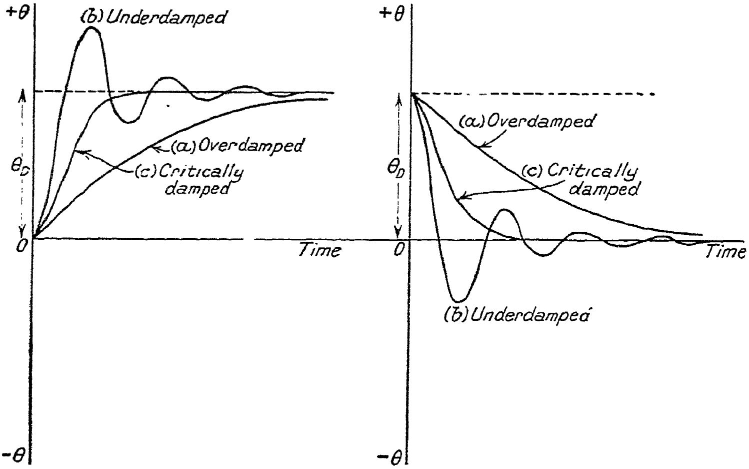


FIG. 189. DEFLECTION-TIME CURVES FOR DIFFERENT DEGREES OF

where I is the final value of the galvanometer current when the coil has attained its steady deflection. The effective damping constant is now, therefore, $\left(\frac{G^2}{R} + b'\right)$, and this now replaces b in the original theory.

If b' is small compared with $\frac{G^2}{R}$ the galvanometer is critically damped when $\left(\frac{G^2}{R}\right)^2 = 4ac$, and since G, a, and c are constants for any particular instrument, critical damping may be obtained by variation of R.

It is important, in order to save both time and trouble in using such galvanometers, that the damping shall be properly adjusted, and also that the sensitivity of the galvanometer chosen for a particular measurement shall not greatly exceed that demanded by the work in hand.

BIBLIOGRAPHY AND REFERENCES

(1) *Methods of Measuring Electrical Resistance,* E. F. Northrup.

(2) *Electrical Measurements,* F. A. Laws.

(3) *Dictionary of Applied Physics,* Vol. II.

(4) F. E. Smith, *British Association Report,* 1906.

(5) *Journal I.E.E.,* Vol. LII, p. 64.

(6) *Phys. Zeits.,* VII, p. 570.

(7) *Bulletin Bur. Stands.,* XI, p. 373.

(8) "Contact Effects Between Electrodes and Dielectrics," B. G. Churcher, C. Dannatt, and J. W. Dalgleish, *Jour. I.E.E.,* Vol. LXVII, p. 271.

(9) "Directions for the Study of Varnish-paper and Varnish-fabric Boards and Tubes," *Jour. I.E.E.,* Vol. LXII, p. 160.

(10) "Directions for the Study of Vulcanized Fibre for Electrical Purposes," *Jour. I.E.E.,* Vol. LXI, p. 964.

(11) "Directions for the Study of Hard Composite Dielectrics," *Jour. I.E.E.,* Vol. LX, p. 565.

(12) "Insulating Oils for High Voltage Cables," T. N. Riley and T. R. Scott, *Jour. I.E.E.,* Vol. LXVI, p. 805.

(13) "A Critical Résumé of Recent Work on Dielectrics," L. Hartshorn, *Jour. I.E.E.,* Vol. LXIV, p. 1152.

(14) "Some Applications of the Unbalanced Wheatstone Bridge," G. W. Stubbings, *Distribution of Electricity,* December, 1934.

(15) "Sensitivity of a Galvanometer as a Function of its Resistance," H. B. Brooks, *Bull. Bur. Stands.,* 4, 297.

(16) "General Design of Critically Damped Galvanometers," F. Wenner, *Bur. Stands.,* Sc. Paper, No. 273.

(17) *Electrical Measurements and the Calculation of the Errors Involved,* Part I, D. Karo.

(18) *Electric Resistance Strain Gauges,* W. B. Dobie and P. C. G. Isaac.

(19) *Resistance Strain Gauges,* J. Yarnell.

(20) *Strain Gauges: Theory and Application,* J. J. Koch, R. G. Boiten, A. L. Biermasz, G. P. Roszbach and G. W. Van Sauten.

(21) "Electrical Weighing," H. I. Andrews, *Jour. I.E.E.,* Vol. 97, Pt. I., p. 98.

(22) "An Accurate Wire Resistance Method for the Measurement of Pulsating Pressures," N. Gross and P. H. R. Lane, *Jour. Sci. Insts.,* Vol. 30, No. 1, January, 1953, p. 1

(23) "A Bridge Method for the Control of Electrolytic Polishing," L. F. Bates and C. D. Mee, *Jour. Sci. Insts.,* Vol. 27, No. 11, November, 1950, p. 317.

(24) "Extended Applications of the Hot-wire Anemometer," S. Corrsin, *N.A.C.A. Tech. Note, No.* 1864.

(25) "A Moving Coil Galvanometer of Extreme Sensitivity," K. Copeland, A. C. Downing and A. V. Hill, *Jour. Sci. Insts.,* Vol. 30, No. 2, February, 1953, p. 40.

(26) "Note on a Hot-wire Speed and Direction Meter," L. F. G. Simmons and A. Bailey, *Aeronautical Research Committee R. amd M,* 1019.

(27) "Hot-wire Type of Instrument for Recording Gusts," L. F. G. Simmons and J. A. Beavan, *Aeronautical Research Committee R and M,* 1615.

(28) "On the Cooling of Fine Wires," L. V. King, *Phil. Trans. Roy. Soc.,* 1914.

(29) "A Shielded Hot-wire Anemometer for Low Speeds," L. F. G. Simmons, *Jour. Sci. Insts.,* Vol. 26, p. 407, 1949.

(30) "A Compensating Circuit for the Simmons Shielded Hot-wire Anemometer," L. L. Fox, P. L. Palmer and D. Whittaker, *Jour. Sci. Insts.,* Vol. 27, p. 289, 1950.

(31) "The Measurement of Low Resistance Earth Connections," G. F. Tagg, *Elec. Times,* Vol. 123, 16 April, 1953.

CHAPTER VIII

POTENTIOMETERS

A POTENTIOMETER is, essentially, a piece of apparatus by means of which e.m.f's are compared. If one of two e.m.f's is known, the other may be determined by comparison with the known one, and thus the potentiometer is used for the measurement of e.m.f's by comparison with a standard e.m.f. It may also be applied to the measurement of current and resistance by methods which will be described below.

Potentiometers for Use with Direct Current. The principle of the potentiometer is illustrated by Fig. 190, which shows the connections of the most elementary form. A battery B sends a current through a slide-wire of uniform cross-section, R being a regulating resistance to limit the slide-wire current. B_1 is a battery whose e.m.f. is to be measured. This is connected in series with a galvanometer G and a key K, *the polarity of* B_1 *being as shown.*

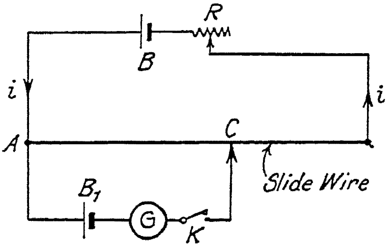


FIG. 190. PRINCIPLE OF THE POTENTIOMETER

Suppose that r is the resistance per unit length of the slide-wire, and that i is the current in it when the key K is open. Then, if the length AC is l, the voltage drop across AC is irl.

If the key K is now closed, a current will flow through the galvanometer in the direction A to C if the voltage drop across the length l of the slide wire is greater than the e.m.f. of battery B_1. (Note that the battery B_1 is connected so as to *oppose* the passage of this current.) If these two e.m.f's are equal no current will flow through the galvanometer.

Suppose, now, that the e.m.f's of two batteries B_1 and B_2 are to be compared. Then, the first battery B_1 is inserted, as shown in Fig. 190, in series with the galvanometer, and the sliding contact C is adjusted until no current flows through the galvanometer. Let the length AC then be l_1. B_1 is then replaced by B_2 and the contact

C again adjusted until no current flows through G. Let the length AC then be l_2.

Then, if E_1 = e.m.f. of battery B_1

E_2 = ,, ,, ,, B_2

(obviously, both E_1 and E_2 must be less than the e.m.f. of the supply battery B)

we have $E_1 = irl_1$ and $E_2 = irl_2$

so that
$$\frac{E_1}{E_2} = \frac{l_1}{l_2}$$

A scale is provided in this elementary form of potentiometer, so that l_1 and l_2 may be read off, and the ratio $\frac{l_1}{l_2}$ gives the ratio of the two e.m.f's as shown above.

If one of the batteries (say B_2) is a standard cell of known voltage, E_2, the e.m.f. of battery B_1, is given by

$$E_1 = \frac{l_1}{l_2} \times E_2$$

Precautions. The supply battery B should be of ample capacity, so that the current i in the slide-wire may remain constant throughout the test. A resistance should be connected in series with the galvanometer, or a universal shunt used, for protection during the initial adjustment of the contact C, this resistance being cut out as balance (i.e. zero deflection) is obtained. Such a resistance is also necessary in order that no appreciable current shall be taken from the standard cell—when inserted in the galvanometer branch—during the preliminary adjustment of C. The e.m.f. of a standard cell cannot be relied upon if it is allowed to give any appreciable current.

It should be noted that when the potentiometer is balanced no current is passing through the battery under test, so that the e.m.f. measured is the open circuit e.m.f. of the battery.

Obviously, in the above elementary form of the potentiometer the accuracy of measurement depends to a large extent upon the accuracy with which the ratio $\frac{l_1}{l_2}$ can be determined. Assuming the same error in reading l_1 and l_2 to be made, no matter what the length of the slide-wire, the longer the slide-wire the less the percentage error in measurement due to these constant errors in l_1 and l_2. In the modern forms of potentiometer designed for precise measurements, the effect of a very long slide-wire is obtained by connecting a number of resistance coils in series with a comparatively short slide-wire, as described

dern Form of Potentiometer. R. E. Crompton first modified the simple slide-wire form of potentiometer described above, the general arrangement of his form of the instrument—which is still in common use—being shown in Fig. 191.

A graduated slide-wire AC is connected in series with fourteen (or more) coils, each of which has a resistance exactly equal to that of the slide-wire (of the order of 10 ohms). There are two moving contacts, P_1 and P_2, sliding over the slide-wire, and the studs of the resistance coils, respectively. B is the supply battery (2 volt), and R_1 and R_2 are two variable resistors, the former consisting of a number of coils for coarse adjustment of the potentiometer current and the latter taking the form of a slide-wire for fine adjustment.

The galvanometer G is connected in series with a key K, and a multiple circuit switch, by means of which either the standard cell S, or other e.m.f.'s to be measured, can be connected in the galvanometer circuit. The terminals to which the apparatus under test is connected are marked positive (+) and negative (–) to avoid the possibility of damage to the potentiometer due to the wrong polarity being used. The standard cell and supply battery terminals are marked similarly.

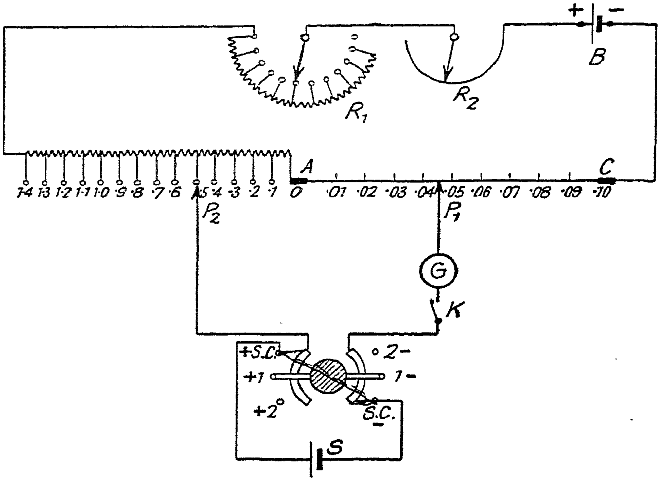


FIG. 191. CROMPTON FORM OF POTENTIOMETER

METHOD OF USE. The potentiometer is first "standardized," i.e. made direct reading by adjustment of the current from the supply battery as follows.

A standard cell—usually of the Weston type, e.m.f. 1·0183 volts —is connected to the terminals marked SC (*being sure to connect with the correct polarity*). The galvanometer is at first heavily shunted, or is connected in series with a high resistance,* for protection. The potentiometer is then set to read, directly, the e.m.f. of the standard cell—corrected to the room temperature, if necessary. If a Weston cell is used, contact P_2 will be placed on stud 1·0 and contact P_1 on 0·0183 on the slide-wire. Resistors R_1 and R_2 are then adjusted until no deflection of the galvanometer is observed with the galvanometer shunt adjusted to give full sensitivity. Leaving the resistors R_1 and R_2 at the settings so obtained, switch over the selector switch to terminals 1, 1, to which the battery whose e.m.f. is to be measured has been connected. Shunting the galvanometer, at first, for protection, adjust P_2 and, finally, P_1, until the potentiometer is again balanced. The reading of the potentiometer will then give the e.m.f. to be measured, directly. If the adjustment of the potentiometer to obtain balance, in this second case, takes any appreciable time, it is advisable to check the initial standardization again. In order to obtain steadiness of the potentiometer current, during a test, it is well to allow the current to flow through the potentiometer for a few minutes before making a measurement.

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* If the galvanometer is shunted it may be necessary to include a series resistance for the protection of the standard cell.

Constructional Details. If a steady current is to be obtained, it is important that the resistance of all portions of the supply battery and slide-wire circuit shall be constant. Since the only resistances in this circuit which are liable to vary are R_1 and R_2, it is necessary that care be taken in their design to ensure that their moving contacts shall not be subject to resistance variations when once set in position.

There is not much difficulty in the design of the resistor R_1 to fulfil this condition. The moving arm and studs must be fitted with care, and the stud contacts kept clean. The resistor R_2, which is of the slide-wire type, consists of a manganin wire wound, in the form of a double spiral, in a groove cut in an ebonite cylinder. Two fixed contacts are fitted, the length of wire included between them being varied by rotation of the cylinder, which is mounted on a spindle having a screw thread cut on it so that rotation causes the cylinder to move up or down in the direction of its axis.

Potentiometers are often designed for a slide-wire current of 10 milliamps. Thus, referring to Fig. 191, if the slide-wire AC and the fourteen resistance coils in series with it have a resistance of 10 ohms each—total 150 ohms—the total resistance of R_1 and R_2 together must be about 50 ohms if a 2 volt battery is used for the supply.

Internal Thermo-electric e.m.f's. It is very important that there shall be no appreciable thermo-electric e.m.f's within the potentiometer itself, as such e.m.f's would obviously affect the readings. For this reason, manganin, which has a very low thermo-electric e.m.f. with copper, is usually chosen as the material for the resistance coils and slide-wire. In some cases the construction is such that all contacts and joints in the potentiometer circuit are included within the case of the instrument. This ensures that all parts are at a uniform temperature and also protects the contacts from the atmosphere. This latter point is important, since any acidity of the atmosphere causes corrosion of the contacts and may set up small voltaic e.m.f's at the joints. To avoid corrosion the contacts are often made of a special gold-silver alloy.

Leakage. In order to avoid leakage between adjacent parts of the potentiometer circuit, the insulation must be good. For this reason the working parts of the instrument are mounted on an ebonite board and the internal connections are spaced so as to be as far apart as possible. A Bakelite cover is often fitted above the ebonite board for protection of the instrument from light and dirt. The knobs operating the moving parts project through holes in this cover, which also carries the graduation marks.

Standardizing Device. To save time, and for greater ease of operation of the potentiometer, the instrument is often arranged so that it is unnecessary, when standardizing the potentiometer, or when checking this standardization during the measurements, to set the moving contacts to any particular value (1·0183 in the case of a Weston standard cell). The simplest way of doing this would be to utilize a standardizing coil in series with the slide-wire as shown in Fig. 192. The resistance of this coil would have to be such that when the standard slide-wire current passed through it, the voltage drop across it would be exactly equal to the e.m.f. of the standard cell. Since this simple series arrangement would necessitate the use of a 4-volt supply in order to obtain the necessary voltage drops across this coil and the slide-wire circuit itself, one of the following alternative arrangements is adopted—

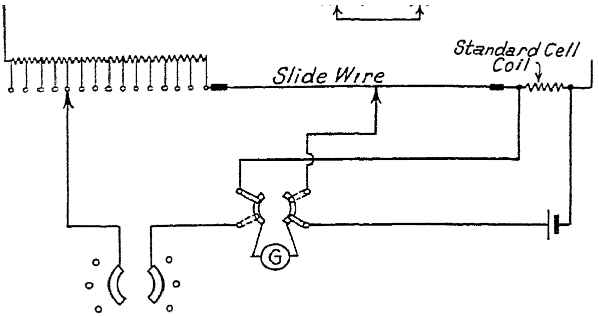


FIG. 192. CONNECTIONS FOR POTENTIOMETER WITH STANDARDIZING DEVICE

(a) The necessary resistance of the standardizing coil is obtained by connecting in series with the slide-wire a coil whose resistance is considerably less than the required value, making up the balance by using a portion of the slide-wire circuit; or

(b) A high resistance is connected in parallel with the slide-wire circuit (i.e. the slide-wire plus its series coils), the standard cell being connected, by the standardizing switch, across a fraction of this resistance. This fraction is such that the voltage drop across it is equal to the e.m.f. of the standard cell when the voltage across the total is exactly equal to the nominal voltage of the slide-wire circuit, e.g. if the nominal voltage drop across the slide-wire circuit is 1·9 and the resistance in parallel with it is 1,900 ohms, the standard cell would be connected across 1,018·3 ohms (the e.m.f. of the standard cell being 1·0183 volts).

In either case, balance of the standard cell e.m.f. is independent of the slide-wire settings. Since the resistance of the standardizing coil bears a direct relationship to the e.m.f. of the standard cell used with the potentiometer, small variations in this e.m.f. due to temperature changes or other causes are often allowed for by making the standardizing coil adjustable over a small range sufficient for the purpose.

Such an adjustable arrangement is called for in the specification of potentiometers for meter-testing stations. (See Electricity Commission Publication, Electricity Supply (Meters) Act, 1936: *Approved Apparatus for Testing Stations*. (Published by H.M. Stationery Office, 1937.))

Other Forms of Potentiometer. VERNIER POTENTIOMETER. Fig. 193 shows the internal connections of a potentiometer manufactured by Messrs. H. Tinsley & Co. This instrument has two ranges—the normal range of 1·90100 down to 0·00001 volt, and a lower range of 0·190100 down to 1 microvolt.

There are three measuring dials. The first dial—on the left—measures up to 1·8 volts (on the normal range), the middle dial reads up to 0·1 volts, while the last dial reads up to 0·001 volt. There is no slide-wire. The resistances of the middle dial shunt two of the coils of the first dial, the moving arm of which carries two contacts, spaced two studs apart, as shown. This follows the principle of the Kelvin-Varley slide (described in the previous chapter). The studs on the two lower dials are staggered to avoid cramping.

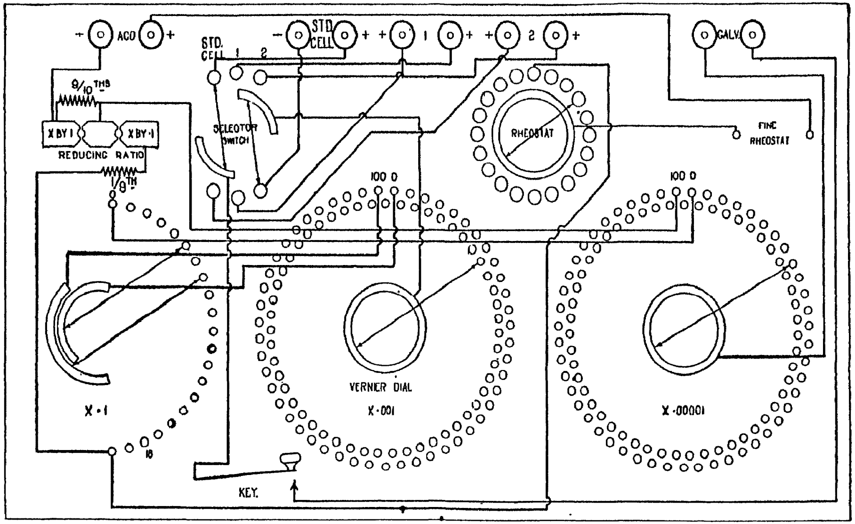
This type of potentiometer is standardized by actually setting the dials to the voltage corresponding to that of the standard cell, there being no series resistance for this purpose.

The lower range of the potentiometer is obtained by moving the plug in the contact blocks marked "reducing ratio" from the position "*X* by 1" to "*X* by 0·1." This inserts a resistance equal to nine-tenths of the total resistance of the working portion of the potentiometer in series with the latter, and at the same time shunts the working portion by a resistance of one-ninth of its total resistance. This has the effect of reducing the current in the measuring dials to one-tenth of its normal value (which is 10 milliamps), whilst keeping the current from the supply battery the same.

Brooks Deflection Potentiometer (Ref. (3)). In this type of potentiometer *the resistance of the galvanometer circuit is kept constant for all positions of the moving slide-wire or dial contacts.* Thus, the deflection of the galvanometer is at all times proportional to the out-of-balance e.m.f. It is largely used in metallurgical work for the thermo-electric measurement of high temperatures, and is especially useful if these temperatures are subject to rapid changes. In using this potentiometer exact balance is not aimed at. This constitutes a great advantage, as the obtaining of an exact balance with the potentiometers already described is difficult if the e.m.f. to be measured is not steady. For the same reason this potentiometer is very satisfactory for the checking of ammeters and voltmeters (in conjunction with a "volt-box" (see page 337) in the latter case, and with shunts in the former case), when a mains supply is to be used.

Fig. 194 gives a simplified diagram of connections of the potentiometer as applied to the measurement of an e.m.f. which may be unsteady but which is less than that of the supply battery *B*—i.e. the e.m.f. to be measured lies within the range of the potentiometer without the use of a volt-box being necessary. In the figure the connections of the standard cell, for the initial standardization of the potentiometer, are omitted for simplicity. Such connections are similar to those used in any other type of potentiometer.

R is the rheostat for adjustment of the supply current from the battery *B* (whose e.m.f. is *E*); *ab* is the resistance representing the



(*H. Tinsley & Co., Ltd.*)

FIG. 193. CONNECTIONS OF VERNIER POTENTIOMETER

dials and slide-wire of the potentiometer, while ef is a resistance in the galvanometer circuit, this, and the moving contacts, c and d, being arranged so that the resistance of the galvanometer circuit is constant.

Theory. Let resistance $ac = r_1$
,, ,, $cb = r_2$
,, ,, $de = r_3$

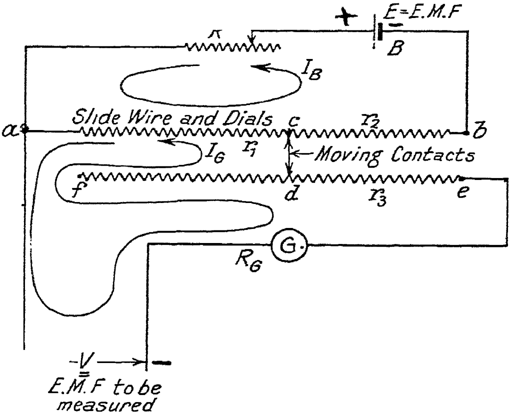


FIG. 194. SIMPLIFIED CONNECTIONS OF THE BROOKS POTENTIOMETER

Let resistance of the galvanometer and its circuit (omitting r_3) $= R_G$.

Then, taking mesh currents I_B and I_G as shown, we have, from Kirchhoff's laws,

$$I_BR + (I_B - I_G)r_1 + I_Br_2 = E$$

or
$$I_B(R + r_1 + r_2) - I_Gr_1 = E \quad . \quad . \quad \text{(i)}$$

and
$$I_GR_G + I_Gr_3 + (I_G - I_B)r_1 + V = 0$$

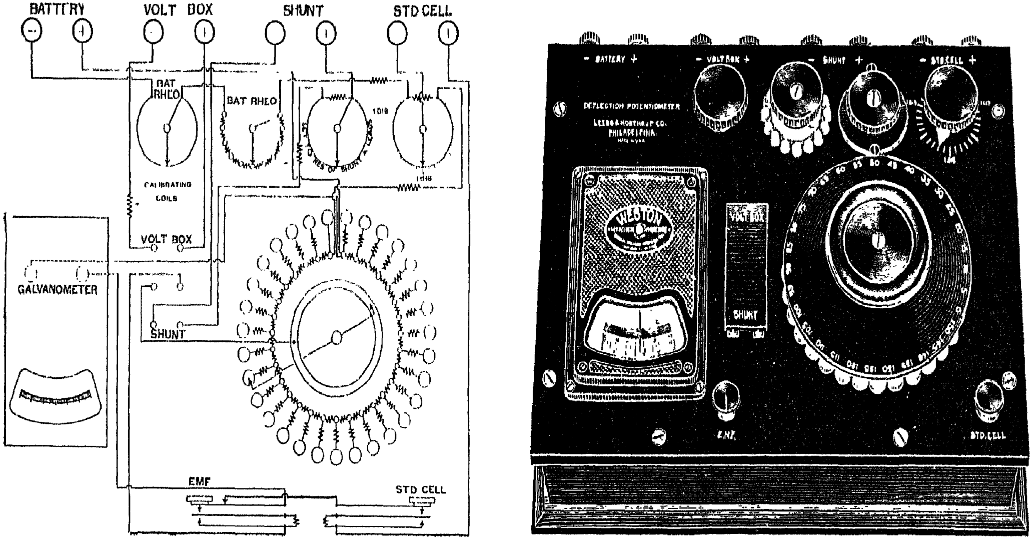
or
$$I_G(R_G + r_1 + r_3) - I_Br_1 + V = 0 \quad . \quad . \quad \text{(ii)}$$

Solving, algebraically, for I_G gives

$$I_G = \frac{\dfrac{Er_1}{R + r_1 + r_2} - V}{R_G + r_3 + \dfrac{r_1(R + r_2)}{R + r_1 + r_2}} \quad ✓ \quad . \quad . \quad (200)$$

At balance, $I_G = 0$

or
$$\frac{Er_1}{R + r_1 + r_2} = V \quad ✓$$



(*Leeds & Northrup Co.*)

FIG. 195. FULL CONNECTIONS AND CONSTRUCTION OF THE BROOKS DEFLECTIONAL POTENTIOMETER

Now, neglecting the internal resistance of the battery, $R + r_1 + r_2$ is the total resistance of the battery circuit, and thus $\dfrac{E}{R + r_1 + r_2}$ is the slide-wire current. Thus $\dfrac{Er_1}{R + r_1 + r_2}$ is $r_1 \times$ slide-wire current, which is the reading of the potentiometer and is marked on the dials and slide-wire (R having been adjusted initially to produce the standard potentiometer current by comparison with the standard cell).

If, now, the e.m.f. V to be measured falls to $(V - dV)$, the galvanometer current—assuming the potentiometer setting to be left the same—is

$$I_G = \frac{\dfrac{Er_1}{R + r_1 + r_2} - (V - dV)}{R_G + r_3 + \dfrac{r_1(R + r_2)}{R + r_1 + r_2}} = \frac{dV}{R_G + r_3 + \dfrac{r_1(R + r_2)}{R + r_1 + r_2}} \quad . \quad (201)$$

Now, $\left[R_G + r_3 + \dfrac{r_1(R + r_2)}{R + r_1 + r_2}\right]$ is the resistance of the galvanometer circuit—the portion r_1 being shunted by $R + r_2$—and is constant for all positions of the moving contact C. Thus, the galvanometer current is directly proportional to the out-of-balance e.m.f.

The galvanometer normally used is of the pivoted moving-coil type, and its scale is graduated in terms of the out-of-balance e.m.f. dV. Thus, the reading of the galvanometer must be added to, or subtracted from, the potentiometer setting—according to the direction of the deflection—in order to obtain the true value, at any instant, of the e.m.f. to be measured. The resistance of the galvanometer circuit should be such that the instrument is, as nearly as possible, critically damped in order to save time in operation.

The full connections of the potentiometer are given in Fig. 195. The instrument is made by the Leeds & Northrup Co.

If currents, or voltages above the normal range of the potentiometer, are to be measured, the method of measurement adopted is similar to that for other types of potentiometer, but the shunts and volt-box used must be specially designed to work in conjunction with a particular potentiometer, and are often included in the potentiometer itself.

DIESSELHORST OR "THERMOKRAFTFREI" POTENTIOMETER (Ref. 4). This potentiometer is designed and arranged so as to eliminate, as far as possible, errors due to thermo-electric e.m.f.'s, set up at junctions of dissimilar metals, and produced also by the heat from the operator's hand during adjustment of the working parts of the potentiometer. It is, particularly, of use in the measurement of very small e.m.f.'s.

Fig. 196 gives a diagram of connections of the potentiometer. B is the supply battery with its rheostat R, while V is the e.m.f. to be measured. S_1 and S_2 are reversing switches. G is the galvanometer which, with its key K, is in series with the unknown e.m.f. and c, c' are two sliding contacts moving over decades of coils as

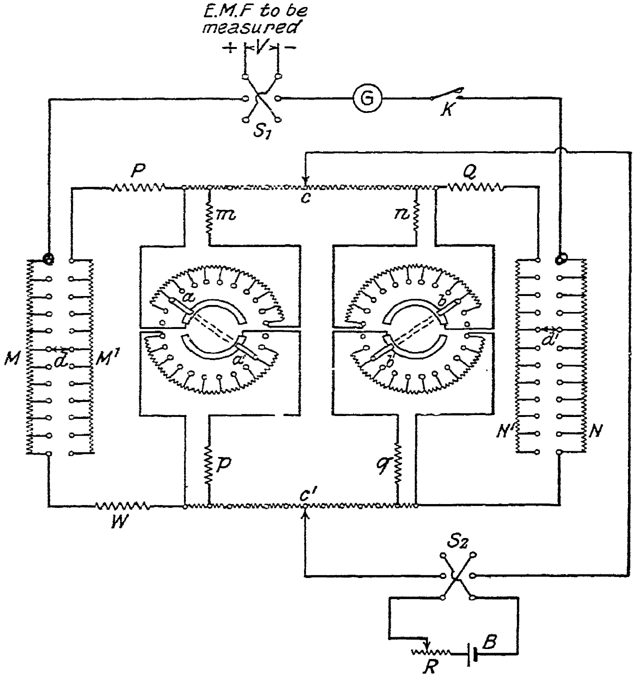


FIG. 196. DIESSELHORST POTENTIOMETER

shown. These contacts are mechanically connected so that a movement of contact c (say) to the left causes an equal movement of c' to the right, and thus maintains the resistances of the paths $cPWc'$ and cQc' constant. At d and d' also, there are sliders, each having two contacts, sliding over decades M and M' and over N and N'. Decade M is similar to decade M', as is decade N to N', and therefore, again, the resistances of the two paths mentioned above remain constant for all positions of d and d' (see diagram). P, Q, and W are series resistances, while m, n, p, and q are four equal fixed resistances, each connected in series with a variable dial-pattern resistance to form, in all, four variable shunts across coils of the decades over which contacts c and c' slide. The sliding contacts a and a', and b and b' are mechanically connected, so that they both give the same reading in all positions. The resistances in these dials are arranged so that a variation of contacts a and a' of one stud alters the value of the total resistance of the path $cPWc'$ by an amount equal to one-hundredth of the resistance of one of the coils in decades M or M'. A similar arrangement is employed in the case of the dials over which b and b' slide.

The resistances of the paths $cPWc'$ and cQc' are arranged so that the current in the former path is ten times that in the latter path.

In operation the potentiometer current is first adjusted to its standard value by the use of a standard cell, as in the case of the simpler forms of potentiometer (the standard cell circuit is omitted in the diagram for simplicity). The sliding contacts cc', dd', aa', and bb' are then adjusted until balance is obtained, when the sum of the readings of the five dials gives the value of the unknown e.m.f. (In the diagram the decades over which c and c' slide are shown separately, but in the actual potentiometer they are arranged in dial form similar to the other four dials, with c and c' moving together as do aa' and bb'.) A second measurement with both switches S_1 and S_2 reversed should give the same value to within very narrow limits.

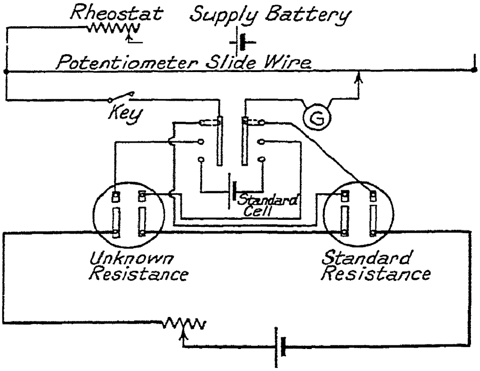
This potentiometer is very largely free from internal thermo-electric effects. Briefly, this is due to the fact that the internal connections of the instrument form a differential arrangement since paths $cPWc'$ and cQc' are in parallel. The thermo-electric e.m.f's of the two paths tend to neutralize one another. The full theory of the apparatus is given by Diesselhorst (*loc. cit.*) and by Laws (Ref. (1)).

A potentiometer of this type is made by H. Tinsley and Co. It measures e.m.f's down to 0·1 microvolt, is subdivided to 1 part in 100,000 and has a maximum volt-drop of 0·11111 volt.

Use of the Potentiometer for the Measurement of Resistance, Current, and Voltage. (*a*) RESISTANCE. The measurement of low resistance by the method of comparison with the resistance of a standard, using the potentiometer for the comparison, was discussed in the previous chapter. The connections to the potentiometer when such a measurement is to be made are shown in Fig. 197, which is self-explanatory. Care must be taken to ensure that the polarity of the connections is correct, and also that the battery which supplies current to the unknown and standard resistances is insulated from the supply battery of the potentiometer slide-wire. As pointed out in the previous chapter, the ratio $\frac{\text{resistance of unknown}}{\text{resistance of standard}}$ is the same as the ratio of the volt drops across the two resistances as measured by the potentiometer. Since only the ratio of the volt drops is required for the determination of the unknown resistance, standardization of the potentiometer by means of a standard cell is not absolutely necessary.

Correction for Thermal e.m.f's. To take into account any thermal e.m.f. which may be set up at the junctions of dissimilar metals within the two resistances, proceed as follows—

Suppose that the voltage drop across the standard resistance has just been measured, and that the galvanometer is giving no deflection under the balance conditions. Then, first of all determine whether an increase of the slide-wire setting causes the galvanometer to deflect to the left or to the right. Next, open the galvanometer key and set the potentiometer slide-wire or dials to zero, and adjust the galvanometer shunt to give the full sensitivity, as used in balancing the potentiometer during the measurement of the voltage drop across the standard resistance. Break the current circuit of the standard resistance and *immediately* depress the galvanometer key, clamping it down to avoid thermal effects due to the operator's hand. Observe the magnitude and direction of the galvanometer deflection, and calculate the value of the thermal e.m.f. by dividing this deflection by the predetermined value of the deflection per microvolt out-of-balance. Whether this thermal e.m.f. (in microvolts) is to be added to or subtracted from the potentiometer reading obtained when measuring the voltage drop across the standard, depends upon the direction of the deflection produced by the thermal e.m.f. If this direction is the *same* as that produced, initially, by *increase* of the potentiometer setting, the thermal e.m.f. has obviously been "assisting" the potentiometer during the measurement, and hence the value of the thermal e.m.f. should be *added* to the potentiometer reading to give the correct value of the voltage drop to be measured. If the thermal e.m.f. produces a galvanometer deflection in a direction *opposite* to that produced by increasing the potentiometer setting, its value should be *subtracted* from the reading of the potentiometer.



197. Measurement of Resistance by Potentiometer

The same procedure should be adopted in the case of the unknown resistance.

It should be noted that these thermo-electric e.m.f's are usually quite small and, if the measurement is not required to be highly accurate, may be neglected.

(b) CURRENT. For the measurement of current by the use of the potentiometer the connections are similar to those of Fig. 197, except that the unknown resistance is, of course, omitted, the standard cell and standard resistance only being required. The magnitude of the standard resistance must be so chosen that the voltage drop across it, when the current to be measured is flowing through it, is of the order of 1 volt, and can thus be easily measured on the potentiometer.

The potentiometer is first standardized, and the voltage drop across the standard resistance is then measured. The value of the current flowing through this standard is obviously given by $\frac{\text{volt drop across standard}}{\text{resistance of standard}}$. The method may be used for the calibration of an ammeter, the ammeter being connected in series with the standard resistance and its readings, for various measured values of the current, noted.

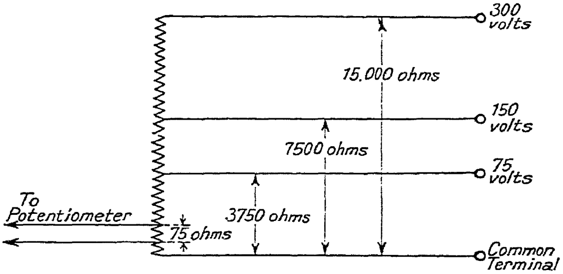


FIG. 198. INTERNAL CONNECTIONS OF VOLT-BOX

(c) VOLTAGE. The use of the potentiometer for the measurement of e.m.f's which lie within its range—i.e. are less than 2 volts—has already been described. A "*volt-box*," or "ratio-box," must be used in conjunction with the potentiometer if the voltage to be measured is above 2 volts. The volt-box consists of a high resistance (from 30 to 50 ohms per volt) having a number of tappings, the resistances between the various pairs of tappings being carefully adjusted. Its function is that of a potential divider.

Fig. 198 gives a diagram of connections. The leads to the potentiometer are taken from two tapping points, which include between them (say) 75 ohms. If a voltage of the order of 150 volts is to be measured, this voltage is connected between the two terminals marked respectively "common" and 150 volts. Thus, if the measured value of the voltage on the potentiometer is 1·25 volts, the actual value of the voltage applied to the volt-box is $1{\cdot}25 \times \frac{7{,}500}{75}$

= 125 volts. Instead of being marked 75 volts, 150 volts, etc., most volt-boxes have the markings "multiply by 50," "multiply by 100," etc., opposite the voltage terminals.

Remembering that, at balance, no current flows in the galvanometer circuit—i.e. no current is taken by the leads from the volt-box to the potentiometer, it is obvious that such a piece of apparatus will give exact subdivision of the applied voltage if the resistances between the tapping points are correctly adjusted.

Muirhead and Co., Ltd. make a voltage-dividing resistance box of 2, 3 or 4 dials with a total resistance of 10,000 ohms. The accuracy on d.c. is ± 0·1 per cent. The maximum permissible input voltage is 200 V r.m.s.

The same manufacturer makes a volt-ratio box of 50,000 ohms total resistance for a maximum voltage of 1,000 V.

Voltage Standardizer. Messrs. H. Tinsley & Co. manufacture an instrument called a "voltage standardizer," which can be used on d.c. circuits for the purpose of maintaining at a constant and known value the voltage applied to a test circuit. Thus, in the calibration of substandard wattmeters, the current in the wattmeter current coil will be measured on a potentiometer; the voltage on the pressure coil could be measured through a volt-box, on the same potentiometer. It is more convenient and more accurate, however, to utilize the voltage standardizer, which consists of a special form of volt-ratio box, the resistance tappings of which are so adjusted that when various standard voltages are connected across the "line" terminals, the potential difference across the low-potential terminals is equal to the e.m.f. of a standard cell. Thus, by connecting a standard cell, through a galvanometer, to these low-potential terminals and maintaining zero galvanometer deflection, one ensures that the "line" voltage is maintained at the correct value.

There is a variable dial for correcting for variation of the standard cell voltage with temperature. This instrument has been approved by the Electricity Commissioners under the Electricity Supply (Meters) Act, 1936, for use in meter testing.

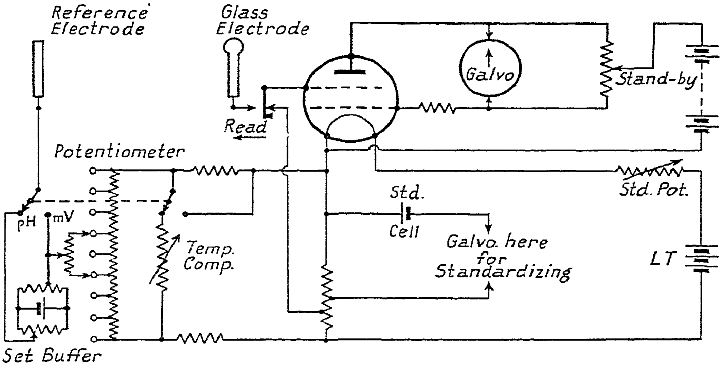
Potentiometers for Special Purposes. In addition to the applications to the measurement of resistance and current described above, potentiometers are also used, in conjunction with auxiliary apparatus, for the measurement of other quantities which are not, essentially, electrical. Such are the measurement of temperature, for which the potentiometer is used together with thermo-couples, and the measurement of the degree of acidity, or alkalinity, of a solution, when a special type of cell, containing the solution under test, is used in conjunction with the potentiometer. The measurement of temperature by potentiometer will be described in Chapter XIII.

For the measurement of acidity of solutions, a special form of cell, with a reference electrode system of calomel and potassium chloride solution, is used. There is a liquid junction between this reference half-cell and the solution under test. An e.m.f. is set up in the cell, and its value depends upon the acidity of the solution.

This e.m.f. is given as

$$E = 0{\cdot}0581p\text{H} + 0{\cdot}2488 \text{ volts at } 20^\circ \text{ C.}$$

where *p*H is a quantity from which the normality of the solution may be obtained. The voltage E may be measured by the potentiometer, and *p*H—and hence the normality—determined therefrom.



(*Marconi Instruments, Ltd.*)

FIG. 199. FUNCTIONAL DIAGRAM OF *p*H METER

Several direct-reading *p*H meters and recorders have been developed. Some of these are self-contained and are portable.

The Cambridge Instrument Company make a direct-reading *p*H indicator which needs standardization against a buffer solution only once a day after which it can be used for routine readings. The *p*H is read directly on a 120-mm scale covering the *p*H range 0 to 14. There is temperature compensation for the solution temperature over the range 10° to 100° C. The instrument is operated from 200- to 240-V a.c. mains.

The General Electric Co., Ltd. make a battery-operated *p*H meter covering the same range.

Both the Marconi *p*H meter (Marconi Instruments, Ltd.) and that made by Muirhead and Co., Ltd. utilize an electrometer valve. In the former, the circuit diagram for which is shown in Fig. 199, the e.m.f. from a standardized potentiometer circuit (standardized against a Weston standard cell within the instrument) is applied to the electrometer valve in series opposition to the e.m.f. to be measured. The valve operates a moving-coil balance indicator and the controlling-dial is calibrated to read *p*H directly.

The Muirhead instrument has an electrometer circuit comprising a pentode operating as a d.c. amplifier and taking a very low and constant grid current. It has a linear grid potential/anode current characteristic. The anode current is directly proportional to the applied e.m.f. and a backed-off meter in the anode circuit reads *p*H directly. Intrinsically the instrument is a millivoltmeter of variable sensitivity.

These two instruments are calibrated by means of buffer solutions having known values of *p*H and tablets are supplied from which the required solutions can be made up. They are both battery-operated.

Electronic Instruments, Ltd. make a *p*H meter and also a *p*H transmitter which is mains operated. It is intended for the automatic recording and controlling of *p*H in chemical manufacturing processes.

Potentiometers for Use with Alternating Current. The potentiometer method is an exceedingly useful one for the accurate measurement of alternating currents and voltages, since such measurements are not easily carried out by other methods.

The principle of the alternating current potentiometer is the same as that of the direct current instrument, the most important difference in operation being that, whereas in the direct current potentiometer only the *magnitudes* of the "unknown" e.m.f. and slide-wire voltage drop must be made equal to obtain balance, in the alternating current instrument the *phases* of these two voltages, as well as their magnitudes, must be equal for balance to be obtained. This condition obviously necessitates modification of the potentiometer as constructed for direct current work, and means that the operation is somewhat more complicated.

C. V. Drysdale, who played a major part in the development of the a.c. potentiometer, has given the history of the development (Ref. (5)), and has described several types using somewhat different methods.

The frequency and waveform of the current in the slide-wire portion of the potentiometer—i.e. of the supply—must, in all a.c. potentiometers, be exactly the same as those of the voltage to be measured, and for this reason the supply for the instrument must be taken from the same source as the voltage or current to be measured. The various forms differ principally in their method of dealing with the question of phase difference between the slide-wire and "unknown" voltages.

There are two general types: (*a*) those which measure the unknown voltage in polar form, i.e. in terms of its magnitude and relative phase; and (*b*) those measuring the rectangular co-ordinates of the voltage under test. Of the potentiometers described below, the Drysdale instrument is of type (*a*), and the Gall and Campbell-en are of type (*b*).

D TINSLEY A.C. POTENTIOMETER. This instrument consists of a potentiometer of the ordinary direct-current type—the coils in which are non-inductively wound—together with auxiliary apparatus. The auxiliaries include—

(a) A Drysdale phase-shifter, or phase-shifting transformer. This consists of a ring-shaped stator within which, fitting closely inside it, is a rotor which carries a winding supplying the potentiometer slide-wire circuit. The stator is wound with either a three-phase or two-phase winding. A rotating field is produced when currents flow in the stator winding, and the phase of the secondary, or rotor, current can be changed, relative to the stator supply voltage, by rotating the rotor through any desired angle, the phase displacement of the secondary e.m.f. being equal to the angle through which the rotor is moved from its zero position. The windings are so arranged that this alteration of phase is not accompanied by alteration of the magnitude of the rotor-induced e.m.f. The phase alteration produced is measured on a divided scale fixed to the top of the instrument. Fig. 200 shows, diagrammatically, the connections of the phase-shifter arranged for operation from a single-phase supply, using a phase-splitting device consisting of a capacitor and resistance as shown. By successive adjustment of the capacitor and resistance exact quadrature between the currents in the two stator windings may be obtained. This method, using a single-phase supply, forms a very convenient means of supplying the stator windings.

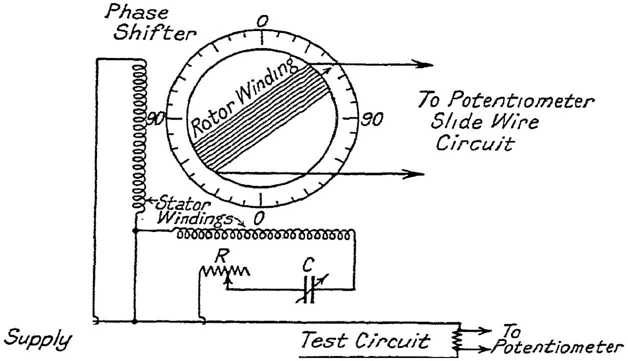


FIG. 200. CONNECTIONS OF THE DRYSDALE PHASE-SHIFTER

(b) A precision type electro-dynamometer ammeter is required for standardization purposes. To standardize the a.c. potentiometer the slide-wire circuit is switched on to a direct current supply, and the standard current is obtained in the ordinary way, using a standard cell. This standard current, required to make the potentiometer direct reading is measured by the precision ammeter which is included in the battery supply circuit of the potentiometer. During operation on alternating current, the ammeter is still included in the supply circuit, and the r.m.s. value of the slide-wire current is maintained at the same value as was required on direct current. This type of ammeter reads correctly on both direct and alternating current, and since the coils of the slide-wire circuit are non-inductively wound, the potentiometer remains direct reading when used with an alternating current supply. A change-over switch, to enable the potentiometer to be used on either direct or alternating current, is also included in the auxiliary apparatus.

Operation with Alternating Current. A simplified diagram of connections of the potentiometer for use with alternating current is given in Fig. 201. The Kelvin-Varley slide principle is employed in the slide-wire circuit as shown. *VG* is a vibration galvanometer—used as a detector for measurements at commercial frequencies. This must be carefully tuned to give resonance at the frequency of the circuit under test (which is also that of the potentiometer supply, since the two are identical). r is a shunting resistance for the reduction of the range of the potentiometer. When this shunt is put in circuit—by the switch S_1—the resistance R is simultaneously connected in series with the slide-wire circuit in order that the resistance of the working portion of the potentiometer may be maintained constant. R' is a rheostat for adjustment of the slide-wire current. A is the precision ammeter mentioned above. The phase-shifting transformer, whose connections are given in Fig. 200, is omitted for clearness.

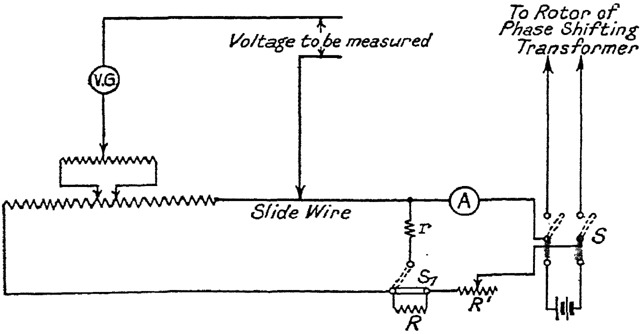


FIG. 201. DRYSDALE-TINSLEY A.C. POTENTIOMETER
DIAGRAM OF

The potentiometer is first standardized by adjusting rheostat R', and the standard current is noted, the switch S being thrown over to the battery side for this standardization, the vibration galvanometer being replaced by a D'Arsonval galvanometer.

The switch S is then thrown over to the alternating supply side, the standard cell and D'Arsonval galvanometer being previously replaced by the alternating voltage to be measured, and the vibration galvanometer, respectively. The stator windings of the phase-shifter are then adjusted to exact quadrature by means of the variable resistance and capacitor, these being adjusted until the alternating current in the slide-wire is constant for all positions of the rotor.

Balance of the potentiometer is obtained by successive adjustment of the sliding contacts of the slide-wire circuit and of the rotor of the phase-shifter. The reading of the potentiometer dials and slide-wire, at balance, give the magnitude of the voltage to be measured, as in the case of direct current measurements, while the reading on the scale of the phase-shifter gives the phase of the voltage being measured relative to the supply voltage. If the voltage measured is that across a standard resistance through which the current in the circuit under test is flowing, the magnitude of this current is obtained by dividing the measured value of the voltage by the value of the standard resistance, while its phase relative to the voltage of the circuit is read off from the scale of the phase-shifting transformer. For accurate results it is necessary that the voltage of the supply shall be steady and that the waveform shall be reasonably sinusoidal.

Constructional details of the potentiometer and phase-shifting transformer are given in Dr. Drysdale's paper (*loc. cit.*), where it is stated that, if the phase-splitting is properly carried out, the angle of rotation of the rotor represents the change in the phase of the rotor e.m.f. within an accuracy of about ± 0·1°.

GALL-TINSLEY A.C. POTENTIOMETER. This potentiometer consists of two separate potentiometer circuits enclosed in a common case. One is called the "in-phase" potentiometer and the other the "quadrature" potentiometer. The slide-wire circuits are supplied with currents which have a phase difference of 90°. On the first of these potentiometers, that component of the "unknown" voltage which is in phase with the current in the slide-wire circuit of the potentiometer is measured. On the other potentiometer the component of the "unknown" voltage in phase with the current in *its* slide-wire circuit is measured. Since the two slide-wire currents are in quadrature, the two measured values are the quadrature components of the unknown voltage. If these measured values are V_1 and V_2 respectively, then the unknown voltage is given by $V = \sqrt{V_1^2 + V_2^2}$, and its phase difference from the current in the "in-phase" potentiometer slide-wire circuit is given by the angle θ where $\tan\theta = \frac{V_2}{V_1}$.

Fig. 202 shows the connections of the potentiometer, simplified somewhat for the sake of clearness. The in-phase and quadrature potentiometers are shown, with their sliding contacts bb' and cc'

and rheostats R and R' for current adjustment. The supplies to these may be from a two-phase alternator, or may be obtained from a single-phase supply by means of a quadrature device used by D. C. Gall, the designer of the potentiometer.* This device is illustrated in Fig. 203.

T_1 and T_2 are two step-down transformers for the purpose of obtaining a 6-volt supply for the potentiometer slide-wire circuits

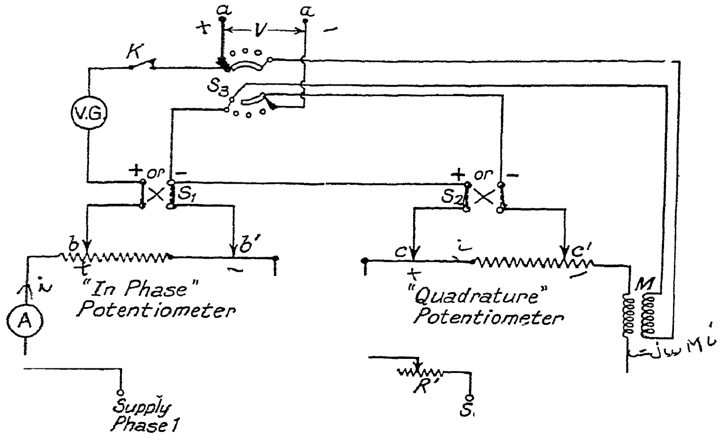


FIG. 202. CONNECTIONS OF GALL-TINSLEY A.C. POTENTIOMETER

and to isolate the potentiometers from the line. r is a variable resistance and T is a transformer for the purpose of phase-splitting. Quadrature is obtained by variation of r. Referring again to Fig. 202, $V.G.$ is a vibration galvanometer (tuned to the supply frequency) with its key K. A is a reflecting dynamometer instrument which is necessary for the maintainance of the currents in the two slide-wires at the standard value (50 milliamperes). S_1 and S_2 are two "sign-changing" switches which may be necessary to reverse the direction of the unknown e.m.f. applied to the slide-wires. The necessity of these switches depends upon the relative phases of the unknown and slide-wire voltages. S_3 is a selector switch by which the unknown voltages to be measured are placed in circuit. There are four pairs of terminals for the application of such voltages, the connections to only one pair—to which an unknown voltage V is applied—being shown in the figure. This selector switch, when in the position shown in the figure—called the "test position"—allows the current in the quadrature potentiometer slide-wire to be compared with that in the in-phase potentiometer wire, utilizing the mutual inductance M for the purpose.

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* In his book (Ref. (9)) Gall enters into a very full discussion of the operation and applications of the potentiometer.

Operation. The current in the in-phase potentiometer wire is first adjusted to its standard value by means of a direct current supply and a standard cell, the vibration galvanometer being replaced by a galvanometer of the D'Arsonval type for this purpose. The dynamometer is of the torsion-head type, and the torsion head is turned to give zero deflection on direct current. This setting is left untouched during the calibration with alternating current, the slide-wire current being adjusted to give zero deflection again. The vibration galvanometer is then placed in circuit and the direct rt supply replaced by the alternating supplies.

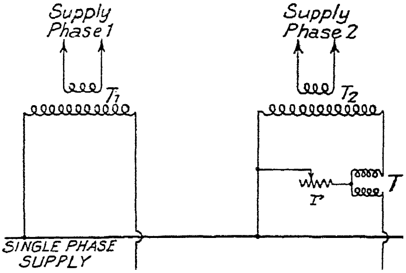

FIG. 203. GALL QUADRATURE DEVICE

the magnitude of the current in the quadrature potentiometer wire must be the same as that in the in-phase potentiometer—namely, the standard value of 50 milliamps. These two currents must also be exactly in quadrature. Rheostat R is adjusted until the current in the in-phase potentiometer wire is the standard value (as indicated on A). The selector switch S_3 is then switched on to the test position (shown in Fig. 202). Now, the e.m.f. induced in the secondary winding of the mutual inductance M—assuming M to be free from eddy current effects—will lag 90° in phase behind the current in the primary winding, i.e. in the quadrature potentiometer slide-wire. Also, if i is the primary current, then the e.m.f. induced in the secondary is $2\pi \times$ frequency $\times M \times i$ where M is the value of the mutual inductance. Thus, for given values of frequency and mutual inductance, the induced e.m.f., when i has the standard value (50 milliamps), can easily be calculated.

E.g. if $f = 50$ cycles per second

and $M = 0{\cdot}0318$ henry

the secondary induced e.m.f.

$$= 2\pi \times 50 \times 0{\cdot}0318 \times 0{\cdot}050$$
$$= 0{\cdot}5 \text{ volt}$$

when i has the standard value.

The slide-wire of the in-phase potentiometer is thus set to this calculated value of induced e.m.f. in the secondary of M (the slide-wire current being maintained at its standard value), and rheostat R' and resistance r (see Fig. 203) are adjusted until exact balance is obtained. For balance, the current in the quadrature potentiometer slide-wire must be both equal to the standard value and also must be exactly 90° out of phase with the current in the in-phase slide-wire. This latter condition follows from the fact that the e.m.f. in the secondary of M lags 90° in phase behind the primary current and, therefore, for this e.m.f. to be *in phase* with the voltage drop across a portion of the in-phase slide-wire, the current in the primary of M must be in *exact quadrature* with the current in this in-phase slide-wire. Any difference in polarity between the two circuits is corrected for by the sign-changing switches S_1 and

made these adjustments the unknown voltage is switched in circuit by means of the selector switch S_3. In this position of S_3 the two slide-wire circuits are in series with one another and with the vibration galvanometer. Balance is obtained by adjusting both pairs of sliding contacts (bb' and cc') together with the reversal of switches S_1 and S_2, if necessary. At balance, the reading of the slide-wire of the in-phase potentiometer, together with the position of S_1, give the magnitude and sign of the in-phase component of the unknown voltage, while the reading of the quadrature potentiometer with the position of S_2 give the magnitude and sign of the quadrature component.

For example, if both S_1 and S_2 are in the positive position and V_1 and V_2 are the in-phase and quadrature components of the unknown voltage V, then the phase of V is as shown in Fig. 204, while its magnitude is $\sqrt{V_1^2 + V_2^2}$.

Errors. The errors which may occur in using this potentiometer may be due to—

(*a*) Slight differences in the reading of the reflecting dynamometer instrument on a.c. as compared with the reading on d.c. Such errors may cause the standard current value on a.c. to be slightly incorrect.

(*b*) Mutual inductance between the various parts of the circuit. An error in the nominal value of the mutual inductance M would cause the current in the quadrature slide-wire circuit to be somewhat different from the standard value.

(*c*) Inaccuracy of the method of measuring the frequency, which again would cause an error in the quadrature slide-wire standard current value.

(*d*) The fact that intercapacitance, earth capacitance, and mutual inductance effects are present in the slide-wire coils and affect the potential gradient.

(*e*) The existence of harmonics in the supply waveform. Standardization of the potentiometer is upon an r.m.s. current basis, while the potential balances on the slide wires are dependent upon the fundamental only.

Fig. 205 gives the vector diagram of the quadrature device shown in Fig. 203. The equivalent circuit, replacing the quadrature potentiometer slide wire and transformer T_2 by a resistance R, is shown in Fig. 205 (*a*), in which E is the supply voltage. The voltage V across the resistance R is to be brought into quadrature with E by adjustment of r.

In the vector diagram the vector OB gives the total voltage drop in r and L_2 together, and when this is combined with the voltage vector ωMI_1, the vector representing the voltage V is obtained. The vector OA represents the total voltage drop in r and L_1 together, and when this is combined with the vector ωMI_3, representing the voltage induced in L_1 by current I_3, the voltage vector E is obtained, this being perpendicular to vector V as required.

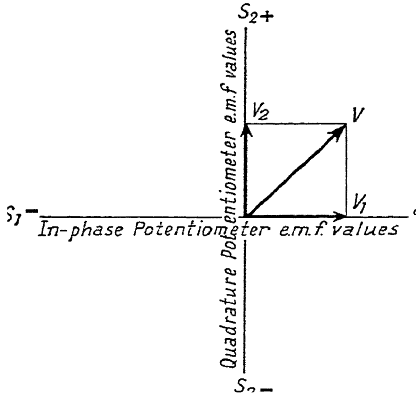
S_2+
Quadrature Potentiometer e.m.f values
V_2
V
V_1
S_1-
In-phase Potentiometer e.m.f values
S_2-

FIG. 204

In a later form of this potentiometer, this phase splitter, which was used to minimize the harmonics in the current waveform, was replaced by the simpler variable capacitor and variable resistor in series with the primary of the quadrature isolating transformer. The difficulty regarding harmonics has been overcome by using nickel-iron cores in the isolating transformers. The result has been a large reduction in the power required for the operation of the potentiometer.

H. Tinsley and Co. also make a precision polar-co-ordinate potentiometer which can be used either as a Drysdale, or as a Gall co-ordinate, potentiometer.

Campbell-Larsen Potentiometer. In this instrument the two rectangular components of the voltage under test are measured in terms of the voltage drop across a slide-wire resistance (for the in-phase component) and the voltage induced in the secondary of a mutual inductance (for the quadrature component). In the original Larsen potentiometer the slide-wire and primary circuit of the mutual inductance were in series and carried the same current, but difficulties in construction of the latter and in the operation of the potentiometer at different frequencies led to Campbell's modification of the instrument (Ref. (10)).

A simplified circuit, as modified, is shown in Fig. 206. *D* is an a.c. detector—either vibration galvanometer or telephones, according to the frequency. While current *I* passes through the primary of the mutual inductance *M*, only a portion of this current, namely I_r, passes through the slide-wire circuit. If the resistance between the movable contacts *ab* is *S* and round the path *adb* is *R*, then $I_r = I \,.\, S/(R + S)$. The setting of *S*, by means of a dial resistance, is arranged to be proportional to the frequency at which the test is being carried out, the dial being calibrated directly in terms of frequency. Since $R + S$ is constant in magnitude, $I_r \propto S \propto$ frequency, so that both the voltage drop in the slide-wire $I_r \,.\, r$ and the voltage in the secondary of the mutual inductance are proportional to frequency. *S* and *M* are chosen so that the settings of *r* and *M* give the two components of the voltage being measured directly in volts.

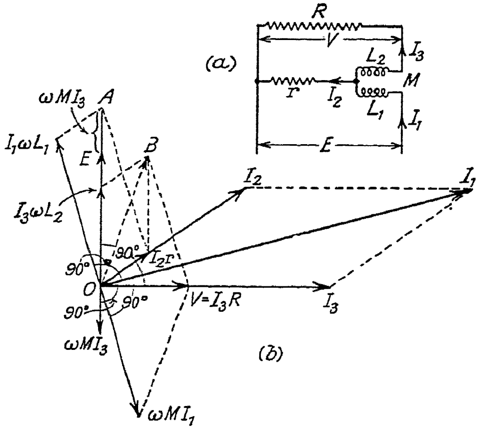

FIG. 205

A special thermal device (Refs. (9), (11)) is used for the a.c. standardization. The preliminary d.c. standardization utilizes a standard cell, and the reference current, indicated on *A*, is thus obtained. The a.c. supply to the potentiometer and the voltage under test must be obtained from the same source, but the instrument is isolated through a transformer.

This potentiometer is manufactured by the Cambridge Instrument Co.

Applications of A.C. Potentiometers. Such applications are numerous, as the a.c. potentiometer is the most universal instrument which exists for alternating current measurements. Only a limited number of applications can be given in the space available here.*

One application—namely the measurement of self inductance—has already been given in Chapter VI. Others are as follows.

(*a*) Voltmeter Calibration. Low voltages—up to 1·5 volts or thereabouts—can be measured directly. Higher voltages can be measured by using a volt-box (for medium voltages) or two capacitors in series (for high voltages) in conjunction with the potentiometer.

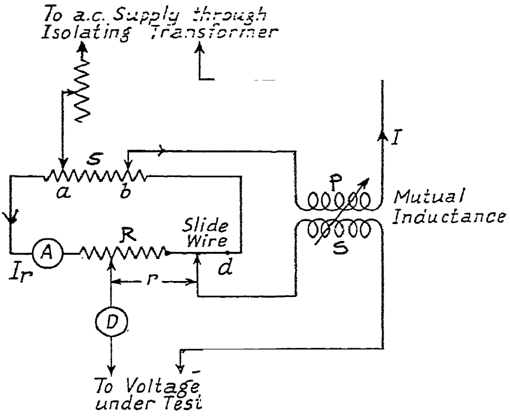

Fig. 206

(*b*) Ammeter Calibration. The measurement of various alternating currents required for such calibration may be made by the use of non-inductive standard resistors with the potentiometer, the method being similar to that adopted when the calibration is to be carried out with direct current.

(*c*) Wattmeter and Energy-meter Testing. Fig. 207 gives a simplified diagram of the connections for such tests, the arrangement being suited to tests at any power factor. The current coil of the wattmeter is supplied through a step-down transformer and the pressure coil from the secondary of a variable transformer whose primary is supplied from the rotor of a phase shifting transformer.

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* The reader should refer to Drysdale's paper (Ref. (5)), to T. Spooner's paper (Ref. (7)), or to D. C. Gall's book (Ref. (9)) for the description of other applications.

The applied voltage to the pressure coil, and current in the current coil, are measured by the potentiometer, using a volt-box and low-resistance standard as shown. The power factor is varied by rotation of the rotor of the phase-shifter, the reading on the dial of which gives the phase-angle between voltage and current. A small mutual inductance M is included to ensure accuracy of measurement at zero power factor (see Drysdale's paper, *loc. cit.*).

Other applications include the measurement of the ratio and phase-angle errors of current transformers, the measurement of core loss and magnetizing current for specimens of sheet steel, the measurement of alternating magnetic fields and the measurement of capacitance.

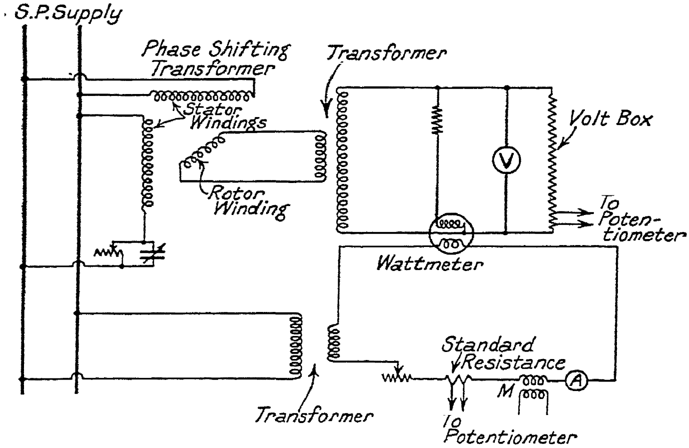

FIG. 207. CONNECTIONS FOR WATTMETER TESTING BY A.C.

A.C. Stabilizer. G. N. Patchett has designed a stabilizer, made by H. Tinsley and Co., to provide a very stable supply, with high purity of waveform, for a.c. measurements such as those with a.c. potentiometers. The requirement for such purposes is a stability of better than ± 0·1 per cent with a total harmonic distortion of not more than 2 per cent and the instrument mentioned fulfils this requirement. On normal supply mains a short-term output stability of about 1 in 10,000 can be obtained and the total harmonic distortion is less than 1 per cent.

The output voltage can be varied over the range 215 to 250 volts and the current ranges are 1 A, and 2 A, at 230 V. The performance of the instrument is unaffected by frequency changes over the range 48 to 52 c/s.

BIBLIOGRAPHY AND REFERENCES

(1) *Electrical Measurements*, F. A. Laws.

(2) *Dictionary of Applied Physics*, Vol. II.

(3) "A New Potentiometer," H. B. Brooks, *Bull. Bur. Stands.*, Vol. II, No. 2 (1906).

(4) "Thermokraftfreier Kompensation Apparatus," H. Diesselhorst, *Zeits. für Instrumentenkunde*, Vol. XXVIII, p. 1 (1908).

(5) "Alternating Current Potentiometers and their Applications," C. V Drysdale, *Jour. I.E.E.*, Vol. LXVIII, p. 339.

(6) "A New A.C. Potentiometer," D. C. Gall, *Electrician* (6th, April, 1923).

(7) "Some Applications of the A.C Potentiometer," T. Spooner, *Jour. Scientific Insts.*, Vol. III, No. 7 (April, 1926).

(8) "Design of Potentiometers," I. Melville Stein, *Jour. Am. I.E.E.*, Vol L, No. 4, p. 1302.

(9) *Direct and Alternating Current Potentiometer Measurements*, D. C. Gall.

(10) "A New Alternating Current Potentiometer of Larsen Type," A. Campbell, *Proc. Phys. Soc.*, Vol. XLI, p. 94.

(11) "Alternating-current Measurements," A. Campbell, *Jour. I.E.E.*, Vol. XXX, p. 889.

(12) "A Review of the Design and Use of Potentiometers," D. C. Gall, *Jour. I.E.E.*, Vol. LXXXV, p. 516.

CHAPTER IX

MAGNETIC MEASUREMENTS

MAGNETIC tests can be divided into two general classes: direct current tests and alternating current tests. Although they may be subdivided to a considerable extent, these are the two most distinctly defined classes of tests. The methods of testing magnetic specimens will therefore be dealt with under these two general headings.

Direct Current Tests. Such tests are most generally made upon solid (as distinct from laminated) materials, the alternating current test methods being used chiefly for laminated materials.

The two most important quantities to be measured in these tests are the flux density in a specimen and the magnetizing force producing this flux density. B. H

Magnetometer Methods. These are fundamentally the simplest of all methods of magnetic testing, and were largely used in the early work on magnetism. Magnetometers are used for the measurement of magnetic field. Often the horizontal component of such a field is measured by them. They may be applied, also, to the measurement of flux density in bar specimens of magnetic material, their advantage for this purpose being that they measure the actual (or static) value of flux density in the specimen as distinct from methods such as those using a ballistic galvanometer, which measure a *change* in flux density. The intensity of magnetization J of a specimen is measured by the magnetometer, and the corresponding flux density B is obtained from the formula

$$B = 4\pi J + H$$

where H is the strength of the magnetic field producing this intensity of magnetization. For full details of such applications the reader should refer to the works given in Refs. (2) and (3).

Magnetometers consist essentially of a suspended magnetic needle or system of needles, the suspension itself having good torsional elastic qualities and exerting, usually, only a small torsional control upon the needle.

Referring to Fig. 208, let ns represent a magnetic needle, suspended at O, and of length l cm. Let n and s be its poles, of strength m units, and suppose that a horizontal control field of strength H exists in the direction XX'. If another horizontal field, of strength F, having a direction at right angles to XX', is made to act upon the needle, a deflection θ is produced. The couples acting upon the needle are $Hml \sin \theta$ clockwise, and $Fml \cos \theta$ anti-clockwise. When the needle is at rest in the deflected position,

$$Hml \sin \theta = Fml \cos \theta$$

from which the strength of the field F is given by

$$F = H \tan \theta$$

If H is the horizontal component of the earth's magnetic field—as it often is—its value can be easily found, from physical tables, for any point on the earth's surface.

If the deflecting field is due to a bar magnet in the neighbourhood of the magnetometer needle, the pole strength of this magnet can be obtained as follows—

Theory. Referring to Fig. 209, let ns be the magnetometer needle

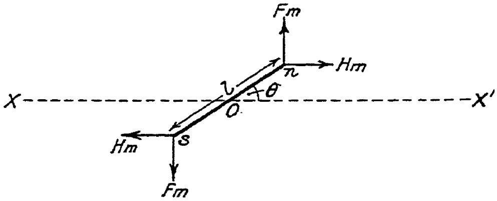

FIG. 208

of length l and pole strength m. Let NS be the bar magnet under test, of length L (between its poles) and pole strength m'. Suppose that both ns and NS are in the horizontal plane and are placed as shown. Then, the force upon pole n of the needle, due to pole N of the bar magnet, when resolved into the direction XX' is $\frac{mm'}{l_1^2} \cos \phi_1$ where l_1 and ϕ_1 are as shown. Pole N will exert the same force upon pole S of the needle, but in the opposite direction to the force on pole n. The pole S will exert forces of $\frac{mm'}{l_2^2} \cos \phi_2$ in direction XX' upon the two poles of the needle in the opposite directions to the forces due to N. Thus, the deflecting moment upon the needle due to the two poles of the bar magnet will be

$$mm'l\left[\frac{\cos \phi_1}{l_1^2} - \frac{\cos \phi_2}{l_2^2}\right]$$

Now, $\cos \phi_1 = \dfrac{r - \frac{L}{2}}{l_1}$ and $\cos \phi_2 = \dfrac{r + \frac{L}{2}}{l_2}$. The deflecting moment is, therefore

$$mm'l\left[\frac{r - \frac{L}{2}}{l_1^3} - \frac{r + \frac{L}{2}}{l_2^3}\right]$$

If the needle is situated in a horizontal control field H in a direction perpendicular to XX', and if the deflection of the needle is θ, then we have

$$Hml \sin \theta = mm'l \left[\frac{r - \frac{L}{2}}{l_1^3} - \frac{r + \frac{L}{2}}{l_2^3} \right] \cos \theta$$

Hence,

$$m' = \frac{H \tan \theta}{\left[\frac{r - \frac{L}{2}}{l_1^3} - \frac{r + \frac{L}{2}}{l_2^3} \right]}$$

from which the ferric induction in the magnet or sample may be determined (see Chapter I).

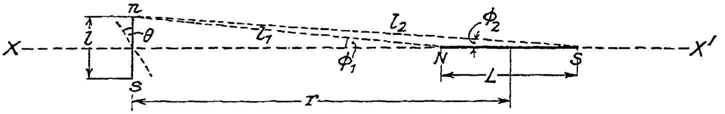

FIG. 209. MAGNETOMETER MEASUREMENT

It is assumed that the length l of the needle is small compared with the distance r.

Instead of using the horizontal component of the earth's field as the controlling field, a stronger control field may be used, this being obtained by using a permanent magnet, or otherwise. The strength of such a field may be measured by comparing it with that of the earth by the *oscillation method*, which forms a simple means of measuring field strength. The magnetic needle is allowed to oscillate freely whilst situated in the field whose strength is to be measured, and the time of one complete oscillation is measured. Let this time be T_1 sec. The needle is then placed in a known magnetic field such as that of the earth, and the time of free oscillation again measured. Let this time be T_2.

Then, from the expression for the time of one complete free oscillation, namely

$$T = 2\pi \sqrt{\frac{I}{MF}}$$

(where I is the moment of inertia of the needle about the axis of oscillation, M is the magnetic moment of the needle, and F the strength of the magnetic field in which it oscillates), we have

$$T_1 = 2\pi \sqrt{\frac{I}{MF_1}}$$

and

$$T_2 = 2\pi \sqrt{\frac{I}{MF_2}} \quad . \quad . \quad . \quad . \quad (202)$$

F_1 and F_2 being the strengths of the two fields, the latter being known.

Hence
$$\frac{T_1}{T_2} = \frac{2\pi\sqrt{\frac{I}{MF_1}}}{2\pi\sqrt{\frac{I}{MF_2}}} = \sqrt{\frac{F_2}{F_1}}$$

or
$$F_1 = F_2 \cdot \frac{T_2^2}{T_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

The magnetometer method has the advantage that it is an absolute method, but the disadvantage that it is susceptible to the influence of external magnetic fields, and also that it requires the samples under test to be in the form either of long, thin rods, or in the form of ellipsoids, owing to the demagnetizing effect of the ends of bar-shaped samples.*

During the second World War several accurate and sensitive magnetometers were developed in Great Britain. These took advantage of the high permeability of alloys, such as Mumetal (see p. 689) to concentrate the magnetic field. Differential arrangements could be made so that a very small movement of a coil removed it from the field to be measured. Another development in magnetometer design is based on the fact that, Mumetal becoming saturated at a relatively low flux density, it is possible to obtain high sensitivity by using this alloy in the region of saturation; a small change in the magnetic field causes a large change in permeability.

A mumetal-wire magnetometer is described by S. Whitehead (Ref. (49)).

The Ballistic Galvanometer. Before proceeding with the description of methods of testing bar- and ring-shaped samples of magnetic material, the ballistic galvanometer and fluxmeter—instruments which are largely used in such tests—will be described.

The ballistic galvanometer is used to measure a *quantity of electricity* passed through it. This quantity, in magnetic measurements, is the result of an e.m.f. instantaneously induced in a "search coil" connected to the galvanometer terminals, when the magnetic flux interlinking with the search coil is changed. Such a galvanometer is usually of the D'Arsonval type, since this type is least affected by external magnetic fields. It does not show a steady deflection when in use, owing to the transitory nature of the current passing through, but gives a "throw" which is proportional to the quantity of electricity instantaneously passed through it. This quantity—and hence the change in the flux producing it—is determined from the calibration of the galvanometer, as will be described later. The proportionality of the throw only holds if the discharge of the electricity through the galvanometer has been completed before any appreciable deflection of the moving system has taken place. For this reason the moving system of such a galvanometer must have a large moment of inertia—often obtained by the addition of weights to the moving system—compared with the restoring moment due to the suspension. This means that the galvanometer has a long period of vibration—usually from 10 to 15 seconds in practice. The damping of the galvanometer should also be small in order that the first deflection (or throw) shall be great.

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* Several forms of magnetometer are described by D. W. Dye in the *Dictionary of Applied Physics*, p. 455, amongst which is F. E. Smith's magnetometer for the measurement of the intensity of the earth's magnetic field.

For convenience in working, a galvanometer which is almost dead-beat is best, but the damping must be electromagnetic, so that it may be determined from the constants of the instrument. Appreciable air damping should not be present, as this is indeterminate. A key by which the galvanometer may be short-circuited saves time in bringing the moving system to rest.

Other important points in the construction of such galvanometers are that the moving coil should be free from magnetic material, and also that the suspension strip should be carefully chosen and mounted to avoid "set." The terminals, coil, and connections within the instrument, should be of copper, throughout, in order to avoid thermo-electric effects at the junctions. In the best instruments the suspension is non-conducting, the current being led into the coil by delicate spirals of very thin copper strip.

Theory. As already stated above, the quantity of electricity must be discharged through the galvanometer in a very short time, during which, the moment of inertia of the moving system being large, the movement from the zero position is negligibly small. The passage of the electricity through the instrument gives to the moving system energy which is dissipated gradually thereafter in friction and electromagnetic damping.

During the actual motion the deflecting torque is thus zero, and the equation of motion is

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = 0$$

a being the moment of inertia of the moving system, b the damping constant, c the control constant, θ the deflection in radians, and t the time in seconds.

As shown on page 261 the solution of this equation is

$$\theta = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t}$$

where A and B are constants and

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\,;\quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The damping, and therefore b, is small so that both m_1 and m_2 are imaginary. Under these conditions, as on page 262 the solution may be written

$$\theta = \varepsilon^{-\frac{bt}{2a}} F \, . \sin\left(\frac{\sqrt{4ac - b^2}}{2a} \, . \, t + \alpha\right)$$

F being a constant which may be evaluated from a knowledge of the initial conditions of the motion. Since b is small a justifiable simplification is

$$\theta = \varepsilon^{-\frac{bt}{2a}} \, . \, F \, . \sin\left(\sqrt{\frac{c}{a}}\, t + \alpha\right)$$

Initial Conditions. When $t = 0$ the deflection $\theta = 0$.

Again, if i is the current in amperes at any instant during the discharge of electricity through the instrument, the torque may be represented by Gi and hence

$$Gi = a\frac{d^2\theta}{dt^2}$$

from which $\int_0^\tau Gi \, dt = \int_0^\tau a \frac{d^2\theta}{dt^2} \, . \, dt$ where $\tau =$ total time of the discharge. Since $\int_0^\tau idt =$ the quantity of electricity discharged $= Q$ coulombs we may write

$$G\int_0^\tau idt = GQ = a\frac{d\theta}{dt}$$

$\frac{d\theta}{dt}$ is the velocity of the moving system at the end of time τ, i.e. at the beginning of the first deflection, since τ is very small.

We may thus write (as a close approximation) when $t = 0$, $\frac{d\theta}{dt} = \frac{G}{a} \, . \, Q$.

Now, differentiating the above expression for θ we have

$$\frac{d\theta}{dt} = -\frac{b}{2a} \, . \, \varepsilon^{-\frac{b}{2a}.t} F \sin\left(\sqrt{\frac{c}{a}}\, t + \alpha\right)$$
$$+ \varepsilon^{-\frac{b}{2a}t} \, . \, F\sqrt{\frac{c}{a}} \cos\left(\sqrt{\frac{c}{a}}\, t + \alpha\right)$$

and, when $t = 0$

$$\frac{d\theta}{dt} = -\frac{b}{2a} \, . \, \varepsilon^{\circ} \, . \, F \sin\alpha + \varepsilon^{\circ} \, . \, F\sqrt{\frac{c}{a}} \cos\alpha$$

Again, since $\theta = 0$ when $t = 0$

$$0 = \varepsilon^{\circ} . F \sin (0 + \alpha) \text{ or, } \alpha = 0$$

$$\therefore \quad \frac{d\theta}{dt} = F\sqrt{\frac{c}{a}} = \frac{G}{a} . Q$$

or,
$$F = \frac{G}{a} . \sqrt{\frac{a}{c}} . Q$$

Substituting in the expression for θ we have

$$\theta = \frac{G}{a} . Q . \varepsilon^{-\frac{b}{2a}t} \sqrt{\frac{a}{c}} \sin\sqrt{\frac{c}{a}} . t$$

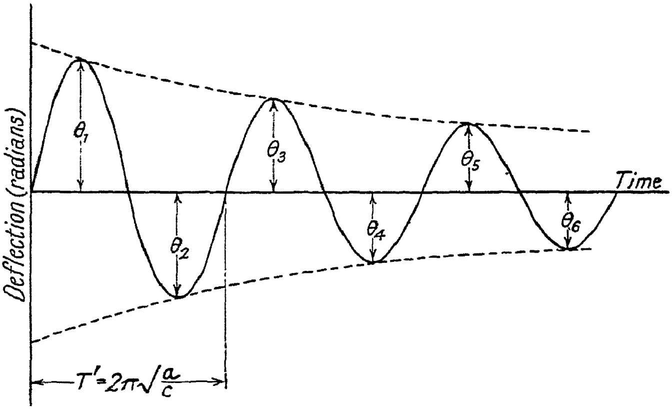


FIG. 210

The deflection at any time is thus proportional to Q and the motion is oscillatory, the frequency of the oscillation being

$$\frac{\omega}{2\pi} = \frac{\sqrt{\frac{c}{a}}}{2\pi}$$

The periodic time of the motion is thus

$$T' = \frac{1}{f} = 2\pi\sqrt{\frac{a}{c}}$$

The graph of deflection against time is shown in Fig. 210, the successive diminishing maxima corresponding to times

$$\frac{T'}{4}, \frac{3T'}{4}, \frac{5T'}{4}, \text{ etc.}$$

Substituting the values of time in the expression for θ gives

$$\theta_1 = \frac{G}{a} \cdot Q \cdot \varepsilon^{-\frac{\pi}{4}\frac{b}{\sqrt{ac}}} \sqrt{\frac{a}{c}}$$

$$\theta_2 = \frac{G}{a} \cdot Q \cdot \varepsilon^{-\frac{3\pi}{4}\frac{b}{\sqrt{ac}}} \sqrt{\frac{a}{c}}$$

$$\cdot \quad \cdot \quad = \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\theta_n = \frac{G}{a} \cdot Q \cdot \varepsilon^{-\frac{\pi}{4}(2n-1)\frac{b}{\sqrt{ac}}} \sqrt{\frac{a}{c}}$$

Without damping the amplitudes would all have been

$$\theta' = \frac{G}{a} \cdot Q \cdot \sqrt{\frac{a}{c}}$$

Now $$\theta' = \frac{\theta_1}{\varepsilon^{-\frac{\pi}{4}\frac{b}{\sqrt{ac}}}} = \theta_1 \varepsilon^{\frac{\pi}{4}\frac{b}{\sqrt{ac}}}$$

and $$\frac{\theta_1}{\theta_2} = \frac{\varepsilon^{-\frac{\pi}{4}\frac{b}{\sqrt{ac}}}}{\varepsilon^{-\frac{3\pi}{4}\frac{b}{\sqrt{ac}}}} = \varepsilon^{\frac{\pi}{2}\cdot\frac{b}{\sqrt{ac}}}$$

Hence $$\sqrt{\frac{\theta_1}{\theta_2}} = \varepsilon^{\frac{\pi}{4}\cdot\frac{b}{\sqrt{ac}}} \text{ and } \theta' = \theta_1 \sqrt{\frac{\theta_1}{\theta_2}}$$

Logarithmic Decrement. The "logarithmic decrement" λ is a constant of the galvanometer which is proportional to the damping and depends upon the resistance of the galvanometer circuit.

From above we have

$$\frac{\theta_1}{\theta_2} = \varepsilon^{\frac{\pi}{2}\cdot\frac{b}{\sqrt{ac}}}$$

so that $$\log_\varepsilon \frac{\theta_1}{\theta_2} = \frac{\pi}{2}\frac{b}{\sqrt{ac}} = \lambda$$

Then $$\theta' = \theta_1 \varepsilon^{\frac{\pi}{4}\frac{b}{\sqrt{ac}}} = \theta_1 \varepsilon^{\frac{\lambda}{2}}$$

$$= \theta_1\left(1 + \frac{\lambda}{2}\right) \text{ approx.*}$$

Obviously, from the equations for θ_1, θ_2, etc.

$$\frac{\theta_1}{\theta_2} = \frac{\theta_2}{\theta_3} = \quad \cdots\cdot = \frac{\theta_{n-1}}{\theta_n} = \varepsilon^{\frac{\pi}{2}\frac{b}{\sqrt{ac}}} = \varepsilon\lambda$$

$$\therefore \quad \frac{\theta_1}{\theta_2} \times \frac{\theta_2}{\theta_3} \times \quad \cdots\cdot \times \frac{\theta_{n-1}}{\theta_n} = (\varepsilon^{\lambda})^{n-1}$$

---

* This follows since

$$\varepsilon^{\frac{\lambda}{2}} = 1 + \frac{\frac{\lambda}{2}}{\underline{|1}} + \frac{\left(\frac{\lambda}{2}\right)^2}{\underline{|2}} + \frac{\left(\frac{\lambda}{2}\right)^3}{\underline{|3}} + \cdots$$

$$\text{or} \quad \frac{\theta_1}{\theta_n} = \varepsilon^{\lambda(n-1)}$$

$$\therefore \log_\varepsilon \left(\frac{\theta_1}{\theta_n}\right) = \lambda(n-1)$$

$$\text{or} \quad \lambda = \frac{1}{n-1} \log_\varepsilon \frac{\theta_1}{\theta_n}$$

From the preceding equation for the ideal undamped deflection θ', namely, $\theta' = \frac{G}{a} . Q\sqrt{\frac{a}{c}}$

$$\text{we have} \qquad Q = \frac{\sqrt{ac}}{G} . \theta' \text{ coulombs}$$

showing that the quantity of electricity to be measured is directly proportional to the undamped deflection θ'.

Since the periodic time $T' = 2\pi\sqrt{\frac{a}{c}}$ we have, by substitution,

$$Q = \frac{\sqrt{ac}}{G} . \theta' = \frac{c}{G}\sqrt{\frac{a}{c}}\, \theta' = \frac{c}{G} \cdot \frac{T'}{2\pi} \cdot \theta'$$

To eliminate the quantities c and G, suppose that a direct current of I_G amperes passed through the galvanometer produces a steady deflection θ, then

$$G . I_G = c . \theta$$

$$\text{or } \frac{c}{G} = \frac{I_G}{\theta}$$

Hence, finally

$$Q = \frac{T'}{2\pi} \cdot \frac{I_G}{\theta} \cdot \theta'$$

$$\text{or} \qquad Q = \frac{T'}{2\pi} \cdot \frac{I_G}{\theta}\left(1 + \frac{\lambda}{2}\right)\theta_1 \quad . \quad . \quad . \quad . \quad (204)$$

This equation may be written shortly as

$$Q = K . \theta_1$$

$$\text{where} \qquad K = \frac{T'}{2\pi} \cdot \frac{I_G}{\theta}\left(1 + \frac{\lambda}{2}\right)$$

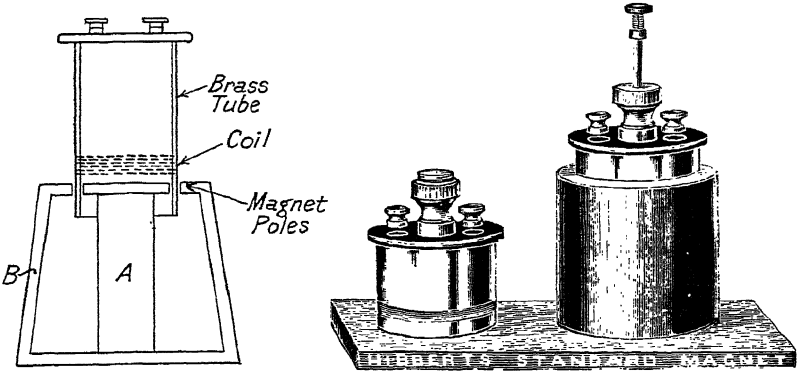
and must be found by calibration.

Since the value of K depends upon the damping and shunting of the galvanometer it is essential that the resistance of the galvanometer circuit during calibration shall be the same as when the galvanometer is being used for testing purposes.

Calibration of the Galvanometer. This may be carried out in a number of ways. Some of the methods used are—

(*a*) BY MEANS OF THE HIBBERT MAGNETIC STANDARD. The principle of this standard is illustrated diagrammatically in Fig. 211 (*a*), and its construction in

Fig. 211 (*b*). It is manufactured by W. G. Pye & Co. The circular bar magnet *A* and iron yoke *B* have a narrow annular air gap between them (about 2 mm. width) as shown. Down through this air gap a brass tube, carrying a single layer coil (of about 1 cm. axial length) in a shallow channel, can slide freely over a support—attached to *A*—which acts as a guide. The brass tube is released from a fixed position by a trigger and falls under gravity, thus ensuring that the coil always cuts through the magnetic field in the air gap at the same rate. The induced e.m.f. per turn on the coil is therefore constant. By the use of this standard the number of "line-turns"—i.e. the product of turns on the coil and lines of force through which these turns cut—which produce an observed throw on the ballistic galvanometer, can be determined in terms of the magnetic flux in the air gap of the standard and the number of turns on the coil. By means of tappings on the coil, the number of turns can be altered to give different numbers of line-turns. The number of turns obtainable is usually from 3 to 100, and the flux in the gap of the order of 20,000 lines, giving a maximum of 2,000,000 line-turns.



(*a*) (*b*) (*W. G. Pye & Co., Ltd.*)

FIG. 211. HIBBERT MAGNETIC STANDARD

Such standards are reliable, and are easy to use, but have the disadvantage that only fixed numbers of line-turns—in multiples of (say) 20,000—can be obtained.

(*b*) BY MEANS OF A CAPACITOR. A capacitor which has been charged to a known voltage, by means of a standard cell, is discharged through the galvanometer. The quantity of electricity discharged can be calculated from the known voltage, and the capacitance of the capacitor. This method is not in general use owing to the difficulty of determining exactly the capacitance of the capacitor under all conditions, and also because of the fact that the damping of the galvanometer during calibration is different from that during testing.

(*c*) BY MEANS OF A STANDARD SOLENOID. This method is most commonly used for calibration purposes. A standard solenoid consists of a long coil of wire wound on a cylinder of insulating and non-magnetic material. There may be one or more layers of wire, but the design is such that the axial length of the solenoid is large compared with its mean diameter. Usually, the axial length is at least 1 metre, while the mean diameter is of the order of 10 cm. The winding must be uniform and the number of turns per centimetre axial length should be such that a strength of field *H* of 100 or more is obtained at the centre of the coil when carrying its maximum allowable current. If the axial length is great compared with the mean diameter, the field strength in the neighbourhood of the centre of the core of the solenoid is uniform and is given by

$$H = \frac{4\pi}{10}\frac{NI}{L}$$

where N = No. of turns on the solenoid,
I = the current in amperes flowing in the winding,
L = the axial length in centimetres.

If this condition as regards axial length and mean diameter is not fulfilled, the field strength at the centre is given by

$$H = k \cdot \frac{4\pi}{10}\frac{NI}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

where $$k = \frac{L}{2d}\log_e\left[\frac{\left(r+\frac{d}{2}\right)+\sqrt{\left(r+\frac{d}{2}\right)^2+\frac{L^2}{4}}}{\left(r-\frac{d}{2}\right)+\sqrt{\left(r-\frac{d}{2}\right)^2+\frac{L^2}{4}}}\right]$$

where d = the radial depth of the winding on the solenoid in centimetres,
r = mean radius of the solenoid in centimetres.

If the radial depth, d, is small then, as seen in Chapter I, the field strength at the centre of the solenoid is given by

$$H = \frac{4\pi}{10}\frac{NI}{L}\cos\theta$$

where θ is the angle subtended at the centre of the coil by the mean radius r at one end of the coil, i.e. $\theta = \tan^{-1}\frac{r}{\frac{L}{2}}$

Thus, $$H = \frac{4\pi}{10}\frac{NI}{L}\frac{\frac{L}{2}}{\sqrt{r^2+\left(\frac{L}{2}\right)^2}}$$

or $$H = \frac{2\pi}{10}\frac{NI}{\sqrt{r^2+\left(\frac{L}{2}\right)^2}} \quad . \quad . \quad . \quad . \quad . \quad (206)$$

At the centre of the solenoid is wound a small secondary coil, usually of several hundred turns of thin wire. The axial length of the secondary coil should be small, and it may either be wound over the solenoid, or placed within, and coaxial with it. In either case its dimensions must be accurately known.

This secondary coil is connected to the ballistic galvanometer, and a measured current is passed through the solenoid from a battery, through a reversing switch (Fig. 212). An e.m.f., producing a throw of the galvanometer, is induced in the secondary coil when the solenoid current is reversed. The number of line-turns producing this throw is obtained from the known value of H at the centre of the solenoid, and from the number of turns and dimensions of the secondary coil. For example, suppose that H at the centre of the solenoid is 60 when a certain current is flowing in it. Suppose, also, that the secondary coil has 400 turns, and that its mean area is 15 sq. cm.

Then, Flux threading through the secondary coil $= 60 \times 15 = 900$ lines

No. of line-turns $= 900 \times 400 = 360{,}000$

Change in the number of line-turns during reversal of the solenoid current $= 720{,}000$

Other methods of calibration of ballistic galvanometers, using a Duddell inductor or standard cell and a known current, are given by T. F. Wall (Ref. (2)) and by D. W. Dye (Ref. (3)).

USE OF THE BALLISTIC GALVANOMETER FOR THE MEASUREMENT OF MAGNETIC FLUX. Referring to Fig. 213, in order to measure the flux in the ring specimen of magnetic material corresponding to a given current I in the magnetizing winding which is uniformly wound on the specimen, a search coil of a convenient number of turns is wound on the specimen and connected, through a resistance and calibrating coil, to a ballistic galvanometer BG as shown. The magnetizing current I is reversed, and the galvanometer throw θ observed, the change in the flux produced by the current reversal being given by $kK\theta'$ where θ' is the undamped deflection (i.e. $\theta_1\left(1 + \frac{\lambda}{2}\right)$) and K is the ballistic galvanometer constant, obtained by the use of the calibrating coil, which forms the secondary coil of a standard solenoid as previously described. k is a constant

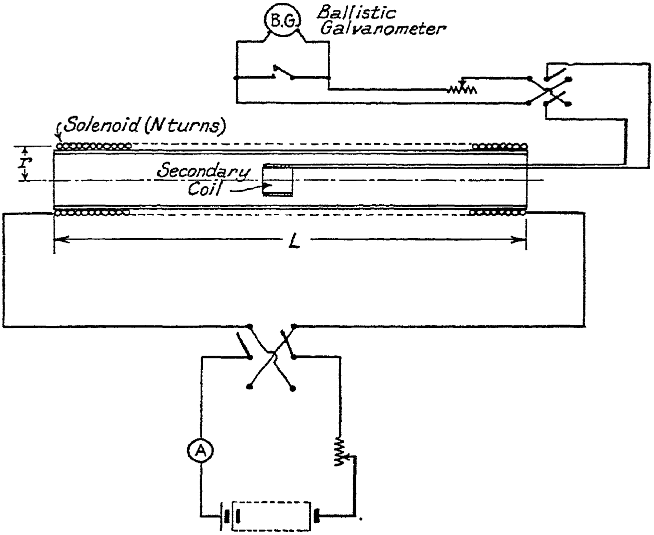


FIG. 212. BALLISTIC GALVANOMETER CALIBRATION BY STANDARD SOLENOID

depending upon the resistance of the galvanometer circuit and number of search coil turns.

Theory.

Let N = No. of turns on search coil.

„ ϕ = flux embraced by the search coil.

„ R = resistance of the ballistic galvanometer circuit.

„ t = the time (in seconds) taken to reverse the magnetizing current (and hence the flux ϕ).

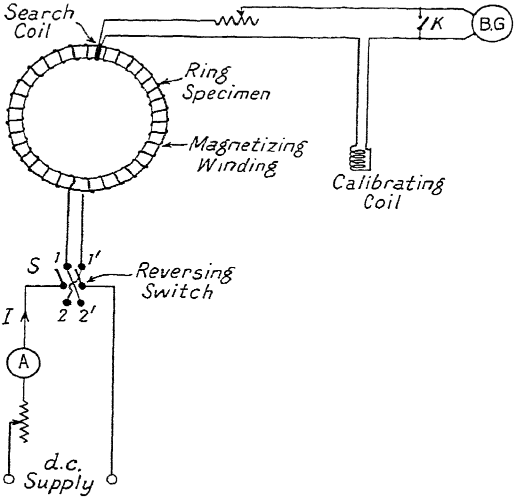


FIG. 213. FLUX MEASUREMENT BY BALLISTIC GALVANOMETER

Then, e.m.f. induced in search coil upon reversal of the flux

$$= N\frac{d\phi}{dt} \times 10^{-8} \text{ volts}$$

Average e.m.f. induced $= N \times \frac{2\phi}{t} \times 10^{-8}$ volts

Average current in the ballistic galvanometer circuit

$$= N \times \frac{2\phi}{tR} \times 10^{-8} \text{ amp}$$

Quantity of electricity discharged through the galvanometer during t sec.

$$= N \times \frac{2\phi}{tR} \times t \times 10^{-8} \text{ coulombs}$$

$$= 2\frac{N\phi}{R} \times 10^{-8}$$

But this equals $K\theta'$.
Hence, the flux in the specimen is given by

$$\phi = \frac{RK\theta' \times 10^8}{2N} \quad . \quad . \quad . \quad (207)$$

so that $$k = \frac{R \times 10^8}{2N}$$

The Grassot Fluxmeter. This instrument is really a special type of ballistic galvanometer in which the controlling torque is very small and the electro-magnetic damping is heavy. The construction is illustrated by Fig. 214.

A coil of small cross-section is suspended from a spring support by means of a single thread of silk, and hangs with its parallel sides in the narrow air gaps of a permanent-magnet system, as shown. Current is led into the coil by spirals of very thin, annealed silver strips. By this construction the controlling torque is reduced to a minimum. The instrument is usually fitted with a pointer (attached to the moving system) and a scale, although it may also be used as a reflecting instrument. The scale is graduated in terms of line-turns.

The instrument is very portable and, although not so sensitive as a ballistic galvanometer, it has the great advantage that the length of time taken for the change in the flux producing the deflection need not be small. The deflection obtained, for a given change of flux interlinking with the search coil connected to the instrument, will, in a good instrument, be the same whether the time taken for the change be a fraction of a second or as much as one or two minutes.

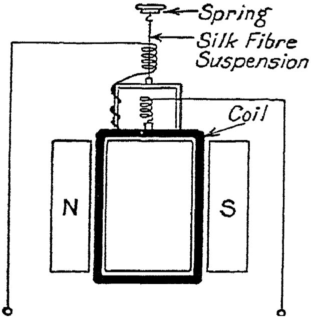


FIG. 214. GRASSOT FLUXMETER

If no controlling torque were present the instrument would remain in its deflected position indefinitely. Actually the pointer returns very slowly to zero, but readings may be taken by observing the difference in deflection at the beginning and end of the change in flux to be measured without waiting for the pointer to return to zero, the scale being uniform. The resistance of the search coil circuit connected to the fluxmeter should be fairly small, although its actual value is not important, a variation in this resistance of several ohms usually having a negligible effect upon the deflection. The inductance of the search coil circuit is also unimportant, and may be quite large, with negligible effect upon the deflection.

Theory of the Fluxmeter. *Assume that the controlling torque is negligibly small, and also that air damping and friction are negligible.*

Let $N.$ = No. of turns on the search coil which is connected to the fluxmeter terminals.

r_s and l_s = the resistance and inductance of the search coil circuit.

R and L = the resistance and inductance of the fluxmeter.

e_s = the e.m.f. induced in the search coil at any instant.

e_f = the e.m.f. being induced at any instant in the fluxmeter coil due to its movement in the permanent-magnet field.

i = the current in the circuit at any instant.

Then,

$$e_s = N \, . \, \frac{d\phi}{dt}$$

where $\frac{d\phi}{dt}$ is the rate of change of flux linking with the search coil.

Also,

$$e_f = K \frac{d\theta}{dt}$$

where *K is a constant depending upon the dimensions of the fluxmeter coil,* its number of turns, and upon the strength of the permanent magnet field; $\frac{d\theta}{dt}$ is the angular velocity of the fluxmeter coil.

The equation connecting the electrical quantities is, therefore,

$$e_s = e_f + (L + l_s)\frac{di}{dt} + (r_s + R)i \quad . \quad . \quad . \quad . \quad (208)$$

The term $(r_s + R)i$ may be neglected if r_s is small, since i is also very small.

Hence,

$$N\frac{d\phi}{dt} = K.\frac{d\theta}{dt} + (L + l_s)\frac{di}{dt} \quad . \quad . \quad . \quad . \quad (209)$$

Integrating with respect to t we have

$$\int_o^T N\frac{d\phi}{dt} . dt = \int_o^T K\frac{d\theta}{dt} . dt + \int_o^T (L + l_s)\frac{di}{dt} .$$

T being the total time taken for the change in the flux.

Thus

$$\int_{\phi_1}^{\phi_2} N d\phi = \int_{\theta_1}^{\theta_2} K d\theta + \int_{i_1}^{i_2} (L + l_s) di$$

ϕ_2 and ϕ_1 are the interlinking fluxes, θ_2 and θ_1 the deflections, and i_2 and i_1 the currents in the search-coil circuit, at the end and beginning of the change in the flux. Since i_2 and i_1 are both zero the value of $\int_{i_1}^{i_2}(L + l_s)di$ is also zero (which means that the inductance does not affect the deflection).

Hence,

$$N(\phi_2 - \phi_1) = K(\theta_2 - \theta_1)$$

or, if ϕ is the change in the flux and θ the change in the fluxmeter deflection

$$\phi = \frac{K}{N} \cdot \theta \qquad . \quad . \quad . \quad . \quad (210)$$

If the fluxmeter permanent-magnet field is uniform for all positions of the moving coil, K is constant, and the change in the flux is directly proportional to the change in the deflection.

Theory when Shunted. If a very large flux is to be measured (e.g. that in one of the poles of a large machine), the number of line-turns may be too great to be measured on the fluxmeter, even though a search coil of only one turn is employed. The range of the fluxmeter can be extended for such measurements by the use of shunts.

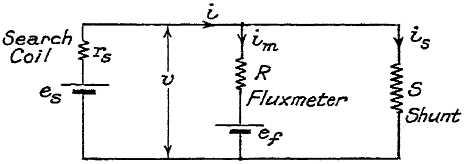


FIG. 215. FLUXMETER CIRCUIT WHEN SHUNTED

Fig. 215 gives a diagram of the circuit when a shunt is used, the induced e.m.f's being shown as batteries, and the fluxmeter being represented by the resistance R. Let the resistance of the shunt be S, i being the current in the search coil, i_m that in the fluxmeter, and i_s that in the shunt.

Then, if v is the p.d., at any instant, across all three branches we have

$$v = i_m R + e_f \qquad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$v = i_s S = (i - i_m)S \qquad . \quad . \quad . \quad . \quad \text{(ii)}$$

$$v = e_s - i r_s \qquad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

$$i = i_m + i_s \qquad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Thus,

$$e_s - i r_s = e_f + i_m R$$

or

$$\begin{aligned} e_s - e_f &= i_m R + i r_s \\ &= i_m R + (i_m + i_s)\, r_s \\ &= i_m(R + r_s) + r_s \frac{(i_m R + e_f)}{S} \end{aligned}$$

(from (i) and (ii)).

$$\begin{aligned} \therefore\ e_s - e_f - \frac{r_s}{S} \cdot e_f &= i_m(R + r_s) + \frac{r_s R}{S} \cdot i_m \\ &= 0 \end{aligned}$$

since the currents and resistances are small.

$$\therefore\ e_s = e_f \frac{(S + r_s)}{S} \qquad . \quad . \quad . \quad . \quad (211)$$

From the previous theory $e_s = N \dfrac{d\phi}{dt}$ and $e_f = K \dfrac{d\theta}{dt}$.

Thus,

$$N \frac{d\phi}{dt} = K \frac{d\theta}{dt} \frac{(S + r_s)}{S}$$

Integrating with respect to t, we have

$$\int_0^T N\frac{d\phi}{dt}\,.\,dt = \int_0^T K\frac{(S + r_s)}{S}\frac{d\theta}{dt}\,.\,dt$$

or

$$N\phi = K\frac{(S + r_s)}{S}\,.\,\theta$$

where θ is the change in deflection caused by a change of ϕ in the flux interlinking the search coil. Hence,

$$\phi = \frac{K}{N}\frac{(S + r_s)}{S}\;\theta \quad . \quad . \quad . \quad . \quad (212$$

Unshunted, the expression is

$$\phi = \frac{K}{N}\,.\,\theta$$

Thus, the deflection for a given change in the flux, when shunted, is to the deflection for the same change when unshunted, as $\frac{S}{S + r_s}$.

It should be noted that it is the *resistance*, r_s, *of the search coil* which is important when shunting is used, and not the resistance of the fluxmeter itself.

MEASUREMENT OF LEAKAGE FACTOR BY MEANS OF THE FLUXMETER. In dynamo-electric machinery the magnetic flux per pole which crosses the air gap—the "useful flux"—is less than the flux in the body of the pole. This is due to the fact that some lines of force—referred to as "leakage flux"—pass from the pole to the adjacent poles without crossing the air gap to the armature. The flux at the root of the pole is called the "total" flux, and the ratio $\frac{\text{Total flux}}{\text{Useful flux}}$ is the "leakage factor" of the pole.

This factor can be measured by means of a fluxmeter, a ballistic galvanometer being unsuitable on account of the high inductance of the field winding, which results in a slow rate of increase of the flux when the voltage is switched on to the field winding.

The total flux may be measured by winding two search coils on the yoke of the machine—in the case of a direct current machine with a stationary field—one on either side of the pole (see Fig. 216). As the yoke carries half of the total flux, these search coils must be connected in series so that the fluxmeter measures the flux embraced by both of them. The flux so measured will be the total flux. Another search coil, placed on the (stationary) armature in such a position that it embraces the useful flux from the pole, is then connected to the fluxmeter and the useful flux measured, the leakage factor being obtained from the two measurements.

It will usually be found that search coils of one turn only will be most suitable, in which case the fluxmeter reading gives the flux directly. In the case of a large machine it may be necessary to use shunts across the search coils, as described previously.

The Chattock Magnetic Potentiometer. Before proceeding with the methods of testing magnetic specimens, this device for the measurement of the magnetic potential between any two points in a magnetic field will be described. The device consists of a uniform helix of thin wire, wound on a thin strip, or rod, of some flexible insulating and non-magnetic material. This can be used, in conjunction with a ballistic galvanometer, to measure magnetic potential differences.

Let the cross-sectional area—assumed uniform—of the strip upon which the helix is wound be A sq. cm., and the number of turns per centimetre length be n. Suppose that, when the helix is situated in a magnetic field, p_{av} is the average magnetic potential difference between the two ends. Then $p_{av} =$, where H is the field strength, at any point within the helix, in the direction of the element of length of path dl, the integral being taken between a certain "average" point on one end of the strip and a corresponding point on the other end.

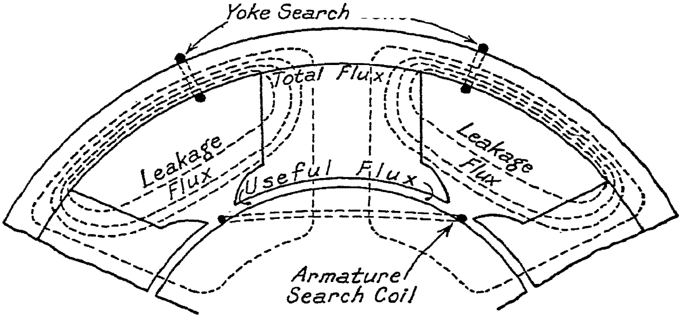


FIG. 216. MEASUREMENT OF LEAKAGE FACTOR

The quantity
$$Ap_{av} = \int p \, . \, dA$$
$$= \int H \, . \, dV$$

where dV is an element of volume within the helix.

Now, if the strip upon which the helix is wound is of non-magnetic material, the permeability is unity and the flux crossing any given cross-section of the helix is given by $\int H \, . \, dA$ where H is the field strength at the point at which the cross-section is taken. Since the flux embraced at any cross-section of the helix is not constant, the e.m.f. induced in the helix when the strength of the magnetic field in which it is situated is altered must be expressed by

$$e = \frac{d}{dt}\int\int HdA \,.\, ndl$$

the integral term giving the effective "flux-turns" of the helix.

Thus,
$$e = \frac{d}{dt}\, n \int HdV$$

or
$$e = \frac{d}{dt}(n \,.\, A \,.\, p_{av}) = nA\,\frac{dp_{av}}{dt} \quad . \quad . \quad . \quad (213)$$

i.e., the induced e.m.f. is proportional to the rate of change of magnetic potential between the two ends of the helix. From the theory of the use of the ballistic galvanometer, given on page 356, it can be seen that the galvanometer deflection, when the magnetic potential is changed, will be proportional to this change, i.e. if Δp be the change in potential

$$\Delta p = K\theta'$$

where θ' is the corrected throw of the galvanometer and K is the galvanometer constant.

The change in potential Δp may be produced by a change in the magnetizing current producing the magnetic field in which the helix is situated, or it may be produced by the rapid movement of one end of the helix from one point in the field to another, the other end being kept in the same position.

Applications of this device are the measurement of the magnetic potential drop across a given part of a magnetic circuit, such as a joint, and the measurement of magnetic leakage. Measurements may be made upon alternating magnetic fields by using a vibration galvanometer instead of a ballistic galvanometer.

From the theory of the potentiometer it can be seen that the results are the same whether the strip upon which the helix is wound is straight or otherwise. This is a great advantage, as the use of a strip of flexible material is very convenient in investigations of this nature.

Other Methods of Measuring Magnetic Field Intensity. Several other methods of exploring magnetic fields, although not in very general use, deserve mention.

(*a*) STANDARD SEARCH COIL. A search coil, consisting of a single layer of (say) 50 to 100 turns of fine silk-covered wire, wound upon a short cylinder of non-magnetic material—usually marble—may be used to investigate the variation in strength, from point to point, of a magnetic field in air. The marble cylinder must be carefully turned so that its cross-sectional area may be uniform throughout its axial length (about 2 or 3 cm.).

The product of cross-sectional area and number of turns for the search coil may be determined to within a few parts in 10,000 by the use of a standard solenoid.

(*b*) BISMUTH SPIRAL. This method depends upon the fact that the resistance of a bismuth wire is increased when it is placed in a strong magnetic field. A curve, showing the order of this increase, is given in Fig. 217, the curve relating to a temperature of 20° C. A flat spiral of pure bismuth wire—about 1 mm. diameter—is used for the exploration of magnetic fields. The resistance of the spiral, when situated at a point in the magnetic field, is measured, and this resistance is compared with the resistance when the spiral is removed from the field, the temperature being the same in both cases. The field-strength is obtained from a curve such as that of Fig. 217, when the ratio $\frac{\text{Resistance of spiral whilst in the magnetic field}}{\text{Resistance of spiral when removed from the field}}$ has been determined.

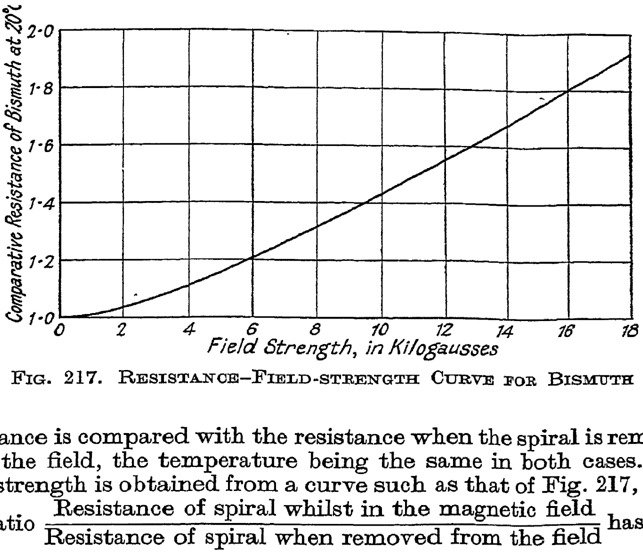


FIG. 217. RESISTANCE–FIELD-STRENGTH CURVE FOR BISMUTH

The disadvantages of this method are that it is rather insensitive —the change in resistance per kilogauss change in field strength being comparatively small—and also that this change in resistance depends very largely upon the temperature.

Its advantages are that it is very simple to use, the resistance being conveniently measured by the Wheatstone bridge method, and that, since the spiral covers a small area—of the order of 1 sq. cm.—the exploration of the magnetic field can be carried out in greater detail than is possible with most other methods.

(*c*) MAGNETIC BALANCE AND DEFLECTING-COIL METHODS. Such methods depend upon the fact that a force is exerted upon a current-carrying conductor when it is placed in a magnetic field, the force depending upon the magnitude of the current in the conductor and upon the strength of the magnetic field.

In the magnetic balance the force upon the conductor is balanced by weights on a pivoted beam, to one end of which the conductor is attached, the weight required for balance being used to determine the strength of the magnetic field in which the current-carrying conductor is situated.

In deflecting-coil methods a narrow coil, some 2 or 3 mm. wide and 1 or 2 cm. long, is used. This coil is suspended by a fine strip suspension from a torsion head, a mirror being attached to the suspension for use in conjunction with a lamp and scale to indicate deflection. A current is passed through the coil, which is placed in the magnetic field to be measured with its plane parallel to the field. If this current is known, the field strength corresponding to a certain deflection may be determined by calibration, using a magnet of known gap-flux density.

The reader is referred to the works given in Refs. (1), (3), and (7) for fuller information regarding such methods.

The Testing of Ring Specimens. Ring specimens are used, in preference to rods or strips, when accurate measurements of permeability, and hysteresis loss, up to a maximum value of field strength (H) of 200 or 250, are required. The use of such specimens has the disadvantage that they are more difficult to prepare than bar specimens, and also are more difficult to wind with the magnetizing winding which, when bar specimens are used, may be a permanently wound solenoid inside which the bar is slipped. The more reliable results obtainable with ring specimens on account of their freedom from self-demagnetizing effects may, in some cases, justify their use.

Such specimens are cut from a representative piece of the iron, the rings being machined so that their dimensions may be accurately determined. The radial thickness of such rings should be fairly small compared with their mean diameter.* If this condition is not fulfilled, most of the flux in the iron passes through the portion of the ring nearest to the inner circumference, thus causing a distribution of flux density across the cross-section of the ring which is far from uniform. The mean value of the flux density (as given by $\frac{\text{Total flux}}{\text{Cross-section}}$ under these conditions) will not correspond to the mean value of H for the cross-section of the ring, and the B-H curve obtained will be erroneous. For accurate results the ratio of outside diameter of the ring to the radial thickness should be at least 15.

If sheet material is to be tested, the ring specimen should be built up of ring punchings taken, if possible, from a number of different sheets. The punchings should be built up with the direction of rolling used in the manufacture of the sheet distributed radially to obtain a uniform distribution of reluctance round the ring. The permeability in a direction perpendicular to that of rolling is only some 75 per cent of the permeability, at the same flux density, in a direction parallel to the direction of rolling. During the shearing of the rings from the sheet, the material near the sheared edges (and for some distance inwards from the edge) is strained, the effect being to reduce its permeability. Unless the rings are to be annealed, after punching, to remove these strains, their radial width should be fairly large—say 2 or 3 cm.—in order that the strained portion shall not form an appreciable percentage of the whole cross-section.

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* This question is fully considered in a paper by E. Hughes (Ref. (8)).

Determination of the Magnetization, or B-H, Curve. (*a*) METHOD REVERSALS. For this test a ballistic galvanometer is used as previously described. Before winding, the dimensions of the ring must be determined. When sheet material is being tested it may be necessary to determine the effective cross-section from the weight of the ring, taking the specific gravity as 7·8 (for soft sheet iron). This is necessary for accurate measurements on account of the air spaces between individual punchings, which cause measurements of thickness to be erroneous.

A layer of thin tape is then wound on the ring, and a search coil consisting of a few turns of thin wire, insulated by paraffined silk, is wound over the tape. The number of search-coil turns depends upon the sensitivity of the ballistic galvanometer. This number of turns must, of course, be noted. The search coil is protected by another layer of tape, over which the magnetizing winding is uniformly wound.

The connections for the test are shown in Fig. 213. Before commencing the test, it is essential that any residual magnetism which may be present in the specimen shall be removed by demagnetization. The short-circuiting key K of the galvanometer is left closed, and the current in the magnetizing winding is given such a value that the magnetizing force H acting upon the specimen is greater than the maximum value to be used in the test. This current is then very gradually reduced—the reversing switch S being continually thrown backwards and forwards meanwhile—in order to pass the iron specimen through as many cycles of magnetization as possible during the process. The minimum value of the current finally reached should give a magnetizing force in the specimen which is well below the smallest test value.

After demagnetization, the test is started by setting the magnetizing current at the lowest test value (such that $H = 1$ (say)). The galvanometer key K being closed, the iron specimen is then brought into a "reproducible cyclic magnetic state" by throwing the reversing switch S backwards and forwards some twenty or more times. The key K is next opened, and the flux in the specimen, corresponding to this value of H, is measured on the ballistic galvanometer as previously described. The change in flux, measured by the galvanometer, when the reversing switch S is quickly reversed, will be twice the flux in the specimen corresponding to the value of H which has been applied.

This value of H is given by $\frac{4\pi}{10} \cdot \frac{NI_1}{l}$ where N = number of turns on the magnetizing winding, I_1 = the magnetizing current, l = the length (in centimetres) of the mean circumference of the ring specimen.

The flux density B_1 corresponding to this value of H is obtained by dividing the value of the flux in the specimen (as measured by the ballistic galvanometer) by the cross-sectional area of the specimen.

In order to check whether the demagnetization of the specimens has been complete and also to determine whether a reproducible cyclic state has been attained, a second measurement of flux density—for the same value of H—may be made after subjecting the specimen to a further number of reversals of magnetization. The second value of B should agree with the first. As further checks upon the measurements, a reversing switch in the ballistic galvanometer circuit (not shown in Fig. 213) may be used as follows: Measurement of flux density for a given value of H may be made, with this reversing switch in one position, first by throwing the reversing switch S over from terminals 11′ to 22′ and the test repeated by throwing over from 22′ to 11′, having carried out a number of reversals of S in between the two measurements. This procedure may be repeated with the ballistic galvanometer reversing switch in its other position, four measurements of B being thus obtained. These should be very nearly equal to one another, the mean giving the value of the flux density.

The whole of this procedure is repeated for various increased values of H up to the maximum testing point, the 20 or more reversals of the magnetizing current at each value of H, before the measurement is made, being important. It should be noted also that if the resistance of the ballistic galvanometer circuit is changed during the test, the logarithmic decrement λ must be determined for each resistance value, in order that the observed galvanometer throws may be properly corrected, the deflection used in determining the flux being given by

$$\theta' = \theta_1\left(1 + \frac{\lambda}{2}\right)$$

where θ_1 is the observed throw.

The B-H curve may be plotted from the measured values of B corresponding to the various values of H.

(*b*) The “Step-by-step” Method is sometimes used. In this method there is no reversal of the magnetizing current, the procedure being as follows. The circuit shown in Fig. 213 is set up in the same way as for the test by the method of reversals, but the direct current supply to the magnetizing circuit is through a potential divider having a number of tappings, as shown in Fig. 218. The tappings are arranged so that the magnetizing force H may be increased, in a number of suitable steps, up to the desired maximum value. The specimen, after the application of the search coil and magnetizing winding, is first demagnetized. The tapping switch S_2 is then set on tapping 1 and the switch S_1 closed, the galvanometer throw corresponding to this increase in the flux density in the specimen, from zero to some value B_1, being observed, and B_1 calculated as previously described. H_1, corresponding to this position of switch S_2, can be determined from the magnetizing current which then flows. The magnetizing force is then increased to H_2 by switching S_2 suddenly on to tapping 2, and the corresponding increase in flux density ΔB determined from the galvanometer throw observed. Then B_2—corresponding to H_2—is given by $B_1 + \Delta B$. This process is repeated for other values of H up to the maximum point, and the complete B-H curve is thus obtained without any reversal of the flux in the specimen.

Determination of the Hysteresis Loop. (*a*) Step-by-Step Method. The determination of the hysteresis loop by this method is carried out simply by continuing the procedure just described for the determination of the B-H curve. Having reached the point of maximum H—when S_2 (Fig. 218) is on tapping 10—the magnetizing current is next reduced, in steps, to zero by moving switch S_2 down through the tapping points, 9, 8, 7, etc. After the reduction of the magnetizing force to zero, negative values of H are obtained by reversing the reversing switch S (Fig. 213) and then moving the switch S_2 (Fig. 218), in steps as before.

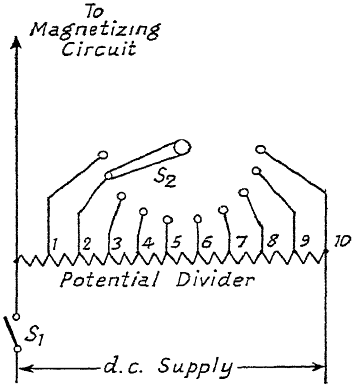


Fig 218 Potential Divider for Step-by-step Method

(*b*) By the Method of Reversals. This test again is carried out by means of a number of steps, but the change in flux density measured at each step is the change from the maximum value $+ B_{max}$ down to some lower value, the iron specimen being passed through the remainder of the cycle of magnetization back to the flux density $+ B_{max}$ before commencing the next step in the test, thus preserving the cyclic state.

The connections for the test are shown in Fig. 219. R_1, R_2, and R_4 are resistances for the adjustment of the resistances of the magnetizing and ballistic galvanometer circuits. R_3 is a variable shunting resistance which is connected across the magnetizing winding by moving over the switch S_2, thus reducing the current in this winding from its maximum value down to any desired value—depending upon the value of R_3.

The procedure is as follows—

The value of H_{max} required to produce the value of B_{max} to be used during the test, is obtained from the previously determined B-H curve of the specimen. The resistances R_2 and R_4 are then adjusted so that the magnetizing current is such that this value of H is obtained when switch S_2 is in the "off" position. (H is, of course, given by $H = \frac{4\pi}{10} \cdot \frac{NI}{l}$, where I is the magnetizing current, N the number of turns on the magnetizing winding, and l the length of magnetic path, or mean circumference, of the specimen.) The resistance R_1 is adjusted so that a convenient deflection of the galvanometer is obtained when the maximum value of the magnetizing current is reversed. R_3 is adjusted to such a value that a suitable reduction of the current in the magnetizing winding is obtained when this resistance is switched in circuit.

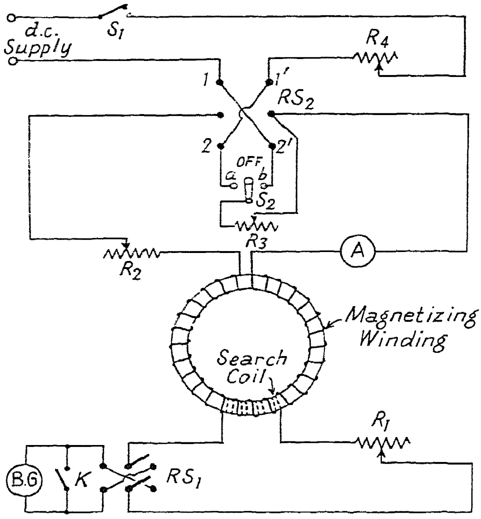


FIG. 219. CONNECTIONS FOR METHOD OF REVERSALS

Switch RS_2 is then placed on contacts 11′ and the short-circuiting key K opened. The magnetization of the specimen—since the maximum magnetizing current is now flowing—corresponds to point A on the loop (Fig. 220).

The next step is to throw switch S_2 quickly over from the "off" position to contact b, thus shunting the magnetizing winding by R_3 and reducing the magnetizing force to H_c (say). The corresponding reduction in flux density, $-\Delta B$, is obtained from the galvanometer throw, and hence the point C on the loop is obtained.

The key K is now closed, and switch RS_2 reversed on to contacts

22′. Switch S_2 is then opened and RS_2 moved back again to contacts 11′. This procedure passes the specimen through the cycle of magnetization and back to point A again, ready for the next step in the test. The section AD of the loop is obtained by continuation of this procedure.

To obtain the section DEF of the loop, with K closed and S_2 in the "off" position, place RS_2 on contacts 11′. Then place S_2 on contact a, open the key K, and rapidly reverse RS_2 on to contacts 22′, observing the corresponding galvanometer throw. From this throw the change in flux $\Delta B'$ (Fig. 220) may be obtained, since the switching operations described cause H to be changed from $+H_{max}$ to $-H_K$ (say). To bring the magnetization of the specimen back to point A, close key K, open S_2 and reverse RS_2 on to contacts 11′.

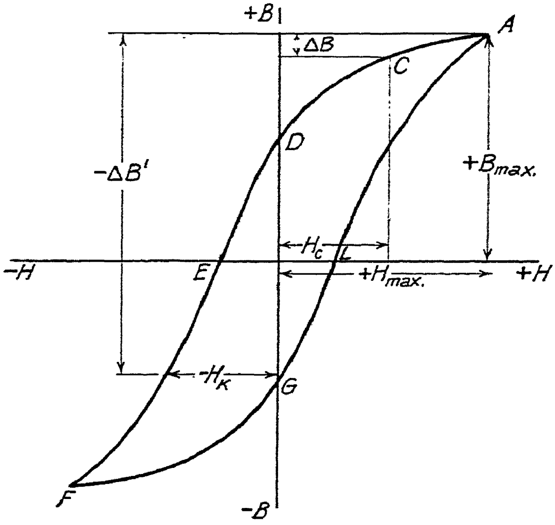


FIG. 220. HYSTERESIS LOOP

By continuing this process, other points on the section DEF of the loop are obtained. The section $FGLA$ of the loop may be obtained by drawing in the reverse of $ADEF$, since the two halves are identical.

By measuring the area of the hysteresis loop so obtained—by means of a planimeter—and expressing this area in B-H units of area, the hysteresis loss for the material may be obtained, since

Hysteresis loss per cycle per cubic centimetre, in ergs

$$= \frac{\text{Area of loop in B-H units}}{4\pi}$$

The Testing of Specimens in the Form of Rods or Strips. It is obviously much easier to prepare a specimen in the form of a machined rod than to prepare a ring specimen as previously described. Such specimens suffer, however, from the disadvantage of "*self-demagnetization.*" When a rod is magnetized electromagnetically, poles are produced at its ends, and these poles produce, inside the rod, a magnetizing force from the north pole to the south which is in opposition to the applied magnetizing force, thus rendering the true value of H acting on the rod a somewhat uncertain quantity. For accurate results, therefore, if the methods of measurement using a ballistic galvanometer as described above, are used, this demagnetizing effect must be corrected for, or, since the effect is least when the ratio of diameter to length of the rod is small, the dimensions of the specimen should be chosen so that the effect is negligible.

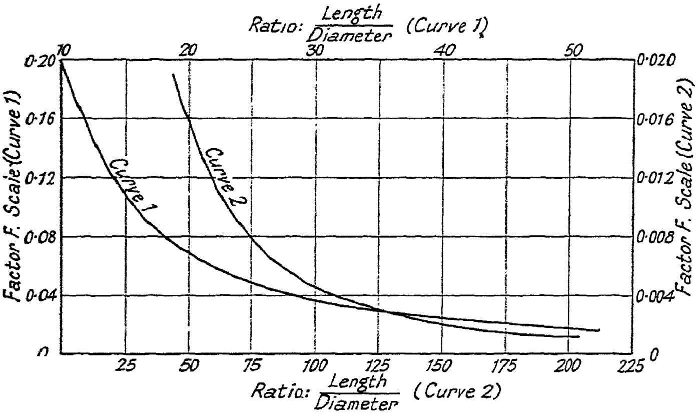


FIG. 221. CURVES OF DEMAGNETIZING FACTORS

The demagnetizing force due to this "end effect" is given by the expression

$$H_d = \frac{F}{4\pi} \cdot B_f \quad . \quad . \quad . \quad . \quad . \quad (214)$$

where B_f is the ferric induction, i.e. the flux density due to the magnetization of the iron itself, and F is a constant which depends upon the relative dimensions of the rod. The expression might also be written $H_d = F \cdot J$, where J is the intensity of magnetization. The value of F for various ratios of length to diameter for cylindrical rods may be determined from a curve such as that given in Fig. 221, which has been plotted from values given by Du Bois (Ref. (5)) and by Thompson and Moss (Ref. (9)).

For an ellipsoid or very long rod, the value of the coefficient F may be calculated from the expression

$$F = 4\pi \left[\frac{1}{2k} \log_\varepsilon \frac{1+k}{1-k} - 1\right]\left(\frac{1}{k^2} - 1\right)$$

where $k = \sqrt{\left(1 - \frac{a^2}{b^2}\right)}$

a = the minor axis of the ellipsoid,
b = the major axis of the ellipsoid.

To obtain the true value of the magnetizing force H acting on a bar specimen H_d must be subtracted from the value of H calculated from the ampere-turns per centimetre length of the magnetizing

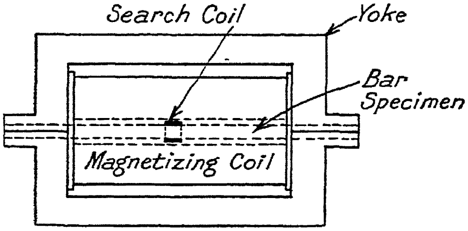


FIG. 222. BAR AND YOKE METHOD

g. From the curves given it can be seen that the ratio $\frac{\text{Length}}{\text{Diameter}}$ of the specimen must be of the order of 25 or more for the effect to have a negligible influence upon the value of H.

On account of this demagnetizing effect the value of H is often measured by means of search coils wound on thin strips of glass and placed with the glass lying flat on the bar specimen (Refs. (2), (3)). The flux density in the air at the surface of the specimen (which is the same as H in the specimen) is measured by this means instead of relying upon calculated values and corrections.

Bar and Yoke Methods. Such methods are commonly used for the testing of bar specimens. They combine the advantages of both ring and bar specimens, the demagnetizing effect being largely eliminated by the use of heavy-section yokes, while the advantages of the bar specimen, as regards preparation and ease of application of the magnetizing force, are retained.

There are a number of such methods which are all essentially modifications of the original yoke method due to Hopkinson.

A search coil is wound upon the bar specimen at its centre, and the bar is then clamped between the two halves of a massive iron yoke, whose reluctance is small compared with that of the specimen, as shown in Fig. 222. The magnetizing winding is fixed inside the yoke, as shown, the specimen fitting inside it.

Let N = No. of turns on the magnetizing winding.
,, I = current in the magnetizing winding.
,, l = length of specimen between the two halves of the yoke.
,, A = cross-section of the specimen.
,, μ_s = permeability of the specimen when the magnetizing current is I.
,, S_y = the reluctance of the yoke.
,, S_g = the reluctance of the two joints between specimen and yoke.
,, ϕ = the total flux in the magnetic circuit.

Then
$$\phi = \frac{\frac{4\pi}{10} . NI}{S_y + S_g + \frac{l}{A\mu_s}}$$

Now, if H is the actual magnetizing force acting on the specimen, and B is the flux density in it,

$$B = \mu_s . H = \frac{\phi}{A}$$

$$\text{Hence, } H = \frac{\frac{4\pi}{10} . NI}{\left(S_y + S_g + \frac{l}{A\mu_s}\right) A\mu_s} = \frac{\frac{4\pi}{10} . NI}{S_y A\mu_s + S_g A\mu_s + l}$$

This may be written

$$H = \frac{\frac{4\pi}{10} . NI}{(1 + m)l}, \text{ where } m = \frac{A\mu_s}{l}(S_y + S_g) \quad . \quad . \quad (215)$$

The quantity m is made small by carefully fitting the specimen into the yoke, and by making the yokes of very heavy section, thus reducing both S_g and S_y to small quantities. In preparing the specimen its dimensions must be very carefully adjusted so that it exactly fits the holes in the yoke to be used. The length of the specimen usually used is about 20 or 25 cm. and the diameter (if of rod form) about 1 cm.

If m is small,

$$H = \frac{4\pi}{10} . \frac{NI}{l}(1 - m) \text{ approx.} \quad . \quad . \quad . \quad (216)$$

which means that the actual value of H in the specimen differs from the value calculated from the magnetizing ampere-turns and length of specimen by the amount

$$\frac{4\pi}{10} . \frac{NI}{l} . m$$

The flux density may be measured by ballistic galvanometer in the usual way.

Permeameters. Permeameters, of which there is a large number of different types, are essentially pieces of apparatus constructed for determination of the B-H curve of magnetic specimens, of bar form, by means of a test which is conveniently simple, and for the performance of which the time required is short. Only a small number of such permeameters can be described in the space available here. In the works given in Refs. (1), (3), (10), (11), (12), and (14), many other forms are described.

EWING DOUBLE-BAR METHOD. In this permeameter two exactly similar bar specimens of the material under test are used, with two pairs of magnetizing coils, one pair of the latter being exactly half the length of the other pair. The number of turns, per centimetre axial length, is the same for both pairs of coil. Two yokes of annealed soft iron, with holes to receive the ends of the bar specimens—the fit being tight—are used. The arrangement of bars and yokes is shown in Fig. 223.

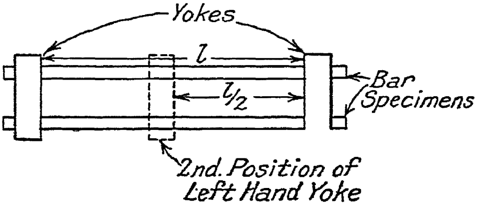


FIG. 223. EWING DOUBLE-BAR METHOD

The object of this arrangement is the elimination of the reluctance of the yokes and air gaps—assumed to be the same, for a given flux density, in all positions of the yokes—by making two tests, one with a length of specimen l and the other with a length $\frac{l}{2}$. The difference in m.m.f. required to produce the same flux density in the two cases being that required for a length of l in each specimen. Thus,

Let n = No. of turns per centimetre length for both pairs of magnetizing coils.

,, I_1 = the current in the coils when the specimen length is l.

,, I_2 = the current in the coils for length $\frac{l}{2}$.

,, H_1 = the apparent magnetizing force for length l.

,, H_2 = ,, ,, ,, ,, $\frac{l}{2}$.

Let a = the m.m.f. required for the yokes and air gaps in each case.

,, B = the flux density in the specimen (the same in each case).

Then
$$H_1 = \frac{4\pi}{10} \cdot \frac{nI_1 l}{l} = \frac{4\pi}{10} \cdot nI_1$$

$$H_2 = \frac{4\pi \cdot nI_2 \frac{l}{2}}{10 \frac{l}{2}} = \frac{4\pi}{10} \cdot nI_2$$

Then, if H be the true magnetizing force in the iron for a flux density B,

$$Hl = \frac{4\pi}{10} \cdot nI_1 l - a = H_1 l - a$$

and
$$H \frac{l}{2} = \frac{4\pi}{10} \cdot nI_2 \cdot \frac{l}{2} - a = H_2 \frac{l}{2} - a$$

Hence,
$$a = l(H_2 - H_1) \quad . \quad . \quad . \quad . \quad . \quad . \qquad (217)$$

and
$$H = \frac{H_1 l - l(H_2 - H_1)}{l} = 2H_1 - H_2 \quad . \quad . \quad . \qquad (218)$$

The flux density corresponding to this actual value of H is measured by means of search coils and ballistic galvanometer in the ordinary way. The complete test is carried out by first obtaining and plotting a B-H curve for the specimens with a length of l—the apparent values of H being plotted. The specimens are then demagnetized, and a second B-H curve is obtained, and plotted, with a specimen length of $l/2$, the two curves being plotted on the same axes. The true B-H curve is obtained from these two, obtaining the true value of H corresponding to any value of B, from the expression above.

The disadvantages of the method are that the joint- and yoke-reluctances are not quite the same for two different positions of the yokes. The test requires, also, two carefully prepared specimens, and is somewhat lengthy in operation.

EWING PERMEABILITY BRIDGE. In this apparatus a standard bar, whose B-H curve is known, is used, the value of H required to produce a certain flux density in the bar specimen under test being compared with that required by the standard bar for the same flux density. The two bars—each of which is placed inside a magnetizing coil—are joined together at the ends by yokes. The number of turns on the coil which magnetizes the standard bar is fixed, and the magnetizing force H which acts upon this bar, for a given current in this coil, is known. The same current flows through both coils, but the number of turns on the coil which magnetizes the test bar can be altered by dial switches, an arrangement for keeping the circuit resistance constant being provided. The adjustment of the number of turns on this winding is continued until there is no difference of magnetic potential between the two yokes, when the flux density is the same in the two bars. The equality of potential of the two yokes is detected by means of a length of iron in the shape of an inverted *U*, having an air gap containing a pivoted magnetic needle at its centre. The two lower ends of this *U* are in contact with the two yokes, and if these are at different potentials flux will pass through the *U* piece and cause the needle to deflect from its zero position. When the magnetizing turns on the test bar coil have been adjusted until this needle ceases to deflect, the flux density in the test bar is the same as that in the standard bar. The value of *H* for the test bar is obtained from the number of turns on its magnetizing winding compared with the number on the magnetizing winding of the standard bar, and the B-H curve of this bar is then constructed from that of the standard. This apparatus is not very widely used owing to its somewhat limited scope, and to inaccuracies which are chiefly due to variations of the reluctance of the joints in the magnetic circuit.

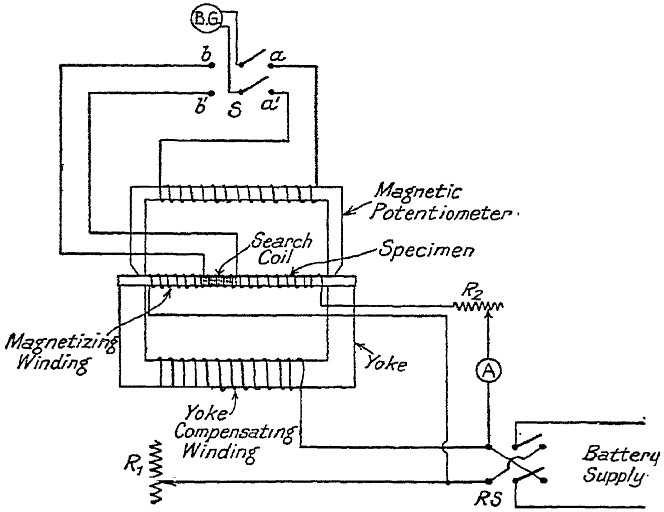


FIG. 224. ILLIOVICI PERMEAMETER

ILLIOVICI PERMEAMETER. The arrangement is shown in Fig. 224. There are two magnetizing windings connected in parallel, one uniformly distributed on the specimen, and the other—which acts as a compensating winding—on the yoke. Each has a variable resistance in series with it, the winding on the specimen having, also, an ammeter in series with it. The operation of the permeameter depends upon the fact that when the magnetomotive forces of the two magnetizing windings—on the yoke and specimen—have been adjusted until they are just sufficient to drive the existing flux through the reluctance of that part of the circuit upon which they are wound, there will be no magnetic potential difference across either of these parts. The magnetic potentiometer is used to indicate when this condition has been attained. Thus, when the magnetic potential drop across the specimen is zero, the m.m.f. in its magnetizing winding is just sufficient to drive the existing flux through its own reluctance. The true value of H in the specimen is then given by the m.m.f. then existing in this winding divided by the length of the specimen between the two arms of the yoke. The effect of yoke reluctance is thus eliminated. The flux density in the specimen is measured by a search coil and ballistic galvanometer in the usual way.

In carrying out a test the bar specimen is placed on the yoke with the search coil and magnetizing winding over it, the magnetic potentiometer being then clamped in position and the specimen demagnetized.

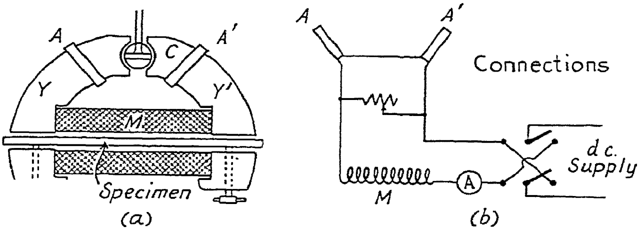
The current in the magnetizing winding of the test bar is then adjusted to give the value of H required, and the throw-over switch S is placed on contacts aa'. Resistance R_1 is then adjusted until no galvanometer throw is observed, when the reversing switch RS is reversed. The value of H in the specimen is then given by

$$H = \frac{4\pi}{10} \cdot \frac{NI}{l}$$

where N is the number of turns on the magnetizing coil on the specimen, l the length (in centimetres) of the specimen between the arms of the yoke, and I is the current indicated by the ammeter. The switch S is now thrown on to contacts bb' and the flux density in the specimen corresponding to this value of H is measured by observing the galvanometer throw when RS is reversed.

Tests up to $H = 400$ can be made with this apparatus, its principal advantages being its simplicity and its independence of the reluctance of the yoke.

KOEPSEL PERMEAMETER. This piece of apparatus will be described because its principle is different from that of most other permeameters of the bar and yoke type. Fig. 225(a) shows the construction, from which it can be seen that the apparatus resembles a D'Arsonval instrument, the permanent magnet being replaced by the bar specimen and the heavy section yokes YY' in which the specimen is clamped. The moving coil C swings in a narrow air gap and is supplied with a known current from a battery, through a milliammeter. This coil carries a pointer moving over a scale, and the deflection, for a given current, is obviously proportional to the flux density in the air gap, which, again, is proportional to the flux density in the specimen. The m.m.f. absorbed in the yokes is compensated for by the two coils AA'. M is the magnetizing coil surrounding the specimen, the proportions and number of turns on this coil being such that H in the specimen is 100 times the current in the coil (in amperes). The moving coil is so designed that the scale gives the flux density B in the specimen, directly, if the current in the moving coil in milliamperes is $\frac{50}{S}$, where S is the area of cross-section of the specimen in square centimetres. The compensating coils AA' are connected in series with coil M, and are shunted by a variable resistance. Before starting a test this resistance is adjusted until the moving coil shows no deflection with a heavy current in M, and with no specimen between the yokes.



(a) (b)

FIG. 225. KOEPSEL PERMEAMETER

A number of different bushings for use in the holes in the yokes allow different sizes of specimens to be used.

In carrying out a test the apparatus is set up so that the axis of the specimen is perpendicular to the magnetic meridian. The specimen is then clamped in position and demagnetized, the resistance shunting coils AA' having been previously adjusted for compensation. The current in M is then adjusted to give the required value of H, and the deflections with this current—both direct and reversed—are observed, the moving-coil current being adjusted in accordance with the cross-section of the specimen. The mean of the two deflections—which may be appreciably different—gives the flux density in the specimen.

The apparatus may be used for the determination of both B-H curves and hysteresis loops. The values of H for given values of flux density were found by C. W. Burrows, when investigating the characteristics of this type of permeameter, to be erroneous by an amount which depended upon the quality and size of the magnetic specimen. Hysteresis loops obtained by its use gave too low a value of residual flux density and too high a value of coercive force.

These errors may be corrected for by checks with standard bars whose magnetization curves are known.

TRACTION PERMEAMETERS. This type of permeameter makes use of the fact that the pull between two magnetized surfaces is given by

$$F = \frac{B^2A}{11,200,000} \text{ lb. wt.}$$

where B is the flux density in lines per square centimetre and A is the cross-section in square centimetres.

In the Thompson form of permeameter the pull required to separate a bar specimen from a yoke is measured by a spring balance and the flux density obtained from the above expression. H is obtained from the constants of the magnetizing coil—which surrounds the bar—and from the current in this coil.

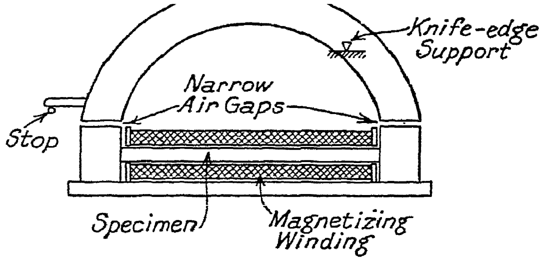


FIG. 226. DU BOIS MAGNETIC BALANCE

Another form of traction permeameter—the Du Bois magnetic balance—consists of a semicircular yoke which is divided into three parts by two air gaps near the lower ends (see Fig. 226). The surfaces at the air gaps are carefully faced to form plane surfaces. The bar specimen is placed inside a magnetizing coil and fits into the two lower fixed parts of the yoke. The weight W is slid along the scale carried by the pivoted upper part of the yoke, until the right-hand side of the yoke is pulled over by the pull across the air gap. The position of the weight, when this occurs, gives a measure of the flux density in the specimen. The apparent value of the magnetizing force acting on the specimen must be corrected by calibration of the apparatus, using a standard bar or otherwise. The correction required depends upon material which is being tested.

The inaccuracy of such permeameters causes them to be little used except for rough tests upon bar specimens.

BURROWS DOUBLE-BAR AND YOKE PERMEAMETER. This permeameter, which was first developed by Dr. C. W. Burrows (Ref. (12)), has been adopted as the standard apparatus for the testing of bar specimens in America, and is used by the Bureau of Standards. The effect of magnetic leakage at the joints between yoke and specimen are eliminated in this permeameter by the use of a number of compensating coils which apply compensating m.m.f's at different parts of the magnetic circuit, these m.m.f's being just sufficient to drive the flux through the reluctance of the part upon which the coils are placed.

Fig. 227 shows the arrangement of the magnetic circuit and coils. S_1 is the bar specimen to be tested, S_2 being a bar of similar dimensions to S_1. These bars are surrounded by magnetizing windings M_1 and M_2, which are uniformly wound along the lengths of the bars. A_1, A_2, B_1, and B_2 are compensating coils for the elimination of leakage effects at the joints between the two bars and the massive yokes YY' into which the bars fit; c and c' are two exactly similar search coils wound at the centres of the two bars, while d_1 and d_2 are two similar search coils wound in the positions shown, on the test bar, and each having exactly half the number of turns of search coil c. Coils d_1 and d_2 are connected permanently in series, as are also the four coils A_1, A_2, B_1, B_2. The dimensions of the coils M_1, M_2 are such that H in the specimen is approximately 100 times the current in the windings, the maximum value of H for which the apparatus is used being from 300 to 400. The dimensions of the bar specimens are approximately 30 cm. long and 1 cm. in diameter.

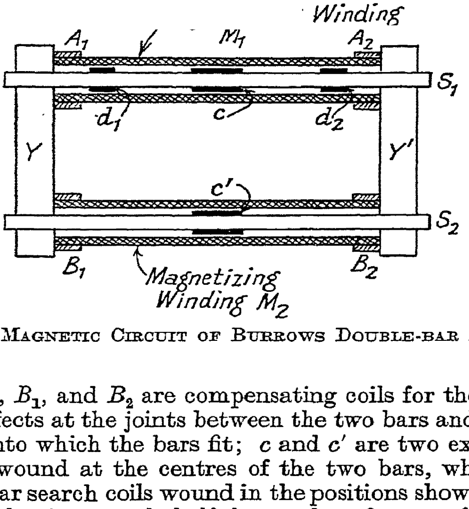


FIG. 227. MAGNETIC CIRCUIT OF BURROWS DOUBLE-BAR AND YOKE

The coils A_1, A_2, B_1, B_2, are supplied from a separate battery supply, coil M_1 from another separate supply, and M_2 from another. In carrying out the test it is necessary, first of all, to ensure that, for a given value of the current in M_1—i.e. of H in the test bar—the flux threading through all four search coils c, c', d_1, and d_2, is the same, the current in the compensating coils, and in M_2, being adjusted until this condition is fulfilled. If the flux threading coils

d_1 and d_2 is the same as that threading c and c', there can be no appreciable leakage of flux through the air in the neighbourhood of the joints. This means that the m.m.f. for the joints is being supplied by the compensating coils, and that the m.m.f. in coil M_1 is used up merely in driving the flux through the bar specimen S_1 inside it. Thus H in the specimen is given by $\frac{4\pi}{10} \cdot \frac{NI}{l}$ where

N = No. of turns on coil M_1
I = current (in amperes) in M_1
l = length of specimen in centimetres

It may, in some cases, be necessary to make corrections for the fact that the magnetizing solenoids are not infinitely long, but such corrections are usually negligible.

The procedure for obtaining equal flux threading all four search coils is as follows.

First, the specimen having been demagnetized, the current in the magnetizing coil M_1 is set at the required testing value. Search coils c and c' are then connected in series, but in opposition, to a ballistic galvanometer. The currents in coil M_1 and M_2 are then simultaneously reversed. A throw will be observed on the galvanometer. The current in M_2 is adjusted until no throw is obtained when the two currents are reversed. Since search coils c and c' have equal numbers of turns, this means that equal fluxes are now threading through them. Next, search coil c is connected in series with, but in opposition to, coils d_1 and d_2, and then to the ballistic galvanometer. The current in the compensating coils A_1, A_2, B_1, B_2, is then adjusted until no galvanometer throw is obtained upon simultaneous reversal of the currents in these coils and in M_1 and M_2. Then, since d_1 and d_2 together have the same number of turns as coil c, the flux threading all three coils is the same.

The flux density corresponding to the value of H in coil M_1 (obtained as described above) can be measured by connecting coil c alone to the ballistic galvanometer, and noting the throw when the currents in the two magnetizing coils and the compensating coils are simultaneously reversed.

Fig. 228 gives a diagram of connections showing how the switching may be arranged for convenience in carrying out the test as described above.*

FAHY SIMPLEX PERMEAMETER. This permeameter, which has been used by the Bureau of Standards, and is being used quite commonly in this country for the routine testing of bar specimens, has the advantage of simplicity both in construction and operation, while its accuracy is of the same order as that of the Burrows apparatus.

An outline of its construction is given in Fig. 229. Two iron

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* Fuller descriptions of the apparatus, testing methods, and the application of corrections, will be found in the works given in Refs. (1), (2), (3), (12).

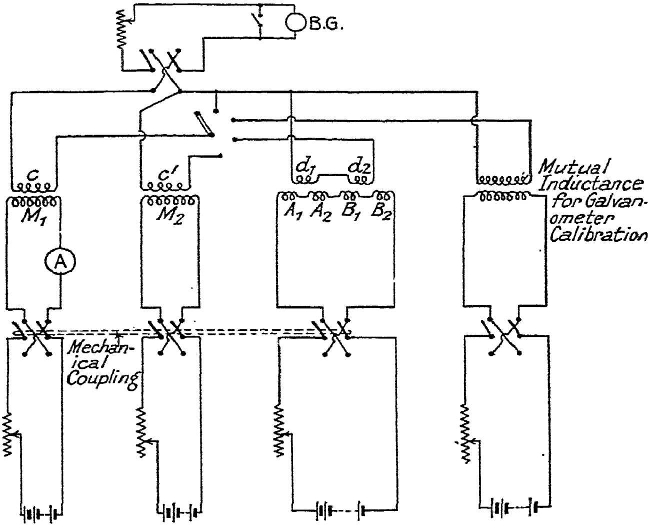

FIG. 228. ELECTRICAL CIRCUITS OF BURROWS DOUBLE-BAR AND YOKE PERMEAMETER

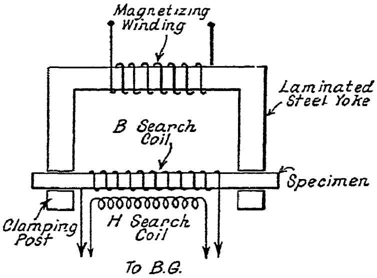

FIG. 229. FAHY SIMPLEX PERMEAMETER

clamping posts clamp the specimen (of cross-section $\frac{3}{4}$ in. by $1\frac{3}{4}$ in.) against a laminated steel yoke. The latter carries the magnetizing winding, and the specimen is surrounded by a search coil for the measurement of the flux density in it. The magnetizing force H acting on the specimen is measured, like the flux density, by a ballistic galvanometer, utilizing an air-cored search coil of several thousand turns located between the two clamping posts as shown. The values of H so measured are corrected by calibration of the H search coil, utilizing a specimen of known magnetic characteristics in place of the test specimen.

DRYSDALE PLUG PERMEAMETER. This permeameter, devised by Dr. C. V. Drysdale, is for the testing of a large mass of magnetic material. A special drill is used to cut a cylindrical hole in the mass of iron to be tested, the drill being so formed that it leaves a thin rod or pin of the metal (about 0·1 in. diameter) standing in the centre of the hole. A rose cutter at the upper end of the drill cuts a conical hole at the surface of the iron, so that after drilling, a section through the hole is as shown in Fig. 230 (*a*). A split conical plug, having a small magnetizing coil and search coil fixed to its lower end, is then pressed into the hole so that it grips the centre pin tightly and also makes good contact with the side of the conical hole in the iron mass (Fig. 230(*b*)). The whole then forms a bar and yoke permeameter on a small scale, the centre pin of material being the bar specimen and the mass of the material, together with the conical plug, forming the yoke. Two holes in the plug allow leads from the magnetizing coil and search coil to be brought out.

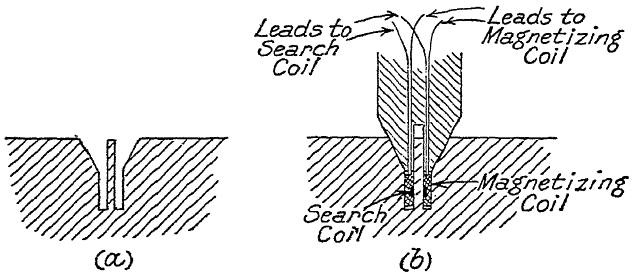

FIG. 230. DRYSDALE PLUG PERMEAMETER

A measured current is passed through the magnetizing coil from a battery through a reversing switch, H being obtained from the characteristics of the magnetizing coil. The corresponding flux density is measured by the search coil and ballistic galvanometer, or fluxmeter, in the usual way.

Magnetic Testing with Intense Fields. The methods of testing so far described have been suitable for testing with values of the magnetizing force H up to 400 or 500 in most cases. For tests at greater field strengths than these special methods have to be used.

Various "isthmus" methods, most of which are modifications of the original isthmus method due to Ewing, are used for this purpose, the title being derived from the fact that the specimen, in such methods, forms a narrow "isthmus" between the two specially-shaped poles of an electromagnet.

EWING'S ISTHMUS METHOD. Ewing carried out tests up to $H = 24{,}500$, and B about 45,000, by the use of the apparatus illustrated in Fig. 231.

The electromagnet carries magnetizing windings as shown. The pole pieces are conical and have a cylindrical seating, so that the specimen may be rotated through 180° during the test. The specimen itself is turned down to a bobbin shape, having a cylindrical portion in the middle and conical ends which abut against faces on the pole pieces, the latter forming a continuation of the conical ends of the specimen. The whole of the flux in the pole pieces is thus forced to pass through the specimen of very much smaller section, thus giving a very high flux density in the specimen. The flux density obtainable depends upon the area of cross-section of the isthmus, or cylindrical part, of the specimen.

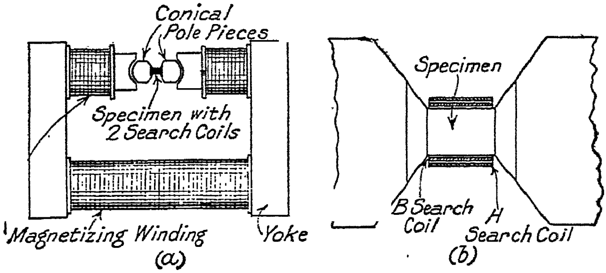

FIG. 231. EWING'S ISTHMUS METHOD

Two search coils, having equal known numbers of turns, and of known cross-sections, are fitted on the cylindrical portion of the specimen, as shown in Fig. 231. The inner coil—for the measurement of B—fits the specimen closely, while the outer coil—for the measurement of H—is separated from the inner coil by a small annular space.

In carrying out a test a magnetizing current is passed through the winding on the electromagnet, the inner search coil being connected to a ballistic galvanometer. The specimen—with its search coils—is then quickly rotated through 180°, this being equivalent, as regards interlinking flux, to a reversal of the magnetizing current. The galvanometer throw is observed and the flux density obtained therefrom. The outer search coil encloses some air flux as well as the flux in the specimen, and the difference between the flux which it encloses, and that enclosed by the inner search coil, is measured. This difference in flux, when divided by the difference in cross-section of the two search coils, gives the flux density in the air surrounding the specimen, and this air flux density is equal to the magnetizing force H in the specimen. The difference in the flux enclosed by the two search coils may be measured by connecting the two coils in series—opposing one another—and to the ballistic galvanometer, when the throw produced by a rotation of the specimen through 180° gives a measure of the flux between the coils. Otherwise, the total flux enclosed by the outer coil may be measured and the difference in enclosed flux obtained by subtraction. The inner search coil must of necessity enclose some air flux as well as the flux in the specimen. This air flux is taken into account, using the measured value of H as the air flux density. The small cross-section of the specimens used in such tests renders this correction necessary in most cases.

The slope of the conical pole and specimen end pieces for maximum concentration of flux is about 60° (between the cone side and axis), whilst the angle for the most uniform field within the specimen is about 39°. A compromise is usually made in practice.

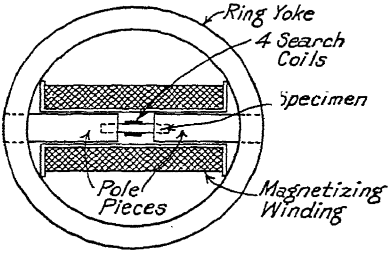

FIG. 232. GUMLICH'S METHOD

GUMLICH'S METHOD. This method is the same in principle as the method described above, but employs an improved method of determining H in the specimen. The arrangement is shown in Fig. 232.

A laminated ring yoke is used with two diametrically opposite pole pieces PP'. These are 25 mm. in diameter, and have a central air gap of 12 mm. The pole pieces are bored centrally to take a specimen of diameter 6 mm. A magnetizing coil M surrounds the pole pieces and specimen, and upon the latter are wound four search coils in four layers. The radial spaces between the layers—and hence between the search coils—are known, and the coils have equal numbers of turns.

The flux density in the specimen is measured by connecting the inner coil—of known cross-section—to a ballistic galvanometer, and reversing the magnetizing current. Then, by connecting the search coils in pairs, opposing one another, to the ballistic galvanometer, and reversing the magnetizing current in each case, the air flux density in inter-coil spaces can be measured. These are plotted vertically on a graph, taking radial distances from the surface of the specimen as abscissae. By extrapolation the flux density at the surface of the specimen—and hence H in the specimen—can be obtained. A correction can be made also, from this graph, for the air flux enclosed by the inner search coil.

This apparatus can be used up to $H = 6{,}000$, and can be adapted to tests upon sheet material by using pole pieces with rectangular holes. Other methods of carrying out tests at high inductions are described in the publications given in Refs. (1), (2), (3), (13).

The Testing of Feebly Magnetic Materials. For measurements upon material which is distinctly magnetic, but whose permeability is low (of the order of unity), the circuit shown in Fig. 233 may be used. The magnetizing winding has a search coil at its centre, this search coil being fixed so that the specimen can be slipped inside it or withdrawn without disturbing the search coil itself. This search coil is connected, through a ballistic galvanometer, to the secondary of a mutual inductance, the primary of which is connected in the magnetizing circuit.

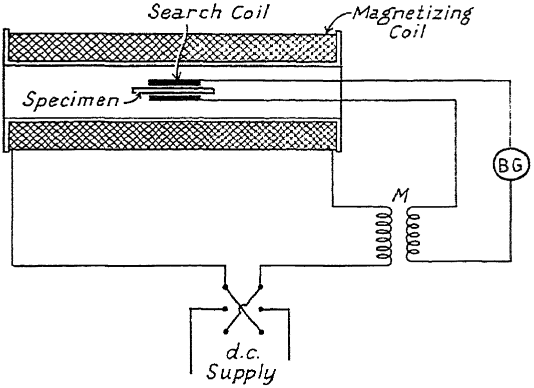

FIG. 233. APPARATUS FOR TESTING FEEBLY MAGNETIC MATERIALS

Since the material under test is only feebly magnetic, it is assumed that there is no demagnetization due to end effect. The procedure in testing is as follows.

Current is passed through the magnetizing coil to give the required field strength H, which is given simply by $H = \frac{4\pi}{10} \cdot \frac{NI}{l}$ (NI being the number of ampere-turns on the magnetizing coil and l its axial length), since the demagnetizing effect of the ends of the specimen is negligible. Before placing the specimen inside the search coil the mutual inductance is adjusted to give no galvanometer throw when the magnetizing current is reversed. The specimen is then inserted, and the galvanometer throw upon reversal of the current observed. The flux density obtained from this throw will be that due to the ferric induction in the specimen. Calling this flux density B_f, we have for the total flux density, when the field strength is H, $B = B_f + H$, and for the permeability $\mu = \frac{B_f + H}{H}$

The Curie balance, and various modifications of it (Refs. (3), (15)), may be used for the measurement of the susceptibility of materials such as ores and rock specimens. A simplified drawing of the arrangement is shown in Fig. 234.

The specimen is placed in a glass tube which is suspended from a light arm which also carries a balance weight and a copper vane, the latter moving in the gap of a permanent magnet for damping purposes. The moving system also carries a mirror and another balance weight, as shown. A permanent magnet of ring shape is mounted on an arm which may be rotated so that the air gap of the magnet may be moved towards or away from the tube containing the specimen.

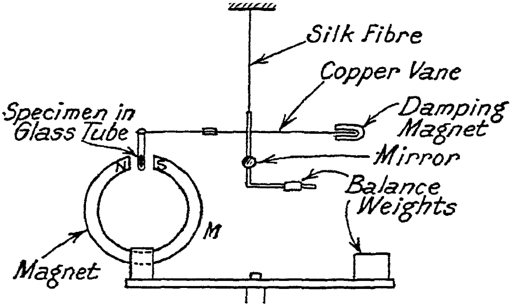

FIG. 234. THE CURIE BALANCE

In carrying out the test a measurement is first made upon the glass tube alone. The magnet M is rotated until maximum deflection of the moving system is obtained, this deflection θ' being observed. The specimen is then placed in the tube and the maximum deflection θ_1 of the moving system with different positions of M again observed. A third measurement is then made with the specimen replaced by some material whose susceptibility is known (distilled water, whose susceptibility is $0{\cdot}79 \times 10^{-6}$ is often used for this purpose).

Let the deflection now be θ_2. Then

$$\frac{\kappa_x}{\kappa_s} = \frac{m_s(\theta_1 - \theta')}{m_x(\theta_2 - \theta')} \quad . \quad . \quad . \quad . \quad . \quad (219)$$

where κ_x and κ_s are the susceptibilities, and m_x and m_s the masses, of the unknown and standard materials respectively.

It should be noted that if θ' is in the opposite direction to θ_1 and θ_2 it must be treated as negative.

W. Sucksmith and R. R. Pearce (Ref. (43)) have described a method, using a balance, for the measurement of magnetic susceptibilities in a controlled atmosphere, or *in vacuo* at temperatures up to 1,500° C.

The Testing of Permanent Magnets. The details of the method of testing permanent magnets, when manufactured, depend upon the shape of the magnet, this varying greatly according to the purpose for which the magnet is to be used. Owing to the self-demagnetizing force, it is necessary, in testing such magnets, to measure H by means of search coils, of small cross-section, laid flat against the surface of the magnet as previously described. These search coils give the value of the air flux density at the surface of the magnet at different points, and thus measure H for these points. The flux density B is measured by search coils in the ordinary way.*

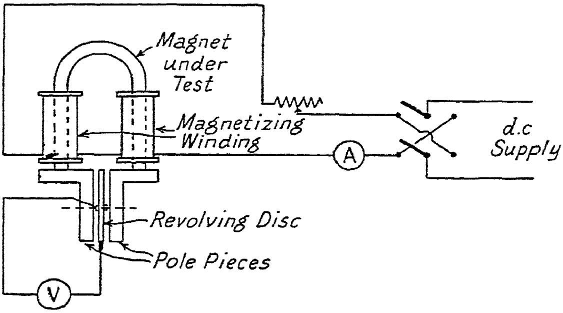

FIG. 235. BETTERIDGE APPARATUS

BETTERIDGE APPARATUS FOR MAGNET TESTING (Ref. (17)). The principle of this apparatus for the commercial testing of permanent magnets may be understood from Fig. 235. The magnet to be tested is placed with its straight ends inside two magnetizing coils and so that its ends press against two pole pieces, in the air gap between which a thin iron disc, plated with copper, is mounted. This disc is mounted in ball bearings and the clearances between it and the pole pieces are small. It is driven round at constant speed by a small motor, and has two small brushes making contact with its spindle and its rim. When a current flows in the magnetizing windings flux crosses the gap between the pole pieces, and the flux density in the gap will be proportional to that in the magnet. An e.m.f. will be induced in the revolving disc, this e.m.f. being proportional to the gap flux density and being measured by the millivoltmeter V.

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* Details of tests upon different shapes of magnets, upon magnet steels, and of tests for the determination of temperature effects upon magnets, will be found in the works given in Refs. (3), (13), (14), and (16).

Thus, the voltmeter reading gives a measure of the flux density in the specimen, while H is obtained by the usual formula from the constants of the magnetizing coils and from the current in these coils.

Some leakage exists between the two arms of the magnet, but with proper design of the apparatus its effect upon the results is small.

British Standard Specification No. 406—1931 gives a complete specification for this apparatus, and also for another apparatus of the moving-coil pattern which uses a fluxmeter of special design.

Magnetic Testing with Alternating Current. When iron is subjected to an alternating magnetic field a loss of power occurs due to hysteresis effects in the iron. Power is absorbed, also, due to eddy currents, which are set up in the iron due to the fact that it is electrically conducting material, and that the flux threading through it is changing. Such eddy currents will be discussed more fully in Chapter XIV.

Although the hysteresis loss per cycle in iron may be determined from the hysteresis loop obtained in a d.c. test, this loss may be somewhat different under the actual alternating magnetization conditions with which it will be used in practice. Also, eddy current losses can only be measured by the use of alternating current. For these reasons, inspection tests upon sheet steel which is to be used in the manufacture of transformers and other a.c. apparatus are very commonly a.c. tests.

Separation of Iron Losses. It is often sufficient, in acceptance tests of sheet material, to measure the total loss in the steel at the standard frequency (50 cycles) and with a maximum flux density of about 10.000 lines per square centimetre. The separation of the losses into their two components—i.e. hysteresis loss and eddy current loss—involves a rather more lengthy test.

Hysteresis loss, as already seen, is given by the expression

$$W_h = k \,.\, f \,.\, B_{max}{}^{1 \cdot 6}$$

where W_h is the loss in watts per cubic centimetre of material, f being the number of cycles of magnetization per second (i.e. the frequency), B_{max} the maximum flux density, and k a constant for any given material. This law holds approximately for values of B_{max} between 1,000 and 12,000 lines per sq. cm.

Eddy current loss, provided the sheets are sufficiently thin for "skin effect" (see Chapter XIV) to be negligible, is given by the expression

$$W_e = k' K_f{}^2 f^2 t^2 B_{max}{}^2$$

where W_e is the loss in watts per cubic centimetre, f the frequency, t the thickness of the sheet, and B_{max} the maximum flux density. K_f is the "form factor" of the alternating wave of flux and depends upon the shape of this wave.

Thus, if the form factor remains constant throughout a test, and the maximum flux density B_{max} is kept constant, the total power loss being measured at different frequencies, then this total loss may be written

$$W_t = Mf + Nf^2$$

where $M = kB_{max}^{1.6}$

and $N = k'K_f^2 \, . \, t^2 \, . \, B_{max}^2$

both M and N being constant for this test.

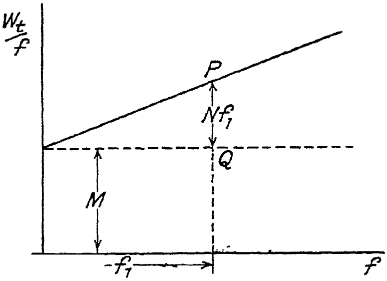


FIG. 236. SEPARATION OF IRON LOSSES

These constants may be determined—and the total loss thus split up into its two components, for any given frequency—by plotting $\frac{W_t}{f}$ against f as shown in Fig. 236.

Then, $\frac{W_t}{f} = M + Nf$, so that the intercept on the vertical axis gives M; and N can be obtained from the slope of the graph. M is the hysteresis loss per cycle. The eddy-current loss for any frequency f_1 is given by the intercept PQ, as shown in Fig. 236. In this way the two components of the total loss for any given form factor and maximum flux density, can be separated.

Again, if the frequency and B_{max} are kept constant and the form factor varied, the total loss being measured for various values of form factor, then we have

$$W_t = C + DK_f^2 \quad . \quad . \quad . \quad . \quad . \quad (220)$$

If W_t is now plotted vertically against values of K_f^2 horizontally the constants C and D may be obtained. The intercept upon the vertical axis (Fig. 237) gives C, and the slope of the line gives D.

Thompson and Walmsley (Ref. (20)) have described a method of measuring and of separating the iron losses of a transformer, using thermionic valve rectification to obtain the form factor of the voltage wave.

The effects of variation of both form factor of the applied voltage and of temperature are considered in this paper.

Methods of Measurement. WATTMETER METHOD. This is perhaps the commonest method of measuring the total loss in sheet steel with alternating current. The sheet material to be tested is arranged in the form of a "magnetic square," of which there are several forms, Epstein being the originator of the arrangement. In this square there are four bundles of strips of the sheet material. These are bound with tape to form four cores to fit inside four magnetizing coils, the individual strips being insulated from one another by thin tissue paper. The ends of the cores, projecting beyond the magnetizing windings, are, in the Epstein apparatus, interleaved and clamped at the corners.

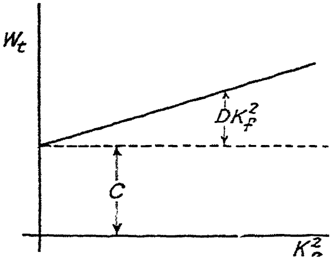


FIG. 237. SEPARATION OF ON LOSSES

Fig. 238 shows a magnetic square as used by Lloyd and Fisher and now in use at the National Physical Laboratory and Bureau of Standards. The strips of material—cut half in the direction of rolling of the sheet during manufacture and half perpendicular to this direction—are about 25 cm. long and 5 or 6 cm. wide. They are built up into four bundles and assembled to form a complete magnetic circuit with the aid of bent corner pieces and clamps, as shown in the figure. These corner pieces should be of the same material as the strips, or at least of material having similar magnetic properties. The overlap at the corners should be only a few millimetres and a correction may be applied, if necessary, to account for the fact that the cross-section of the magnetic circuit is doubled at the overlapping points.

A very small number of strips is shown in the figure for the sake of clearness. The bundles of strips are placed inside four similar magnetizing coils of heavy wire, connected in series to form the primary winding. Each of these coils has, underneath it, two single-layer coils of thin wire and having equal numbers of turns. These secondary coils are connected in series in groups of four —one on each core—to form two separate and similar secondary windings.

The primary winding is connected either directly, or through a transformer having a variable secondary, to an alternator having a waveform which is as nearly as possible sinusoidal. By this means regulation of the magnetizing current by means of resistance, with consequent alteration of waveform, is avoided. If the total loss in the sample is to be measured, the connections are as shown in Fig. 239, the wattmeter pressure coil being supplied from one secondary winding and an electrostatic voltmeter from the other.

The iron specimen must be weighed and its cross-section determined before assembly. The magnetizing current is adjusted to give the value of B_{max} required, the frequency of the supply being previously adjusted to the correct value. The wattmeter and voltmeter readings are observed.

Theory of the Method. Then, the voltage induced in secondary winding S_2, whose r.m.s. value is measured by the voltmeter, is given by the expression

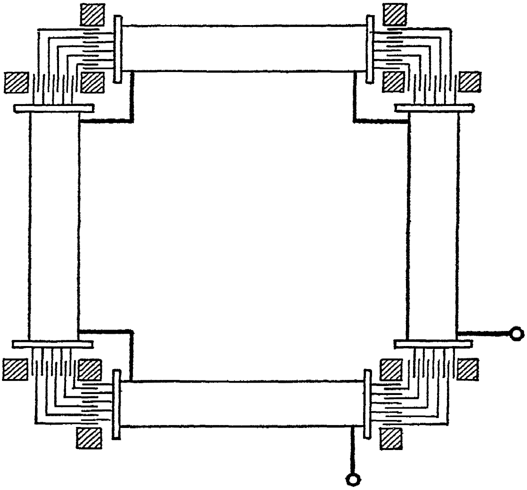


FIG. 238. LLOYD-FISHER MAGNETIC SQUARE

$$E = \frac{4K_f \,.\, B_{max} A \,.\, f \,.\, N_2}{10^8} \text{ volts} \quad . \quad . \quad . \quad (221)$$

where K_f is the form factor, A the cross-section of the specimen, in which the maximum flux density is B_{max}, f the frequency, and N_2 the number of turns on secondary winding S_2.

Hence the value of B_{max} may be obtained,

$$B_{max} = \frac{E \times 10^8}{4\,K_f \,.\, A . f . N_2} \quad . \quad . \quad . \quad . \quad . \quad (222)$$

It may be necessary to correct this expression, especially at high values of B_{max}, for the fact that the coil S_2 encloses some air flux as well as the flux in the sample, since the cross-sectional area of the coil must be greater than that of the sample itself.

Let A_c = the cross-sectional area of the coil.
,, A_s = the cross-sectional area of the sample.
,, H_{max} = the maximum magnetizing force (equal to the flux density in the air space within the coil).
,, B_{max} = the actual maximum flux density in the sample.

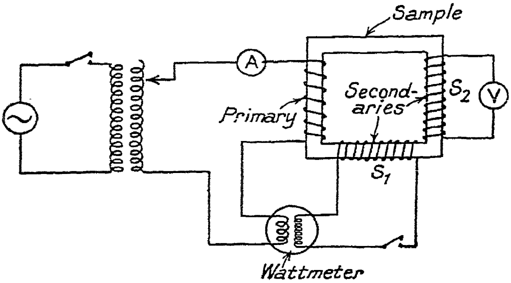


FIG. 239. CONNECTIONS FOR WATTMETER METHOD OF IRON-LOSS MEASUREMENT

Then the total flux within the coil is

$$B_{max} A_s + H_{max} (A_c - A_s) = B'_{max} A_s$$

where B'_{max} is the apparent maximum flux density in the sample. Thus,

$$E = 4K_f \,.\, fN_2 \,[B_{max} A_s + H_{max} (A_c - A_s)] \times 10^{-8}$$
$$= 4K_f \,.\, f \,.\, N_2 \,.\, A_s B'_{max} \times 10^{-8} \text{ volts}$$

where $$B'_{max} = B_{max} + H_{max} \left(\frac{A_c - A_s}{A_s}\right)$$

H_{max} may be determined from the permeability curve of the sample.

As regards the power loss in the iron—

Let W_t = total iron loss.
,, W = wattmeter reading.
,, V = voltage applied to wattmeter pressure coil.
,, E = voltmeter reading
= voltage induced in coil S_1, since S_1 and S_2 have equal numbers of turns.
,, r_p = resistance of wattmeter pressure coil.
,, r_c = resistance of coil S_1.
,, i_p = current in the pressure coil circuit.

Then $$E = i_p (r_p + r_c)$$
$$V = i_p r_p$$

Hence, power loss in the iron, together with the copper loss in the winding S_1 and in the wattmeter pressure coil

$$= W \cdot \frac{E}{V} = W \frac{i_p (r_p + r_c)}{i_p r_p} = W \frac{(r_p + r_c)}{r_p}$$

$$= W \left(1 + \frac{r_c}{r_p}\right)$$

Again, the copper losses in r_p and r_c are

$$i_p^2 (r_p + r_c) = \left(\frac{E}{r_p + r_c}\right)^2 (r_p + r_c) = \frac{E^2}{r_p + r_c}$$

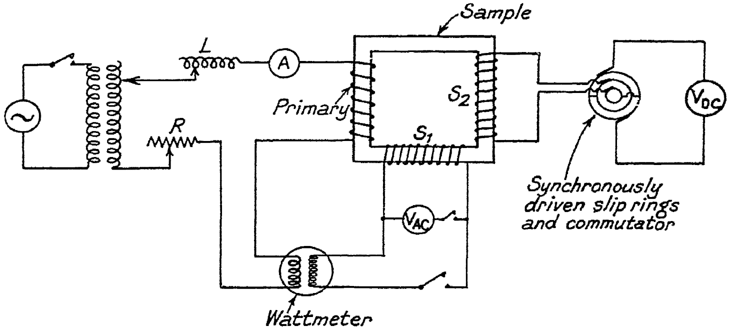


FIG. 240. MEASUREMENT OF POWER LOSS AT DIFFERENT FORM FACTORS

Therefore, $$W_t = W \left(1 + \frac{r_c}{r_p}\right) - \frac{E^2}{r_p + r_c} \quad . \quad . \quad . \quad (223)$$

As mentioned previously, the hysteresis and eddy current components of the loss can be graphically determined from the results of power measurements such as the above, at different frequencies.

Fig. 240 gives the connections for the measurement of power loss at different form factors, the form factor being obtained from measurements of r.m.s. and mean voltages induced in the two windings S_1 and S_2, which have equal numbers of turns. Variation of form factor is obtained by adjustment of R and L—variable resistance and inductance—and of the number of turns on the secondary of the supply transformer. The voltmeter V_{AC} measures the r.m.s. voltage, and voltmeter V_{DC}, in conjunction with a synchronously-driven two-part commutator and slip rings, measures the mean, or average, voltage.

RICHTER APPARATUS. This apparatus, designed for the testing of complete sheets of steel instead of small strips cut from them, does not involve a different method of measurement but only a difference in the arrangement and assembly of the sample. The wattmeter method of measurement of losses is used. The sample consists of four sheets of steel having dimensions 100 cm. wide and 200 cm. long approximately. These sheets are placed inside a hollow wooden drum having a lattice arrangement inside to hold the sheets, and wooden clamps to clamp the ends of each sheet together. The drum is wound with about 100 turns of thick copper wire to carry a heavy magnetizing current.

The disadvantages of the apparatus are that corrections for the air space flux must be made for all values of B_{max}, the copper losses are large, and the sheets are in a strained condition during the test, which means that the losses measured do not give the true losses under unstrained conditions.

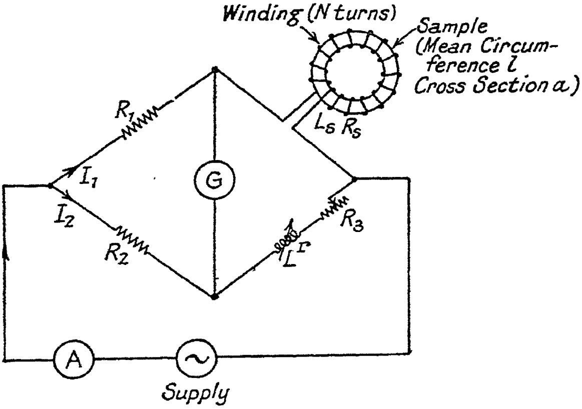


FIG. 241A. MAXWELL BRIDGE FOR IRON-LOSS MEASUREMENTS

A.C. BRIDGE METHODS. The a.c. bridge network can be adapted to the measurement of iron loss and effective permeability of magnetic samples. Such methods are very useful when the available samples are small, and when the test is to be carried out at commercial- or audio-frequencies, and with fairly low values of flux density. Fig. 241A shows the connections of the Maxwell bridge as applied to iron loss and permeability measurements.

R_1, R_2, and R_3 are resistances, the latter being variable. In series with R_3, in the same arm, is a variable inductance L and resistance r. It may be necessary to connect R_3 in series with the winding on the sample if the resistance of the latter is small. The specimen, in ring form, is wound with a winding whose inductance is L_s and effective resistance R_s, this effective resistance containing an iron-loss component. R_w is the actual resistance of the winding on the ring. G is a vibration galvanometer or telephone, and A an ammeter. The supply is from an alternator having a pure sine waveform.

Balance of the bridge is obtained by adjustment of L and R_3.

Theory. At balance

$$I_1R_1 = I_2R_2$$

and $$I_1(R_s + j\omega L_s) = I_2[(r + R_3) + j\omega L]$$

using the symbolic notation. $\omega = 2\pi \times$ frequency.

Then $$\frac{I_1}{I_2} = \frac{R_2}{R_1}$$

$$I_1R_s = I_2(r + R_3).$$

and $$I_1L_s = I_2L$$

from which $$\frac{R_2}{R_1} = \frac{r + R_3}{R_s}$$

or $$R_s = (r + R_3)\frac{R_1}{R_2}$$

and $$L_s = L\frac{R_1}{R_2}$$

The iron loss in the sample is given by $R_sI_1^2 - R_wI_1^2$.

Now, the current $I = I_1 + I_2$ (I_1 and I_2 being in phase)

Thus $$I = I_1 + \frac{R_1}{R_2}I_1 = I_1\left(\frac{R_1 + R_2}{R_2}\right)$$

or $$I_1 = \frac{R_2}{R_1 + R_2} \cdot I$$

Hence, the iron loss W_t is given by

$$W_t = I^2\left(\frac{R_2}{R_1 + R_2}\right)^2 (R_s - R_w) \quad . \quad . \quad (224)$$

If N = No. of turns on the specimen,
l = length of mean circumference of specimen (in centimetres),
a = cross-section of specimen in square centimetres,
μ = effective permeability of specimen,

then, the inductance is

$$L_s = \frac{4\pi \, . \, NI_1}{10^9 \dfrac{l}{a\mu}} \times \frac{N}{I_1} = \frac{4\pi N^2 a}{10^9 l}\,\mu \text{ henries}$$

from which μ may be calculated when L_s has been measured.

CAMPBELL BRIDGE METHOD. Fig. 241B shows the connections of a bridge method due to Campbell (Ref. (19)), this method being one of the best known for the purpose of iron-loss measurement by a bridge network.

The ring specimen carries two windings, a primary and a secondary, having N_1 and N_2 turns respectively, and having the same cross-sectional area. M is a variable mutual inductance connected as shown. R_1 and R_2 are variable resistances, and A is an ammeter.

G is a vibration galvanometer or telephone according to the supply frequency. The waveform of the supply voltage should be sinusoidal, and R_1 should be made sufficiently large to ensure that the current waveform is also sinusoidal. M and R_2 are adjusted until the galvanometer G shows no deflection or until the telephone gives minimum sound. Under balance conditions the vector diagram is as shown in the figure: ϕ is the flux in the sample and I the current in the primary winding, leading the flux by a small angle on account of iron loss; e_1, e_2, and e_m are the *induced* voltages in the primary winding, the secondary winding, and in the secondary of the mutual inductance M respectively. Balance is obtained when the vector sum of e_m and e_2 is equal, and opposite, to the vector IR_2 representing the voltage drop in resistance R_2.

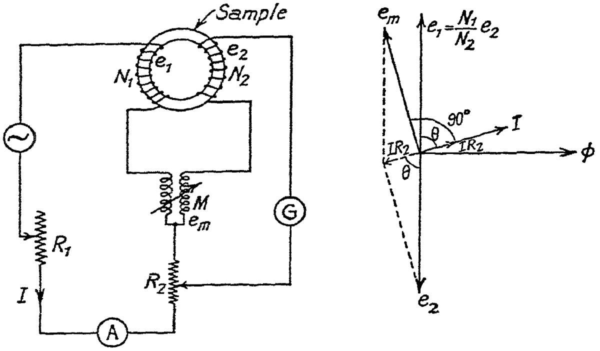


FIG. 241B. CAMPBELL BRIDGE METHOD

From this vector diagram (since $e_m = \omega MI$ and $e_2 = \omega mI$, m being the mutual inductance between the primary and secondary windings on the specimen), we have,

$$e_m = e_2 \text{ (very nearly)}$$

or

$$m = M$$

and also, the iron loss

$$W_t = e_1 I \cos\theta = e_1 I \cdot \frac{IR_2}{e_2} = I^2 R_2 \frac{e_1}{e_2}$$

Now $e_1 = \frac{N_1}{N_2} e_2$, therefore the iron loss in the specimen is given by

$$W_t = \frac{N_1}{N_2} R_2 \cdot I^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Let a be the cross-section of the specimen and l its mean circumference, a' being the cross-section of the windings on the specimen.

Then, flux per ampere flowing in the primary winding, linking the two windings

$$= \frac{4\pi N_1 I}{10\left(\dfrac{l}{a\mu + a' - a}\right)I}$$

or

$$\phi = \frac{4\pi N_1(a\mu + a' - a)}{10l}$$

where μ is the permeability of the specimen. Hence, the mutual inductance

$$m = \frac{4\pi N_1 N_2(a\mu + a' - a)}{10^9 l} = M \quad . \quad . \quad (226)$$

If the current I is measured as an r.m.s. value, and is of sinusoidal waveform,

$$H_{max} \text{ in the specimen} = \frac{4\pi \,.\, N_1 \,.\, I_{max}}{10l}$$

$$= \frac{4\pi N_1 I\sqrt{2}}{10l}$$

If, also, B_{max} is the maximum flux density in the specimen, the flux threading the secondary, per ampere in the primary, is

$$\frac{aB_{max} + (a' - a)H_{max}}{\sqrt{2}I}$$

or

$$m = \frac{N_2\,[aB_{max} + (a' - a)H_{max}]}{\sqrt{2}I \times 10^8} \quad . \quad . \quad (227)$$

For the description of other bridge methods of iron-loss measurement, see Ref. (3).

Measurements on Iron Powders. T. H. Oddie (Ref. (44)) has described researches on the permeability and magnetic loss coefficients of carbonyl- and hydrogen-reduced iron powders, at frequencies from 200 c/s to 40 Mc/s. The method used, over the range 200 to 100,000 c/s, employs the resonance bridge circuit shown in Fig. 242. The powders under test were moulded in the form of (*a*) cylindrical cores with an axial hole or (*b*) toroidal cores of rectangular cross section.

The resonance bridge network is used in combination with a beat-frequency type oscillator-analyser. The apparent resistance and inductance of the coil containing the test core are obtained from the settings of the standard capacitor, of the variable resistances and the potential drop across R_3 for balance. These apparent values for the coil have to be corrected for losses in the mica standard capacitor, the self capacitance of the coil (found from the apparent increase of inductance with frequency) and the small inductance due to the air path in the magnetic circuit in parallel with the core.

The corrected value of inductance

$$L_m = 4{\cdot}61 \times 10^{-9} . \mu_m . N^2 . h \log_{10} \frac{d_1}{d_2}$$

where N = number of turns on the coil, d_1 and d_2 are external and internal diameters of the core, h is the length of the core.

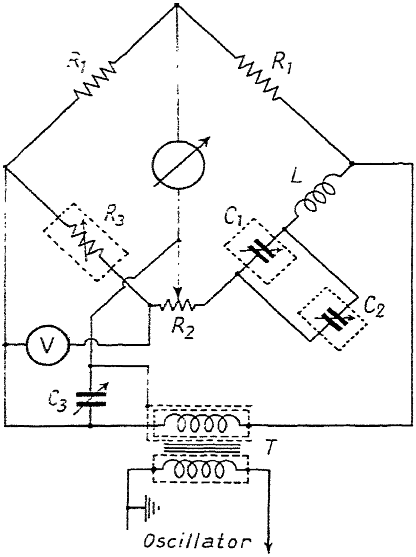


FIG. 242

C_1 = air capacitor, C_2 = mica capacitor, C_3 = trimmer capacitor, L = test coil, R_1 = ratio arms, R_2 = fine adjustment slide wire, R_3 = resistance box, T = input transformer, V = valve voltmeter.

The permeability of the core μ_m can be found from this expression.

The maximum flux density

$$B \qquad 2{\cdot}83 \times 10^8 . I . L_m$$

$$- d_2)$$

where I is the r.m.s. current (in amperes) in the windings.

The magnetic losses in the core are found from analysis of the measured resistance R into component losses by methods given in Refs. (44), (45), and (46).

Measurement of Iron Loss by A.C. Potentiometer. The a.c. potentiometer, which has already been described, may be used for the measurement of the power loss in samples of iron at low flux densities and forms a very satisfactory method. The connections are given in Fig. 243, in which the potentiometer is assumed to be of the Tinsley-Gall pattern. This potentiometer is very suitable for such measurements, since it measures the magnetizing- and iron-loss components of the exciting current separately.

The sample—of ring form—carries two windings. The primary winding has N_1 turns and the secondary N_2 turns, the supply to the former being through a regulating transformer, from an alternator

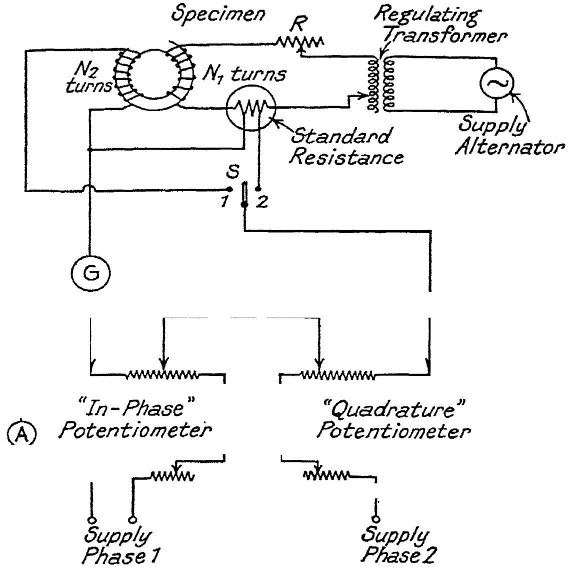


FIG. 243. IRON-LOSS MEASUREMENT BY A.C. POTENTIOMETER

which also supplies current to the two potentiometer slide-wire circuits. The alternator should have a sinusoidal voltage waveform in which case the current waveform will be approximately sinusoidal, if the flux density in the sample is low. A variable resistor R and a standard resistor are connected in series with the primary winding, the latter being for the purpose of measuring the current in the primary winding by measuring the magnitude and phase of the voltage drop across it. G is a vibration galvanometer.

The flux density B_{max} in the sample is obtained from the expression

$$B_{max} = \frac{E \times 10^8}{4K_f \,.\, A \,.\, f \,.\, N_2}$$

where E is the e.m.f. (r.m.s. value) induced in the secondary winding, A is the cross-section of the sample and f the supply frequency. If the form factor K_f is 1·11—i.e. if the supply voltage wave is sinusoidal—B_{max} is given by $\frac{E \times 10^8}{4 \cdot 44\, Af}$.

The voltage E is measured by moving the switch S on to contact 1, setting the "quadrature" potentiometer at zero, and adjusting the "in-phase" potentiometer until the vibration galvanometer shows no deflection, it being assumed that the necessary standardizing adjustments of the potentiometer have first of all been made as described in Chapter VIII. The setting of the "in-phase" potentiometer for balance then gives the value of the voltage E directly.

The switch S is then thrown over to contact 2, and both the "in-phase" and "quadrature" potentiometers adjusted to give zero galvanometer deflection. The reading of the "in-phase" potentiometer then gives the value of I_eR_s, where I_e is the loss component of the current in the primary winding and R_s is the value of the standard resistance. The reading of the "quadrature" potentiometer gives the value I_MR_s, where I_M is the magnetizing component of the primary current.

Hysteresis Loss in Small Specimens. In development work on electrical sheet steels, it may be necessary to determine both the rotational and alternating hysteresis loss in the sheet when the available samples are quite small. Owing to the considerable variation in magnetic properties in different directions relative to the direction of rolling, it is also convenient to be able to carry out tests upon the same sample with different directions of magnetization. F. Brailsford (Refs. (39) and (40)) has described methods of measurement of such rotational and alternating hysteresis losses, utilizing for the purpose a modified torque magnetometer. The samples used are in the form of discs of 1¼ in. diameter, three such discs being used as one sample. These are mounted on a brass rod which is suspended, by a phosphor-bronze suspension, in the air gap of an electromagnet with which field strengths up to $H = 450$ can be obtained. Since the method of testing does not involve continuous rotation, eddy-current losses are absent.

The reader is referred to Brailsford's papers for a full description of the apparatus and for the theory of the method.

BIBLIOGRAPHY AND REFERENCES

(1) *Properties and Testing of Magnetic Materials,* T. Spooner.
(2) *Applied Magnetism,* T. F. Wall.
(3) *Dictionary of Applied Physics,* Vol. II.
(4) *Electrical Measurements,* F. A. Laws.
(5) *The Magnetic Circuit,* H. du Bois.
(6) *Magnetic Induction in Iron and Other Metals,* J. A. Ewing.

(7) "The Measurement of Flux Density in the Air Path of a Magnetic Circuit," W. P. Conly, *Jour. I.E.E.*, Vol. LXI, p. 161.

(8) "Errors in the Magnetic Testing of Ring Specimens," E. Hughes, *Jour. I.E.E.*, Vol. LXV, p. 932.

(9) S. P. Thompson and E. W. Moss (*Proc. Phys. Soc.*, Vol. XXI, p. 630).

(10) "A New Universal Permeameter," A. Illiovici, *Bulletin de la Société Internationale des Electriciens* (1913), Vol. III, p. 581.

(11) "An Improved Form of the Picou Permeameter," R. V. Picou, *Revue Génerale de l'Electricité* (1926), Vol. XX, p. 346.

(12) "The Determination of the Magnetic Induction in Straight Bars," C. W. Burrows, *Bull. Bur. Stands.* (1909), Vol. VI, p. 31.

(13) "The Magnetic Testing of Bars of Straight or Curved Form," A. Campbell and D. W. Dye, *Jour. I.E.E.*, Vol. LIV, p. 35.

(14) "Precision Permeability Measurements on Straight Bars and Strips in the Region of High Permeability," C. E. Webb and L. H. Ford, *Jour. I.E.E.* Vol. LXVII, p. 1302.

(15) P. Curie, *Proc. Phys. Soc.*, 1909–10, XXII, p. 343.

(16) "Propriétes Magnétiques des aciers trempés," Mme. Curie, *Comptes Rendus* (1879), CXXV, 1165–1169; "Magnetic Properties of Tempered Steels," Mme. Curie, *Electrical Review* (1899), XLIV, 40–42, 75–76, and 112–113.

(17) "Apparatus for the Commercial Testing of Permanent Magnets," Betteridge, *Electrician*, No. 17 (1916).

(18) B.E.S.A. Standard Specification No. 406 (1931), "Apparatus for Workshop Testing of Permanent Magnets."

(19) A. Campbell, *Proc. Phys. Soc.* (1910), XXII, 24.

(20) "Notes on the Testing of Static Transformers," J. L. Thompson and H. Walmsley, *Jour. I.E.E.*, Vol. LXIV, p. 505.

(21) "Testing Magnetic Sheet Steels," B. G. Churcher, *World Power*, Vol. IX, No. XLIX.

(22) "An Accurate Method of Testing Bent Permanent Magnets," C. E. Webb and L. H. Ford, *Jour. I.E.E.*, Vol. 68, p. 773.

(23) "The Power Losses in Magnetic Sheet Material at High Flux Densities," C. E. Webb, *Jour. I.E.E.*, Vol. XLIV, p. 409.

(24) "A Flux Voltmeter for Magnetic Tests," G. Camilli, *Jour. Am. I.E.E.*, October, 1926, p. 989.

(25) "The Magnetization of Iron at Low Inductions," L. W. Wild, *Jour. I.E.E.*, Vol. LII, p. 96.

(26) "Investigation by Electrolytic Means of the Magnetic Leakage of Salient Poles," A. A. Ahmed, *World Power*, December, 1924, p. 338.

(27) "Iron Losses in D.C. Machines," E. Hughes, *Jour. I.E.E.*, Vol. LXIII, p. 35.

(28) "The Magnetic Behaviour of Iron under Alternating Magnetization of Sinusoidal Wave-form," N. W. McLachlan, *Jour. I.E.E.*, Vol. LIII, p. 809.

(29) "Representation of the Total Losses in Iron, due to Alternating Magnetization, by an Expression of the Form $W = cB^n$," N. W. McLachlan, *Jour. I.E.E.*, Vol. LIV, p. 350.

(30) "The Effect of the Time of Passage of a Quantity of Electricity on the Throw of a Ballistic Galvanometer," F. Wenner, *Phys. Rev.* 25, 139.

(31) "On the Theory and Measurement of the Magnetic Properties of Iron," D. C. Gall and L. G. A. Sims, *Jour. I.E.E.*, Vol. LXXIV, p. 453.

(32) "Alternating-Current Permeability and the Bridge Method of Magnetic Testing," C. E. Webb and L. H. Ford, *Jour. I.E.E.*, Vol. LXXVI, p. 185.

(33) "Magnetic Characteristics of Nickel-Iron Alloys with Alternating Magnetizing Forces," E. Hughes, *Jour. I.E.E.*, Vol. LXXIX, p. 213.

(34) "The Variation of the Magnetic Properties of Ferro-magnetic Laminae with Frequency," C. Dannatt, *Jour. I.E.E.*, Vol. LXXIX, p. 667.

(35) "Nickel-Iron Alloys of High Permeability, with special reference to Mumetal," W. F. Randall, *Jour. I.E.E.*, Vol. LXXX, p. 647.

(36) "A New Magnetic Flux Meter," G. S. Smith, *Trans. A.I.E.E.*, Vol. LVI, p. 441.

(37) "A New Photo-electric Hysteresigraph," R. F. Edgar, *Trans. A.I.E.E.*, Vol. LVI, p. 805.

(38) "Use of Bismuth Bridge Magnetic Fluxmeter for A.C. Fields," G. S. Smith, *Trans. A.I.E.E.*, Vol. LVIII, p. 52.

(39) "Rotational Hysteresis Loss in Electrical Sheet Steels," F. Brailsford, *Jour. I.E.E.*, Vol. LXXXIII, p. 566.

(40) "Alternating Hysteresis Loss in Electrical Sheet Steels," F. Brailsford, *Jour. I.E.E.*, Vol. LXXXIV, p. 399.

(41) "The Grassot Fluxmeter," E. W. Golding, *Electrician*, 27th October, 1933.

(42) *Permanent Magnets*, F. G. Spreadbury.

(43) "The Paramagnetism of the Ferromagnetic Elements," W. Sucksmith and R. R. Pearce, *Proc, Roy. Soc.*, Vol. 167, 1938, p. 189.

(44) "Magnetic Measurements on Iron Powders," T. H. Oddie, *Jour. Sci. Insts.*, Vol. 21, No. 9, p. 154, Sept., 1944.

(45) *Bell Syst. Tech. J.*, V. E. Legg., Vol. 15, p. 39, 1936.

(46) *Electronic Engineering*, V. G. Welsby, pp. 16, 96, 149, 191, 230 and 281, 1943.

(47) "Measurements of Magnetic Permeability," Measurements Section Discussion Meeting, 20th January, 1953, *Proc. I.E.E.*, Part II, Vol. 100, No. 75, June, 1953.

(48) "Some Developments and Simplifications in Permeameters," A. M. Armour, A. J. King, and J. W. Walley, *Proc. I.E.E.*, Vol. 99, No. 20, Part IV, Monographs, 1952, p. 74.

(49) "British Achievements in Electrical Measuring Instruments," S. Whitehead, *Proc. Joint Engineering Conference*, 1951, p. 473.

(50) "Some Aspects of the Theory of Iron Testing by Wattmeter and Bridge Methods," N. F. Astbury, *Jour. I.E.E.*, Vol. 95, Part II, p. 607.

CHAPTER X

ILLUMINATION

In the subject of illumination we are concerned with the light which is obtained from incandescent bodies of various forms. The light emitted from such bodies depends very largely upon their temperature. A body which is hotter than the medium surrounding it radiates energy into that medium. This energy is radiated in the form of heat waves only, if the temperature of the body is below that at which it becomes "red-hot." At the "red-hot" temperature the body is emitting light waves in addition to heat waves. The wave-length of the heat waves emitted at lower temperatures is comparatively great, but, as the temperature rises, the wave-length of the shortest wave which is being emitted diminishes until it enters the band of wave-lengths which are classed as light waves and gives the impression of red light. At higher temperatures still, the wave-length of the shortest wave emitted from the body gives the impression of white light, when the body is said to be "white-hot." Throughout, heat waves are emitted as well as light waves. The velocity of all these radiated waves—heat and light alike—is the same, namely, 3×10^{10} cm. per sec. The wave-lengths of the visible light waves are between 0·0008 mm. and 0·0004 mm. The ratio

$$\frac{\text{Energy radiated in the form of light}}{\text{Total energy radiated by the body}}$$

is called the *radiant efficiency* of the body, and obviously depends upon the temperature of the body. The maximum radiant efficiency is obtained at such a temperature that the wave-length of the shortest wave radiated by the body is 0·0004 mm., since further increase in temperature will only increase the total energy radiated without increasing the amount of energy which is radiated in the form of light.

Before proceeding with the consideration of the laws of illumination, some definitions of the quantities to be dealt with must be given. These have recently been revised (see Ref. (15)).

Definitions. *Light* is defined as that part of the energy radiated from a body which produces the sensation of light upon the human eye. Light is thus *energy*.

Luminous Flux (Symbol F or Φ) is the *light energy radiated per second* from a luminous body. Luminous flux is therefore a form of *power*.

The *Luminous Intensity* (Symbol I) in any given direction is the luminous flux radiated per unit solid angle (i.e. the solid angular density of flux) measured in the direction in which the intensity is required. Referring to Fig. 244, if O is a point source of light and OX is the direction in which the intensity is required, then the luminous flux contained within the solid angle $\delta\omega$, divided by $\delta\omega$, is the intensity in the direction OX. The angle $\delta\omega$ is a very small solid angle containing the direction OX. If a is the area, at radius r, which subtends the solid angle $\delta\omega$ at O, then $\delta\omega = \frac{a}{r^2}$. If F is the flux within the solid angle, then intensity in direction OX is $\frac{F}{\delta\omega} = \frac{F}{a}r^2$.

The *Lumen* (abbreviation lm). The unit of flux is the *lumen*,

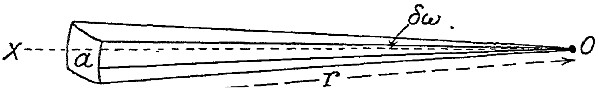


FIG. 244

which is the flux emitted in unit solid angle from a point source having a uniform intensity of one candela.

The *Candela* (abbreviation cd) is defined as the unit of luminous intensity "such that the luminance of a full radiator at the temperature of solidification of platinum is 60 units of luminous intensity per square centimetre."

Candle-power is "the light-radiating capacity of a source in a given direction, in terms of the luminous intensity expressed in candelas."

[In the past the lumen was defined as the flux, per unit solid angle, from a point source of 1 candle-power so that the total flux from the source was 4π lumens.]

Luminance (symbol L) which now replaces the old term "Brightness" is defined, for a point of a surface and in a given direction, as "the quotient of the luminous intensity in the given direction of an infinitesimal element of the surface containing the point under consideration, by the orthogonally projected area of the element on a plane perpendicular to the given direction."

The recommended unit of luminance is the *Foot-lambert* (abbreviation ft-L) which is "the luminance of a uniform diffuser emitting one lumen per square foot."

The *Mean Horizontal Candle-power* (m.h.c.p.) of a source of light is the average value of the candle-powers in all directions in a horizontal plane passing through the source of light.

The *Mean Spherical Candle-power* (m.s.c.p.) of a source is the mean of the candle-powers in all directions from the source of light, and is the same as the candle-power of a source of light which radiates the same total flux uniformly in all directions.

The *Mean Hemispherical Candle-power* (m.h.s.c.p.) of a source may be measured for the space either above or below a horizontal plane, through the source. It is, therefore, the mean of the candle-powers taken in all directions within the hemisphere, either above or below the horizontal plane containing the source of light.

The *Candle-hour* is a unit of quantity of light, and is the quantity of light emitted in one hour by a source of unit mean spherical candle-power (i.e. emitting unit flux in a unit solid angle).

The *Lumen-hour* is also a unit of quantity and is the light emitted when a flux of one lumen continues for one hour.

The *Reduction Factor* of a source of light is the ratio

$$\frac{\text{Mean spherical candle-power}}{\text{Mean horizontal candle-power}}$$

for the source.

The *Illumination* (symbol E) of a surface is measured by the flux received per unit area of surface, in lumens per square foot (abbreviation lm/ft.^{2}).

This unit (lm/ft.^{2}) is now preferred to the previous unit, the foot-candle, which is its equivalent. The definition of the foot-candle was "the illumination of the inner surface of a sphere of 1ft. radius when a point source of light of 1 candle-power is placed at the centre of the sphere." Another unit of illumination, the *Lux*, is one lumen per square metre.

Laws of Illumination. (1) The illumination of a surface is inversely proportional to the square of the distance of the surface from the source of light, provided the latter is a *point* source or is sufficiently distant from the surface to be regarded as such.

This is obviously true, since the solid angle subtended at the point source by the surface is $\frac{A}{r^2}$ (where A is the area of the surface and r its distance from the source) and hence, for a given flux per unit solid angle from the source, the flux per unit area $\propto \frac{1}{r^2}$

(2) LAMBERT'S COSINE LAW. This law states that, if a surface is inclined at an angle $90 - \theta$ to the direction of the luminous flux, then the illumination of the surface is reduced from that given according to law 1 (above) in the ratio $\frac{\cos\theta}{1}$. From Fig. 245 it can be seen that if F is the luminous flux falling upon the surface when in position 1, then the flux falling upon the surface when in position 2 will be $F \cos\theta$ where θ is the angle between the two positions. Hence, if A is the area of surface, the illumination in position 1 will be $\frac{F}{A}$, and in position 2, $\frac{F\cos\theta}{A}$

Standards. The earliest standard was the simple candle which was carefully specified but which was incapable of providing the accuracy which was demanded as lighting techniques developed. It was superseded by flame standards—two of which are briefly described below. These flame standards are retained as standards of reference but for most practical purposes they have been replaced by electric lamps with tungsten filaments.

THE PRIMARY STANDARD. Proposals were frequently made that a specified area of a full radiator (black body) at a definite temperature should be used as a primary standard. This temperature was finally accepted as that of the melting point of platinum (2,042° K) which is accurately reproducible, and in 1931 a standard on this basis was realized at the Bureau of Standards in America. J. W. T. Walsh (Ref. (17)) describes the form of this standard, in which the platinum is melted in a high-frequency induction furnace, and makes the suggestion that the use of iridium, with a melting point of over 2,700° K would overcome a disadvantage of the melting platinum, namely that its comparatively low temperature causes it to have a considerable colour difference from that of modern light sources.

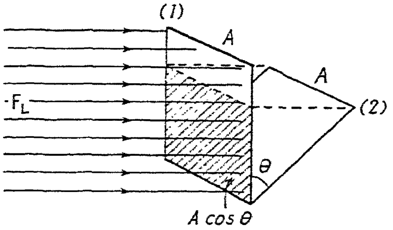


FIG. 245. LAMBERT'S COSINE LAW

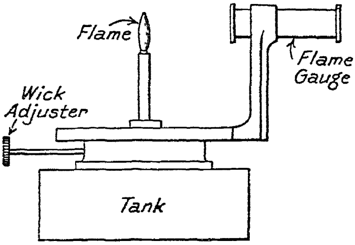


FIG. 246. HEFNER STANDARD LAMP

As already stated on p. 412, the unit of luminous intensity (the candela) is defined in terms of this platinum standard, the radiator having, at the melting point, an intensity of 60 candelas/cm.². In the old system of units this source has a brightness of 58·8 candles/cm.² but the round number (60) was chosen for the new unit.

FLAME STANDARDS. There are two flame standard lamps which merit description. These are the Hefner and the Harcourt pentane lamp.

Hefner Standard Lamp. This lamp was adopted as the German standard of candle-power. Its construction is illustrated in Fig. 246. It consists of a brass tank containing the fuel, which is a specially pure grade of amyl acetate. The height of the flame—which should be 4 cm. from the tip to the top of the tube—is adjusted by means of a screw which adjusts the height of the wick.

The flame height is observed through a gauge containing a lens, and a ground glass screen with a horizontal hair line to fix the tip of the flame. It is necessary to protect the flame from draught, as the candle power varies directly with the height of the flame within limits of a few per cent above or below the standard height.

The formula for candle-power is

$$\text{c.p.} = 1 + 0{\cdot}0055(8{\cdot}8 - e) - 0{\cdot}00015(760 - b) \qquad (228)$$

where the candle-power is in Hefner units, e the humidity in litres of water vapour per cubic metre of moist air, and b the atmospheric pressure in millimetres of mercury. The correction for carbon dioxide in the atmosphere is made by adding $0{\cdot}0072(0{\cdot}75 - k)$ to the above expression for candle-power, k being the number of litres of carbon dioxide present in one cubic metre of air, but this correction is somewhat uncertain and should be avoided if possible.

Under standard conditions, the candle-power (i.e. 1 Hefner candle-power) of this lamp is 0·9 British candle-power. The British candle-power was adopted as the International unit before the acceptance, in 1948, of the "candela" as the International unit. The disadvantages of the lamp as a primary standard are the unsteadiness of its flame, the colour of the flame—which is reddish—and its low candle-power. It has the advantages, however, of simplicity, both in construction and operation, while it is easily reproducible and is durable.

The Harcourt Pentane Lamp. This lamp, the construction of which is shown in the simplified sketch of Fig. 247, was adopted as the British standard of candle-power. Its candle-power under the standard conditions of temperature, atmospheric pressure, and humidity, is 10.

There is no wick in this lamp. The burner, which is made of steatite, and has thirty holes drilled in it, is fed with pentane vapour from a tank containing liquid pentane. Stop-cocks on the air inlet tube to the tank, and on the vapour outlet tube, regulate the rate of flow of vapour and so control the height of the flame. A window in the pentane tank serves as a gauge for the level of liquid in the tank, which should be maintained about half full. Air, which enters the tank, passes over the liquid pentane and carries pentane vapour with it to the burner through a rubber tube. The chimney tube, held over the flame, has a small mica window with cross-bar, the latter being 3·8 cm. above the bottom end of the chimney. A cylindrical wooden gauge is used to adjust the level of the bottom of the chimney to 4·7 cm. above the top of the burner. When the lamp is burning the flame tip should reach a level half-way between the window cross-bar and the bottom of the window. This is the standard height of the flame, and is obtained by adjustment of the fuel stop-cocks. Air passes both up the centre of the burner and outside the burner through the guiding tubes shown. A tube surrounding the chimney carries air for the supply to the centre of the burner. A blackened screen, surrounding the top of the burner, protects the flame from draught.

In using the lamp the hollow supporting column should be vertical, levelling screws at the base being provided for adjustment purposes. The chimney should, also, be exactly central over the burner, and this condition also is obtained by adjusting screws (not shown). In making photometric measurements with the lamp, the photometer should be at a distance of 1 metre from the centre of the burner.

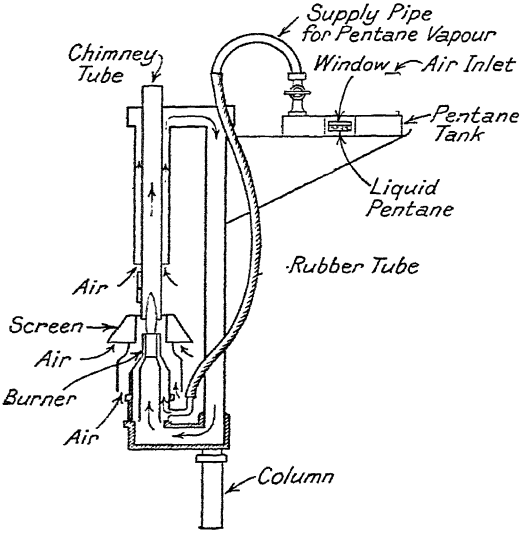


FIG. 247. HARCOURT PENTANE LAMP

The equation for the candle-power of the lamp, allowing for temperature, pressure, and humidity values, but neglecting atmospheric carbon dioxide, as given by Walsh (Ref. (1)) is

$$\text{c.p.} = 10\,[1 + 0{\cdot}0052(8 - e) + 0{\cdot}001\,(15 - t) - 0{\cdot}00085(760 - b)] \quad (229)$$

where e is the humidity in litres of water vapour per cubic metre of moist air, b is the atmospheric pressure in millimetres of mercury, and t is the temperature in degrees centigrade.

This lamp has the advantages, as compared with the Hefner, of higher candle-power, better colour, and greater steadiness of flame. It is also less affected, as regards candle-power, by the height of flame. Its disadvantages are its lack of portability, complicated construction, and its bulk.

The Carcel Lamp was used in France as a flame standard, but is not used now to any appreciable extent owing to the difficulty of obtaining a sufficient standard of purity in its fuel—colza, or rapeseed oil, supplied to the wick by a pump driven by clockwork.

1 Carcel candle = 9·66 British candle-power units

INCANDESCENT LAMPS. Incandescent filament lamps have been used, during recent years, as working standards of candle-power, and are very convenient for the purpose. No lamp which can be used as a primary standard has so far been constructed. Standard lamps are preserved and used at the National Physical Laboratory in this country, and in similar institutions abroad. One such lamp used at the National Physical Laboratory (Ref. (1)) consists of a cylindrical bulb of 10 cm. diameter, containing a single loop of carbon about 10 cm. long. It is used with the plane of the filament perpendicular to the direction in which the candle-power measurement is being made. The distance of this lamp from the photometer is kept fixed, so that the illumination at the photometer is 10 metre-candles. The candle-power of this lamp is about 15.

Sub-standards. Standards of candle-power for use in general laboratory measurements must be conveniently portable and robust. Incandescent lamps are now very largely used for this purpose, their candle-power being first of all obtained by comparison with some primary standard and thereafter checked from time to time against such a standard.

The principal requirement of such sub-standard lamps is that their candle-power shall not vary appreciably with time, and that the light which reaches the photometer from the lamp shall be of a uniform character.

A special construction is adopted for sub-standards which are to be used in precise work. The filament is of tungsten, and is mounted in a single plane, this plane being at right angles to the axis of the photometer bench when measurements are being made. This is necessary in order to fix, definitely, the direction in which the candle-power of the lamp is that measured when the lamp was standardized. For the same reason these lamps are permanently fixed in a special holder which can be attached to the photometer bench, and which ensures that the directions in which the light is radiated by the lamp to the photometer "head," or measuring device, is always the same as when the lamp was standardized. This precaution is necessary, since the candle-power of the lamp will not be the same in all directions, due to the effects of the supports for the filament, to slight differences in thickness of the glass bulb, and to other causes. The leads to the terminals of the lamp are also permanently connected to avoid unsteadiness of contact. Further, in order to ensure constancy of candle-power, the joints between filament and leading-in wires are welded, the bulb is made specially large in order to reduce the density of any blackening deposit upon the glass and, after manufacture, the lamps are "aged" before standardization by being run for 100 hours or more. In use, these lamps must be run at a specified potential, which must not be exceeded by even a small percentage, if the candle-power is to remain constant.

In sub-standard lamps of low candle-power the filament may be a single loop of tungsten or carbon.

In order that sub-standard lamps shall not be used any more than is absolutely necessary, comparison lamps are often used for laboratory purposes, these lamps being such that their candle-power remains reasonably constant. Such lamps may be checked at frequent intervals against the sub-standard.

Acetylene Sub-standard. In cases where the use of electric lamps is inconvenient, an acetylene sub-standard of the type shown in Fig. 248 (Eastman-Kodak type) may be used. The burner, which gives a flame some 5 cm. high and 3 or 4 mm. diameter, is of the Bray, air-mixing type. A metal cylinder, having a wedge-shaped opening in one side, surrounds the flame. The height of this opening is adjusted to a little below the middle of the flame, where the intensity is steadied. The lamp burns gas at a pressure of 9 cm. of water. The colour of the light is very nearly the same as that of a vacuum tungsten lamp.

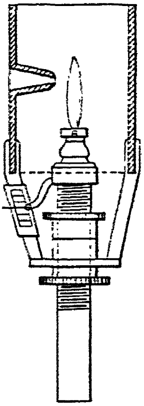


FIG. 248

ACETYLENE SUB-STANDARD

The Measurement of Candle-power. For this purpose, apparatus consisting of a photometer bench and a "photometer head" is used, in conjunction with a standard or sub-standard lamp. The principle upon which most of the methods of measurement are based is the Inverse Square Law. Fig. 249 represents, diagrammatically, two lamps S and T set at a distance apart, with some type of screen in line with, and between, them. Let S be a standard lamp (of known candle-power) and T the lamp whose candle-power, in the direction of the line joining it to the standard lamp, is required. If the screen is moved in between the lamps until the illumination is the same on both sides of it, then, from the Inverse Square Law,

$$\frac{\text{Candle-power of } T}{\text{Candle-power of } S} = \frac{l_2^2}{l_1^2}$$

where l_1 and l_2 are the distances of the screen from S and T respectively.

The screen or detecting device, used for determining the point at which the lamps give equal illumination, is called the "photometer head," and the graduated bench upon which it slides is called the "photometer bench." Many different types of photometer head can be used in conjunction with one photometer bench. A number of these will be described, but, first of all, the photometer bench merits description.

THE PHOTOMETER BENCH. The most usual type of photometer bench consists of two steel rods, some three or four metres long, which carry the stands or saddles holding the two lamps to be compared, the carriage for the photometer head, and for any other apparatus which may be used in making measurement. One of the bars carries a brass strip which bears a scale graduated in millimetres. The main requirements of such a bench are that it shall be rigid—in order that the lamps being compared may be free from vibration—and that the carriage which holds the photometer head may be moved along smoothly and with very little effort. Since the method of measurement involves the *square* of distances, it is necessary that these distances shall be measured very accurately if the results of the candle-power measurement are to be accurate. The carriages which slide upon the bench have, except in the case of the one which carries the photometer head, a circular table which can be rotated in a horizontal plane and clamped in any position. This table carries a scale, graduated in degrees, round its edge, for measurement of the angle of rotation. The carriages also carry sockets, etc., for the attachment of the lamp, under test. Since it is very important that no light, other than that from the lamps under comparison, shall reach the screen of the photometer head, a system of opaque black screens is often used. For the same reason the photometry room is usually darkened and its walls are painted dead black.

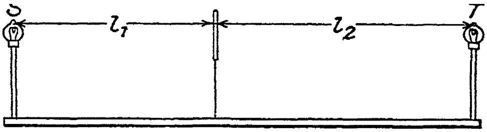


FIG. 249. MEASUREMENT OF CANDLE-POWER

In using the bench, the photometer head is moved along until the illumination from the two sources is the same as nearly as can be judged. To obtain the distance more exactly, two points are found at which there is a perceptible difference in the illumination from the two sides, and the point half-way between them is then taken as the position for equal illumination.

PHOTOMETER HEADS. Most photometer heads consist of some device by means of which the illumination of two surfaces, side by side—one illuminated by the standard lamp and the other by the test lamp—may be compared under exactly similar conditions and without movement of the eye. Each of these surfaces should receive light from only one of the two sources, and they should be separated from one another only by some sharply-defined boundary. The heads most commonly used are the Bunsen and the Lummer-Brodhun (contrast type), there being little difference between these two types from the point of view of accuracy of measurement. When the lamps to be compared give lights of the same, or very similar, colour, these two photometer heads are the best for the purpose, but if the colours of the two lights are appreciably different a Flicker photometer will probably give better results.

Bunsen Head. This device consists essentially of a piece of thin opaque paper which has a translucent "spot" at its centre, this "spot" being obtained by treating the centre portion of the paper with oil or wax. The device is often called the "Grease Spot Photometer." If the paper is placed between the two lamps to be compared, with its plane perpendicular to the line joining them, the opaque part of the paper will be illuminated from one source only, while the transparent part will be illuminated from both sources. The positions at which the transparent spot is no longer perceptible are then obtained, viewing the paper first from one side and then from the other. The candle-power of the test lamp may then be obtained from the expression

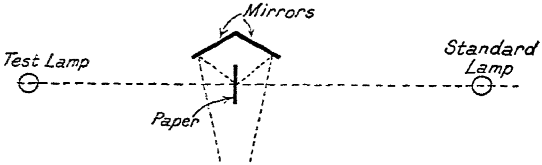


FIG. 250

$$(\text{c.p.})_T = (\text{c.p.})_S \cdot \frac{l_1 l_1'}{l_2 l_2'}$$

where $(\text{c.p.})_T$ and $(\text{c.p.})_S$ are the candle-powers of the test lamp and standard respectively, l_1 and l_1' being the two distances of the photometer head from the former and l_2 and l_2' the two distances of the head from the latter.

A second method of using the Bunsen head is to illuminate one side of the paper from a fixed lamp, the photometer head also being kept in a fixed position, and then measuring the distances, on the opposite side of the paper, at which, first the test lamp, and then a standard lamp, must be placed in order to cause the grease spot to disappear. If l_1 and l_2 are these distances, then

$$\frac{(\text{c.p.})_T}{(\text{c.p.})_S} = \frac{l_1^2}{l_2^2}$$

A third, and perhaps the best, method of using the photometer head is to use two mirrors, placed behind the paper as shown in Fig. 250, so that the two sides may be viewed at the same time.

The position of the head for equal *contrast* in illumination between the opaque and transparent portions of the paper on the two sides is then determined, and the candle-power of the test lamp calculated from the same expression as before. It may be necessary to reverse the photometer head and repeat the experiment, taking the geometric mean of the two results as the correct value.

Lummer-Brodhun Photometer Head. This head may be considered as an improved "grease spot" photometer in which the spot is completely transparent. There are two types of this head, the equality-of-brightness type and the contrast type, the latter being the better, and being very generally used in photometric measurements.

Fig. 251 illustrates the equality-of-brightness type. In the photometer head there are two mirrors, M_1 and M_2, a plaster-of-paris screen S, and a compound prism P. This compound prism is made up of two right-angled glass prisms. The principal surface of one of these prisms is spherical, but has a small flat portion at its centre, this flat surface making optical contact with the flat surface of the other prism as shown. Light entering the head from the two lamps illuminates both sides of the screen S. The light from the test lamp (say) is reflected by the screen surface, upon which it falls, to mirror M_1 which again reflects it to the compound prism. Only that portion of this reflected light which falls on the flat central portion of the spherical prism is allowed to pass through to the telescope, the rest of the light being reflected back. Again, light from the standard lamp, after being reflected from S and M_2, falls on the compound prism. That portion of its light which falls on the surface of contact of the two prisms passes through the compound prism, the rest being reflected through the telescope as shown. The effect in the telescope is to show the centre portion of a circular area illuminated by the test lamp and a surrounding area illuminated by the standard (Fig. 252). Thus the arrangement gives the same effect as the "grease spot" photometer. The photometer head is moved until the dividing line between the centre portion and surrounding ring

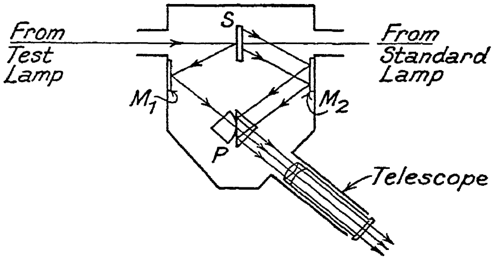


FIG. 251. LUMMER-BRODHUN PHOTOMETER HEAD: EQUALITY-OF-BRIGHTNESS TYPE

of light disappears. This will only disappear completely if the two lights being compared are of the same colour.

In the "contrast" type of Lummer-Brodhun head the compound prism is arranged as shown in Fig. 253 (*a*). The two joining surfaces of the component right-angled prisms are flat, but the left-hand one has its hypotenuse surface etched away at *a*, *b*, and *c*, so as to form a pattern as shown in Fig. 253(*b*). The light falling on the compound prism from the two sides of the screen passes through the unetched portions of the joining surface and is reflected at the etched surfaces. This means that, with the weaker source of light on the right and the stronger on the left (the sources referred to being the illuminated surfaces of the screen in the photometer head), the etched surface will be less brightly illuminated than the unetched portion, as shown by the shaded and unshaded portions in Fig. 253 (*b*). *G* and *G′* are sheets of glass which give a little reflected light for the purpose of maintaining some *difference* between the illumination of the etched and unetched areas in all positions of the photometer head. This difference is about 8 per cent when the photometer head is in the balance position. In this balance position

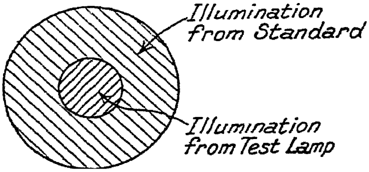
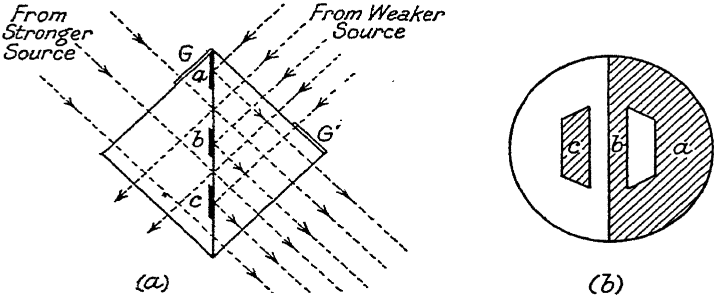


FIG. 252



(*a*) (*b*)

G. 253. LUMMER-BRODHUN PHOTOMETER HEAD: CONTRAST TYPE the *difference* in illumination between the etched and unetched portions should be the same on each half of the circular area. Upon moving away from the balance position the contrast in illumination between area *c* and its surrounding area will become less (say), and the contrast between the illumination of *ab* and the inner trapezium will become greater. The balance position is, therefore, that which gives *equal contrast,* and not equal brightness.

An accuracy of within 1 per cent can be obtained with this type of head when comparing lights of similar colour, and for such a purpose it is the most accurate photometer head available.

There are many other types of photometers, amongst which are several *polarization* photometers, in which the two fields of light which are being compared are made equal by weakening one of them by the use of polarizing prisms. The Martens photometer is an instrument of this type. For descriptions of this and other types of photometers, Refs. (1), (2), (3) should be consulted.

Flicker Photometers. These photometers depend upon the fact that, if two illuminated surfaces are presented to the eye alternately, the alternations being rapid, flicker disappears when the surfaces are of equal brightness. Colour differences between the two lights under comparison do not affect these photometers to the same extent as they do photometers of the steady comparison type, since the colour difference between two alternating fields of light disappears at a lower speed of alternation than does a difference of brightness. The speed of alternation used should be the lowest speed with which disappearance of flicker can be obtained over the smallest possible range of variation of brightness, this speed depending upon the difference in colour of the two lights.

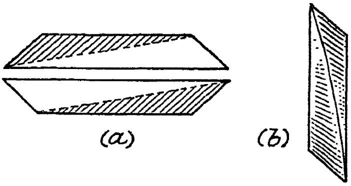


Fig. 254. Rotating Disc of Simmance-Abady Flicker Photometer

Fig. 254 shows the form of rotating disc used in the Simmance-Abady flicker photometer. The disc is of plaster-of-paris, and is in the form of a double-truncated cone. In constructing this disc, two cones are truncated by a cut in a plane making an angle with the base of the cone, and passing through a point on the circumference of the base. These truncated portions, shown shaded in Fig. 254(*a*), are then fitted together to form the disc shown in Fig. 254(*b*). This disc is placed with its axis along the line joining the two lamps under comparison, and is driven round at the required minimum speed by a small motor. Since each half of the disc is illuminated from one source only, the eye is presented with the two fields of light to be compared, alternately. Candle-power measurements are made in the ordinary way, the rotating disc being moved to such a position that flicker disappears when the two halves of the disc are illuminated equally.

The line dividing the halves of the disc should have no thickness, and the two parts should be visible for equal lengths of time per revolution, if accurate results are to be obtained.

H. E. Ives and E. T. Kingsbury (Ref. (7)) have investigated the subject of flicker photometers very fully, and have stated the conditions necessary for their satisfactory use.

Distribution of Candle-power. In the above the candle-power of a lamp in one given direction only has been considered. It is often important that the candle-power distribution in a number of directions round the source shall be known. The distribution measurements usually made are for the determination of the mean horizontal candle-power and the mean spherical candle-power.

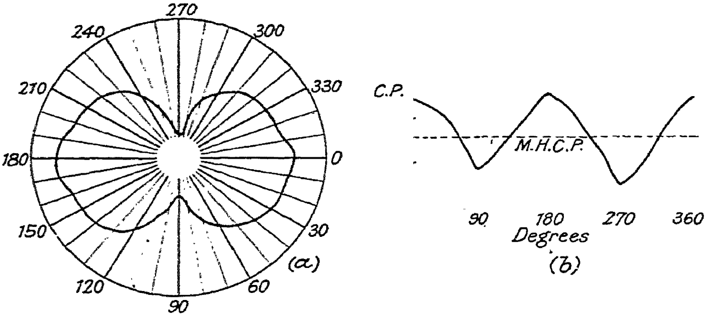


FIG. 255. DETERMINATION OF MEAN HORIZONTAL CANDLE-POWER

MEAN HORIZONTAL CANDLE-POWER. This may be determined for any lamp by turning the lamp about a vertical axis (by means of the rotating table on the photometer bench carriage) and measuring the candle-power in the direction of the line joining the test lamp and standard after every 10° or 15° rotation. A polar curve may be plotted from these measurements, this curve being the *horizontal distribution curve* of candle-power (see Fig. 255(*a*)).

To obtain the mean horizontal candle-power the candle-power may be plotted against degrees rotation on rectangular axes, as shown in Fig. 255 (*b*). The mean height of this curve gives the mean horizontal candle-power. The m.h.c.p. may also be determined from a single measurement by using a piece of apparatus which spins the lamp round at about 200 revolutions per minute. The candle-power is measured by a photometer in the ordinary way. If the speed of rotation is high enough there is no appreciable flicker on the photometer head screen.

MEAN SPHERICAL CANDLE-POWER. This may be obtained from the candle-power distribution curve in a vertical plane. The vertical distribution curve of a lamp may be obtained by tilting the lamp in a vertical plane, using a specially constructed apparatus which keeps the centre of the test lamp stationary during the rotation through 180°. Measurements of candle-power are then made in the ordinary way every 10° or 15°. The assumption is made that the candle-power of the lamp is not altered by movement from its normal position. If this assumption is not justifiable in a particular case, measurements of candle-power must be made by using an arrangement of mirrors in conjunction with a bodily movement of the lamp in a vertical direction (see Ref. (1)).

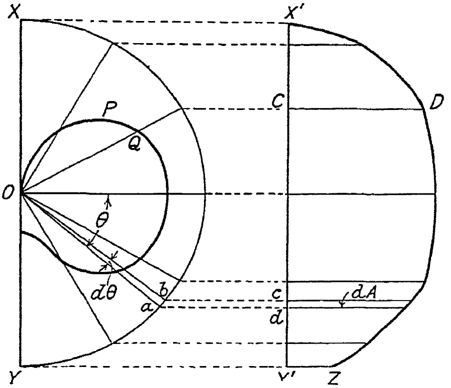


FIG. 256. ROUSSEAU'S CONSTRUCTION

It may be necessary to make measurements in two or more vertical planes, and to obtain the curve of mean vertical distribution from these measurements, but in many cases measurements in one vertical plane only are sufficient.

The mean vertical distribution curve of the lamp having been obtained, the mean spherical candle-power may be obtained graphically by one of several constructions.

Rousseau's Construction. This construction is illustrated in Fig. 256. The polar curve P is the mean vertical distribution curve for the lamp. A semicircle of any convenient radius is drawn with the pole O of the polar diagram as centre, XY being its diameter. The line $X'Y'$ is drawn equal and parallel to this vertical diameter, and from this line are set up ordinates which are equal to a corresponding radius on the polar curve (e.g. ordinate CD is made equal to radius OQ, and so on), these ordinates being set up along projections from the ends of radii of the polar curve. The curve joining the ends of these ordinates (i.e. curve $X'DZ$) is the Rousseau curve. The mean height of this curve $\left(\text{i.e. } \frac{\text{Area } X'DZY'}{\text{length } X'Y'}\right)$ gives the mean spherical candle-power of the lamp to the same scale as that to which the polar curve is drawn.

Theory.

$$\text{Mean spherical candle-power} = \frac{\text{Total flux radiated}}{\text{Total solid angle}}$$
$$= \frac{\text{Total flux}}{4\pi}$$

(Solid angle being defined by $\frac{\text{Area subtending the angle}}{\text{Radius}^2}$, for a sphere the solid angle is $\frac{4\pi r^2}{r^2} = 4\pi$.)

Now, consider the candle-power, or luminous intensity, in a direction making an angle θ with the horizontal, as in Fig. 256. Let this intensity be I. This intensity will apply to the surface of a zone of a sphere produced by the rotation of the small length ab about the line XY. Length ab is $rd\theta$, where r is the radius of the semicircle whose diameter is XY and $d\theta$ a very small angular element. Now, the area of this zone is $2\pi r \cos\theta \, . \, rd\theta$ or $2\pi r^2 \cos\theta d\theta$, so that the solid angle subtended by it at O is $\frac{2\pi r^2 \cos\theta d\theta}{r^2}$, i.e. $2\pi \cos\theta \, . \, d\theta$.

The flux contained within this solid angle is, by definition of luminous intensity,

$$I \times \text{the solid angle}$$

or

$$2\pi I \cos\theta \, . \, d\theta$$

Thus, the total flux radiated from the lamp is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi I \cos\theta \, . \, d\theta$$

and the m.s.c.p. $$\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi I \cos\theta \, . \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I \cos\theta \, . \, d\theta$$

Now, the small length cd is equal to $rd\theta \, . \cos\theta$, and hence the area dA (to scale) is

$$Ir \cos\theta \, . \, d\theta$$

Thus the total area under the Rousseau curve is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Ir \cos\theta \, . \, d\theta$$

and its mean height

$$\frac{1}{2r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Ir \cos\theta \, . \, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} I \cos\theta \, . \, d\theta$$

which is the expression derived above for the mean spherical candle-power of the lamp.

The mean hemispherical candle-power of the lamp may be obtained by using Rousseau's construction for either the upper or lower half of the polar curve of mean vertical distribution.

The Integrating Sphere. This piece of apparatus is now commonly used for the measurement of mean spherical candle-power. (See B.S. 354 : 1945. Photometric integrators.) The method of measurement consists essentially of a measurement of the total flux of light radiated by the lamp, from which the m.s.c.p. may be obtained by dividing by 4π. The candle-powers of the lamp in *all* directions from the source are taken into account in this apparatus, which thus gives a better value of m.s.c.p. than the method previously described, which is based upon the assumption that the candle-power distribution is the same in all vertical planes—an assumption which may not always be justifiable.

The integrating sphere consists of a hollow sphere, a metre or more in diameter, the inside surface of which is painted with a white paint which is as perfectly diffusing as possible. The sphere contains a small window of translucent glass which is illuminated by reflection from the inner surface of the sphere when a lamp is placed inside. Light is prevented from reaching the window directly from the lamp by the interposition of a small screen in between the two.

It can be shown that the illumination of every part of the internal spherical surface is the same (Refs. (1), (2), (3)). This illumination is proportional to the total flux of light emitted by the lamp, and hence to the m.s.c.p. of the lamp.

To measure the mean spherical candle-power of a lamp, the lamp is placed inside the sphere and the brightness of the glass window measured. A sub-standard lamp, whose m.s.c.p. is known, is then substituted for the test lamp, and the brightness of the window again measured. The mean spherical candle-powers of the two lamps are proportional to the corresponding brightnesses of the sphere window.

The measurements of window brightness may be made by the use of some form of illuminometer to compare the brightness of the window with that of a surface whose illumination can be varied and is known. A Lummer-Brodhun photometer head can also be used for the purpose. A mirror is placed so as to reflect light from the sphere window into the compound prism of the photometer and balance is obtained, using a lamp on the other side of the photometer for comparison purposes. Balance is obtained (by moving the comparison lamp) first with the sub-standard lamp in the sphere and then with the test lamp inside, the squares of the corresponding distances of the comparison lamp from the photometer head being inversely proportional to the mean spherical candle-powers of the lamps.

Ulbricht, in 1900, first suggested this use of the internally illuminated sphere, and it is therefore named, after him, the Ulbricht sphere.

The arrangement, together with the photometer for brightness measurement, is shown in Fig. 257.

Illumination Photometers. These photometers are used for the measurement of the illumination of surfaces. They may be required to measure the illumination in rooms or in streets, and should, therefore, be, in general, portable instruments. The general principle of such instruments is that of the comparison of the illumination of a white matt surface placed at the point at which the illumination is to be measured, with the illumination of another surface forming part of the instrument, whose illumination is known and variable.

THE TROTTER ILLUMINATION PHOTOMETER. This was the earliest form of "tilting screen" illumination photometer—a type which depends, for its operation, upon Lambert's cosine law. The construction is shown in Fig. 258A. A small lamp L—supplied from a battery—throws light on to a mirror M, from which it is reflected on to a white diffusing surface A, of matt celluloid, which can be tilted. The top of the box containing the lamp and tilting surface carries another white celluloid surface Q which has a slot F cut in it, through which the surface A can be viewed.

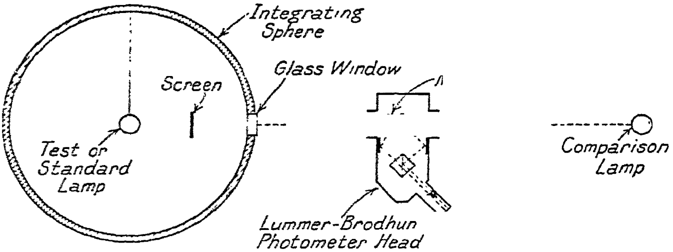


FIG. 257. MEASUREMENT OF MEAN SPHERICAL CANDLE-POWER BY MEANS OF THE INTEGRATING SPHERE

In use the apparatus is placed with the surface Q at the place at which the illumination is to be measured. The surface A—illuminated, by reflection, from the lamp L—is tilted by means of the cam C, through a roller attached to the tilting surface, until its illumination, as viewed through the slot F, is the same as that of the surface Q. A pointer attached to the shaft of the cam then gives the illumination of Q directly from a scale S_2 on the side of the box, over which the pointer P moves.

This scale is calibrated by means of a standard lamp which is used to produce a number of known illuminations upon the surface Q.

The cam C is for the purpose of producing a more uniform scale than would be obtained by mounting the pointer on the spindle carrying surface *A*. The lamp *L* must be "aged" to give a constant candle-power, although the latter need not be known. During calibration this lamp can be moved up and down until the best position—observed from scale S_1—is obtained.

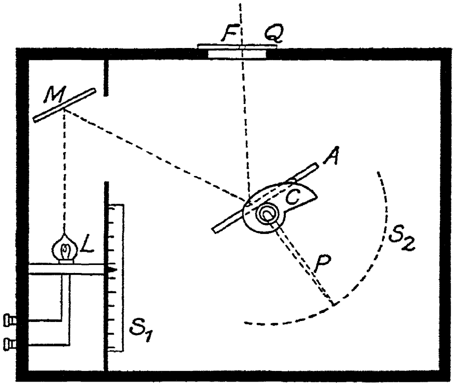


FIG. 258A. TROTTER ILLUMINATION PHOTOMETER

A plumb-bob and tilting arrangement for the box of the instrument are usually provided so that the measurement of illumination of sloping surfaces may be carried out. The range of the instrument is usually 0·01 to 4 foot-candles.

TROTTER DAYLIGHT ATTACHMENT. For the measurement of daylight illumination, Trotter devised an attachment which can be used with almost any form of portable illumination photometer. This consists of a straight tube *A* (Fig. 258B) having a branch tube *B* leading into it as shown. The tube *A* is blackened inside and has a circular opening *D* in the cover at its upper end. Its lower end is rigidly attached to the photometer box lid, and covers the surface *Q*, no light being admitted except through the opening *D*. The surface *Q* is viewed from the branch tube *B*, and its illumination measured. No light must be admitted through the observation tube *B*.

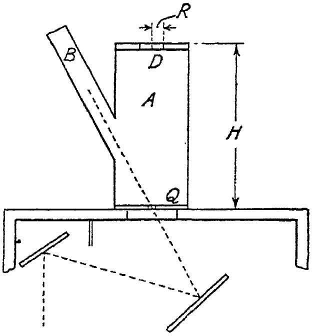


FIG. 258B. TROTTER DAYLIGHT ATTACHMENT

Then, if *b* is the brightness of the sky, *H* the height of tube *A*, and R the radius of its circular opening D, the total illumination of the surface in the neighbourhood of the instrument may be obtained as follows—

For a sky brightness b, the open daylight illumination is πb (Ref. (1)). The illumination of surface Q is $\frac{\pi R^2 . b}{H^2}$. Hence, the open daylight illumination is obtained by multiplying the measured illumination by $\frac{H^2}{R^2}$.

MACBETH ILLUMINOMETER. The operation of this instrument is based upon the Inverse Square Law, and it is one of the most accurate instruments of this type. The construction of the instrument is shown in Fig. 259.

A Lummer-Brodhun compound prism H is mounted in a box and viewed through a telescope T. On the opposite side of the prism from the telescope is a tube P, which is pointed towards a test plate of opal glass, situated at the point at which the illumination is to be measured. A is a screen of opal glass, which is illuminated by a lamp L in an enclosure having an aperture in the side nearest the screen A. The lamp is moved along a tube which is at right angles to the telescope, by a rack and pinion R, until balance is obtained at the photometer head. A scale S then gives the illumination of the test plate directly in foot-candles. The lamp L is supplied from a dry battery or accumulator, and a milliammeter and rheostat are provided with the instrument for adjustment of the lamp current to its correct value (i.e. that which makes the scale S direct reading).

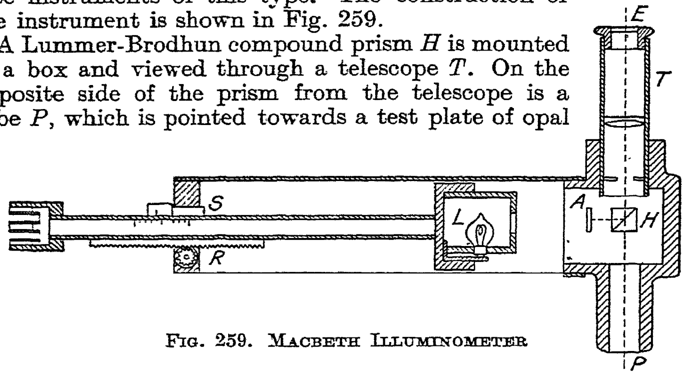


FIG. 259. MACBETH ILLUMINOMETER

A reference standard, for the purpose of checking the calibration of the instrument, is also supplied, the whole apparatus being made up in a portable form which includes rheostats, milliammeter, reference standard, and the instrument itself in a small case.

The construction of the reference standard is shown in Fig. 260 The standard is placed with its base on the test plate whose illumination is to be measured. The sighting aperture P of the illuminometer is inserted in the branch tube at *E* as shown. The lamp in the standard is supplied from the same battery as the illuminometer lamp, and its current is adjusted to give some known value of illumination (corresponding to the lamp current) on the test plate. The illuminometer lamp is then moved until its scale reading is that corresponding to the test-plate illumination, and its current is adjusted until balance is obtained at the photometer head with the illuminometer lamp in this fixed position. The current so obtained is then maintained in the illuminometer lamp during the illumination measurements. The normal range of this illuminometer is 1 to 25 foot-candles.

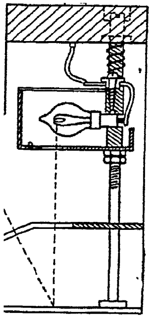


FIG. 260. REFERENCE STANDARD

HOLOPHANE LUMETER. Fig. 261 (A) shows the schematic arrangement of the "Lumeter" while Fig. 261 (B) gives a view of the inside of the instrument itself. When the photometer is used the brightness of an internal white comparison surface, illuminated by light from a lamp inside the box, is adjusted, by rotating a disc having a sector-shaped opening, until it matches the brightness of the test surface. The light from the lamp, whose output is maintained at a fixed value by a resistance, is diffused by an opal glass window. Colour filters may be interposed between this window and the comparison surface to match illumination [illegible] yellow or blue-green light.

One scale of the instrument is marked to read the illumination of the test surface in lumens per square foot (foot-candles) and another

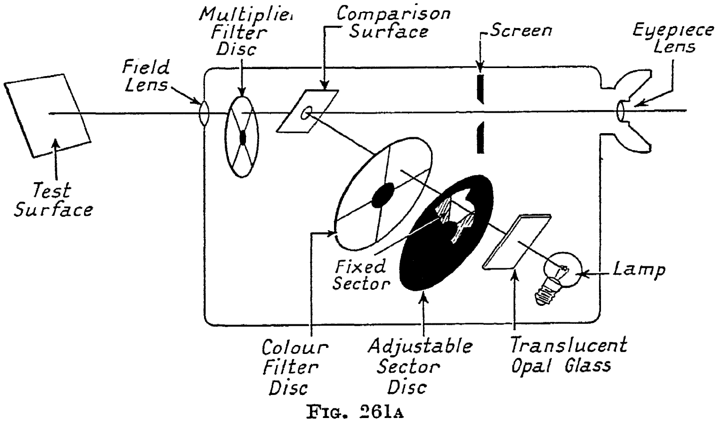
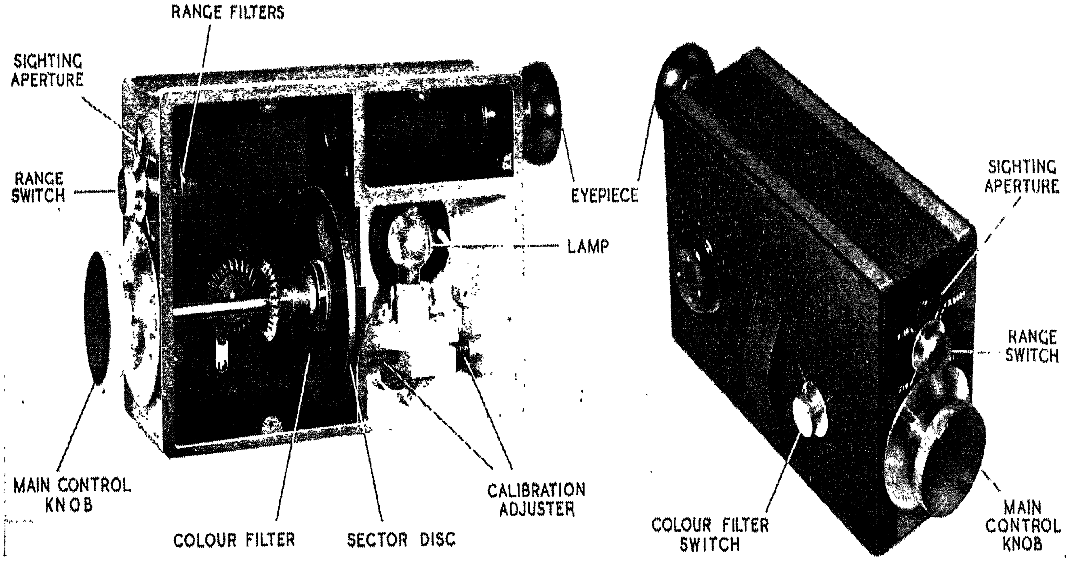


FIG. 261A



(*Holophane Ltd.*)

FIG. 261B. HOLOPHANE "LUMETER"

is graduated to read the luminance (brightness) of this surface in foot-lamberts. The range of illumination which can be covered is from 0·002 to 8,000 lumens per square foot.

Physical Photometry. Photometric measurements which do not depend upon the human eye for the judgment of equality of brightness, utilize a photo-electric cell.

A photo-electric cell is a piece of apparatus which, when connected in an electric circuit, allows a current to pass only when light falls upon the cell. The magnitude of the current varies also with the intensity of the light.

A difficulty experienced with many of the earlier types of photo-electric cells was that the relationship between the photo-electric current and the luminous flux falling on the cell was not linear, nor did it remain constant over a long period. More recently the construction of such cells has been improved to overcome this difficulty, and photo-electric methods of testing lamps for commercial purposes have been developed.

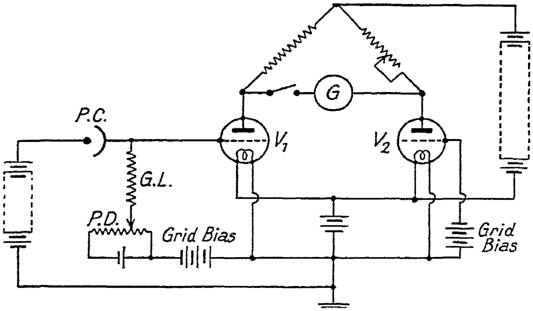


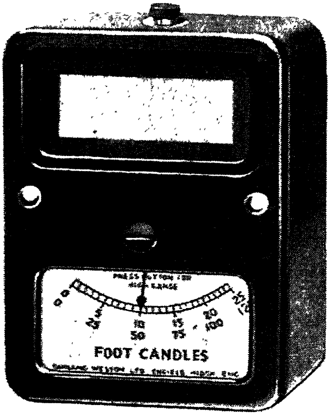
FIG. 262

A circuit for the measurement of luminous flux by a bridge, or null, method is shown in Fig. 262. *P.C.* is a photo-electric cell and V_1 and V_2 are amplifying valves. The bridge being initially balanced with no light falling on *P.C.*, the introduction of light will produce a voltage across the grid leak *G.L.* which unbalances the bridge. Balance is restored by applying a potential from the potential divider *P.D.*, this voltage being directly proportional to the photo-electric current, and therefore to the luminous flux falling on *P.C.* (assuming *P.C.* to have a linear current/luminous flux characteristic). *P.D.* is thus calibrated in lumens. The lamp under test is placed inside an integrating cube, the aperature of which allows light (directly proportional to the total lumens of the lamp under test to fall on the photo-cell which is contained within a small adjoining integrator.

For consistent results the insulation of the grid circuits of V_1 and V_2 must be kept very high and constant and the apparatus must be screened from stray electrostatic and electromagnetic disturbances.

Fig. 263 shows a conveniently small photometer (Weston Model S85) which contains a photo-electric cell coupled directly to a sensitive indicating instrument. It weighs less than 6 oz. and has two ranges; 0 to 25 and 0 to 125 foot candles.

Weston exposure meters and "Eel" photometers (Evans Electroselenium, Ltd.) use "Photo-voltaic" or barrier layer photo-cells.



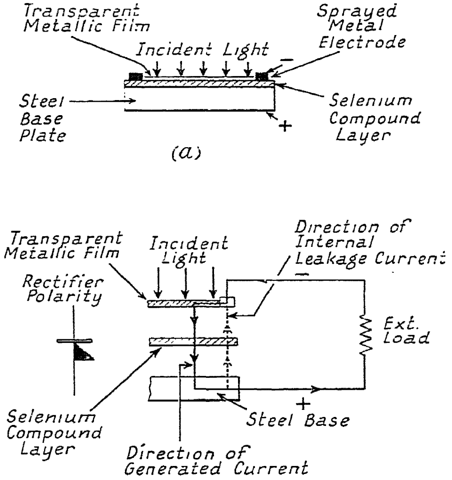
(*Sangamo Weston, Ltd.*)

FIG. 263. THE WESTON PHOTOMETER

The construction and principle of the Weston "Photronic" cell of the barrier layer, selenium-rectifier type are shown in Fig. 264 (*a*) and (*b*). The cell, which is fitted behind a glass window, consists of a layer of a selenium compound prepared on a metal base plate. A transparent metallic conducting film is deposited on the layer which is first heat-treated to convert it into a sensitive crystalline state. A metallic ring is sprayed on to the top surface for ease of contact. The conducting film, within this ring, is finally sprayed with a thin film of transparent lacquer.

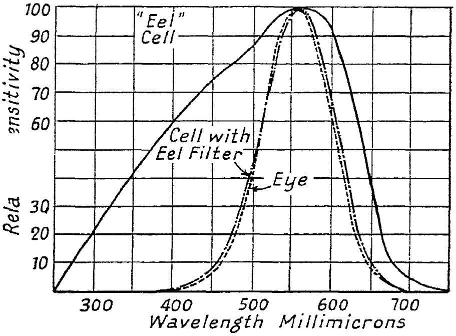
Light, falling on the cell, causes a movement of electrons from the selenium compound surface which acts as a photo-sensitive cathode. The electrons are collected by the transparent metal film—the anode—and passed to the external circuit. This is equivalent to an electric current flowing into the ring as shown in Fig. 264 (*b*).

Such cells have a non-linear voltage/current characteristic and act as rectifiers. The steel base acts as the positive pole of the cell and there is internal leakage current as shown. The e.m.f. generated in the cell is directly proportional to the intensity of illumination but the output current is proportional to this illumination only



(b) (*Sangamo Weston, Ltd.*)

FIG. 264



(*Evans, Electro-selenium, Ltd.*)

FIG. 265. SPECTRAL RESPONSE OF THE "EEL" PHOTO-CELL

when the external resistance is low compared with the internal resistance. A range of cells gives an output, at 40 ft-candles and 1,600 ohms external resistance, of 80–90 μA to 130–140 μA.

Since the barrier layer has a large capacitance (about 0·1 this type of cell cannot be used in pulsating light over 10,000 cycles per second; the external resistance must, in any case be low with pulsating light to avoid serious reduction of response due to the shunting effect of this capacitance.

The spectral response of the "Eel" cells is shown in Fig. 265. With a correction filter, having a transmission factor of about 50 per cent for white light, the response can be made very closely in agreement with that of the eye.

Descriptions of other photo-electric methods of measurement will be found in the publications mentioned in Refs. (1), (2), (3), (5), (9), (11), (12), (13), (14).

BIBLIOGRAPHY AND REFERENCES

(1) *Photometry*, J. W. T. Walsh.

(2) *Electrical Photometry and Illumination*, H. Bohle.

(3) *Illuminating Engineering*, F. E. Cady and H. B. Dates.

(4) *Modern Electrical Illumination*, C. Sylvester and T. E. Ritchie.

(5) *Photo-electric Cells*, N. R. Campbell and D. Ritchie.

(6) J. E. Petavel, *Roy. Soc. Proc.*, 65 (1899), p. 469, and *Electrician*, 44 (1900), pp. 747, 827, and 863.

(7) H. E. Ives and E. F. Kingsbury, *Phil. Mag.*, Vol. XXVIII, p. 708 (1914), and Vol. XXXIII, p. 18 (1917).

(8) C. H. Sharp and P. S. Millar, *Elect. Rev.* (*N.Y.*), Vol. LII, p. 141 (1908).

(9) "A Direct-reading Photo-electric Photometer for the Commercial Measurement of Incandescent Electric Lamps," G. T. Winch, *Jour. I.E.E.*, Vol. LXVIII, p. 533.

(10) "Illuminating Engineering," J. W. T. Walsh, *Jour. I.E.E.*, Vol. LXV, p. 733.

(11) "The Development of Commercial Photo-electric Photometers," G. T. Winch and C. T. Harper, *G.E.C. Jour.*, August, 1932.

(12) *Photocells and their Application*, V. K. Zworykin and E. D. Wilson.

(13) "The Selenium Rectifier Photo-cell: Manufacture, Properties, and Use in Photometry," J. S. Preston, *Jour. I.E.E.*, Vol. LXXIX, p. 424.

(14) "The Design of Precision Commercial Photo-electric Photometers," G. T. Winch and C. F. Machin, *G.E.C. Journal*, Vol. VI, p. 205, November, 1935.

(15) "Glossary of Terms used in Illumination and Photometry," *British Standard* 233: 1953.

(16) "Portable Visual Photometers," *British Standard* 230: 1945.

(17) "The Contribution of Research to Progress in the Lighting Art," J. W. T. Walsh, *Jour. I.E.E.*, Vol. 100, Pt. II, No. 73, Feb., 1953, p. 9.

(18) "The New Unit of Light," J. W. T. Walsh, *Light and Lighting*, June, 1940, p. 96.

(19) *Illuminating Engineering*, W. B. Boast.

(20) *I.E.S. Lighting Handbook* (2nd Edn., 1952).

(21) *Photo-Electric Multipliers*, S. Rodda.

CHAPTER XI

HIGH-VOLTAGE MEASUREMENTS AND TESTING

General Classification. The rapid advance in the use of high voltages for transmission purposes, during the last few years, has necessarily been accompanied by extensive research upon high-voltage phenomena. The subject of high voltage testing is, therefore, a very important one at the present time, and includes a large variety of testing methods.*

These may be classified as follows—

1. Sustained low-frequency tests.
2. Constant direct-current tests.
3. High-frequency tests.
4. Surge, or impulse tests.

Purpose of Various Test Methods. 1. SUSTAINED LOW-FREQUENCY tests are the commonest of all high-voltage tests. The frequency used is almost always the standard frequency—50 cycles per second in Britain and most European countries, and 60 cycles per second in America.

Voltages of 2 or 3 kilovolts (50 cycles) are used for routine pressure tests upon motors, switchgear, and other apparatus after manufacture, and in some cases after installation, in accordance with specifications of the British Standards Institution.

British Standard Specification 116 : 1952, "Oil Circuit Breakers for Alternating Current Systems," states that when such apparatus is tested at the maker's works, the test-voltages for the various components shall be as given in Table XI.

"The test voltage shall be alternating, of any frequency between 25 and 100 cycles per second, and approximately of sine-wave form."

Other tests are specified for the apparatus after erection on site.

Low-frequency tests are made upon specimens of insulation material for the determination of their dielectric strength and dielectric losses, for the routine testing of supply mains, and for works tests upon high-voltage transformers, porcelain insulators, and other apparatus. Voltages of 2,000 kilovolts and upwards are used for research work and for the testing of porcelain insulator strings and of high-tension cables.

2. CONSTANT DIRECT CURRENT TESTS. To transmit large amounts of power efficiently, high transmission voltages are necessary. Overhead lines have been erected with a working pressure of 380 kV, while cables for 275 kV have been made and tested.

The electricity supply regulations state that no extra-high-pressure main shall be brought into use "unless, after it has been

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* Owing to the impossibility of covering, fully, the whole ground of the subject in a single chapter, an extensive list of recent papers on high-voltage measurements is given at the end of the chapter. These should be referred to for more detailed information.

TABLE XI

Component	Rated Service Voltage	Test Voltage		
		All Circuit Breakers for Systems up to 88 kV	Circuit Breakers for Non-effectively Earthed Systems of 110 kV and above	Circuit Breakers for Effectively Earthed Systems of 110 kV and above
	kV	kV	kV	kV
Circuit-breaker including any series coils	Up to 0·66	2	—	—
	3·3	9·5	—	—
	6·6	17	—	—
	11	27	—	—
	22	52	—	—
	33	76	—	—
	44	100	—	—
	66	150	—	—
	88	200	—	—
	110	—	250	200
	132	—	300	250
	165	—	365	300
	220	—	485	400
	275	—	—*	485
Neutral earthing circuit breaker	1·91	6·5	—	—
	3·81	10·5	—	—
	6·35	16	—	—
	12·7	31	—	—
	19·1	45	—	—
	25·4	60	—	—
	38·1	88	—	—
Operating coils and small wiring	—	2	2	2
Operating motors	In accordance with B.S. 168 and B.S. 170.			

* All 275-kV systems are considered for this specification to be effectively earthed.

placed in position, and before it is used for the purposes of supply, the insulation of every part thereof has withstood the continuous application, during half an hour, of pressure exceeding the maximum pressure to which it is intended to be subjected in use, that is to say, in the case of every electric line to be used for a pressure not exceeding 10,000 volts, twice the said maximum pressure, and in the case of a line to be used for a pressure exceeding 10,000 volts, a pressure exceeding the said maximum pressure by 10,000 volts; and the undertakers shall record the results of the tests of each main or section of a main."

If such pressure tests were carried out with an alternating high-tension supply, the transformer for the purpose would have to be of inconveniently large capacity, owing to the heavy charging current taken by such lengths of cable. Transport difficulties would be encountered, also, in moving the testing transformer to the site of the cable system, and for this reason high-tension direct current is used for testing purposes. The plant for such a supply need only be of small capacity, and may be constructed in a portable form.

High-tension direct current is also used in research work upon insulation and upon X-rays and electrical precipitation.

3. HIGH-FREQUENCY TESTS. Such tests, at frequencies varying from several thousand cycles per second to a million or more cycles, are important in work upon insulators for wireless purposes where such frequencies are usual. It has been found, also, that in the case of porcelain insulators in service on transmission lines, breakdown, or flashover, occurs in most cases as a result of high-frequency disturbances in the line, these being due either to switching operations or to external causes. Thus, insulators which may be satisfactory under high voltage of low frequency are not necessarily satisfactory under the different conditions existing when the frequency is of a much higher order.

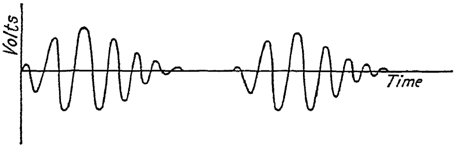

FIG. 266. DAMPED OSCILLATIONS

It has been found that undamped high-frequency oscillations—i.e. oscillations of approximately constant amplitude—cause failure of insulators at comparatively low voltages, due to high dielectric loss and heating. Such oscillations do not, however, occur in power systems in practice. Damped high-frequency oscillations having a waveform such as that shown in Fig. 266 do occur in practice, due to switching operations, or to arcing grounds.

For this reason, high-voltage tests at high frequencies are made in insulator manufacturing works, in an endeavour to obtain a design of insulator which will satisfactorily withstand all conditions in service.

4. SURGE, OR IMPULSE TESTS. These tests are carried out in order to investigate the influence of "surges" in a transmission line upon breakdown of the line insulators, and of the end turns of transformers connected to the line. In order to appreciate the importance of such tests and to realize what are the requirements of the testing plant, the nature of these surges must be understood.

Such surges are usually produced by lightning in the neighbourhood of the transmission line, and are the result of the sudden discharge of the electricity on the suspended particles of a thunder cloud, which takes the form of a lightning stroke. A direct lightning stroke on the line is comparatively rare, but a charged cloud above, and near to, the line, will induce charges, of opposite sign to that of its own charge, in the line. These charges are "bound" (i.e. held in that portion of the line nearest the cloud) so long as the cloud remains near without discharging its electricity by a lightning stroke.

If, however, the cloud is suddenly discharged—as it is when the lightning stroke occurs—the induced charges in the line are no longer bound, but travel, with the velocity of light, along the line to equalize the potential at all points of the line.

This means that a steep-fronted voltage wave travels along the line, its form being as shown in Fig. 267.* Due to this travelling wave the potential of any point along the line will rise very suddenly from its normal value by an amount equal to the amplitude of the wave. The time taken for this voltage rise is of the order of one millionth of a second. The effect is to impose very severe stresses upon the line insulators and upon the transformer windings (if the wave is allowed to reach them), the stresses depending upon the steepness of the wave front.

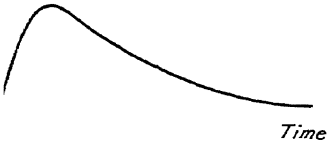

FIG. 267. SURGE WAVEFORM

Violent rupture of insulators is often caused by such surges, and the investigation of their effect upon different types of insulators and other apparatus, as well as the testing of surge-absorbers designed for transformer protection, is obviously of great importance.

The shapes of voltage waves for impulse testing have now been standardized in several countries. Thus in Britain, in B.S.S. 923 : 1940 the standard testing wave is expressed as "1/50 microsec." which indicates that it is to rise to its peak in 1 microsecond and is to fall to one half of its peak value in 50 microseconds.

The testing plant for this purpose must be capable of producing a high, uni-directional voltage, suddenly applied to the apparatus under test.

Testing Apparatus. Certain apparatus is common to all types of high-voltage tests. Special apparatus is required, in addition, for direct current, high frequency, and impulse tests. For all tests, the following apparatus is usually essential.

(*a*) A high-voltage transformer.
(*b*) Apparatus for voltage regulation.
(*c*) Control gear and high-tension connections, including safety protective devices.
(*d*) Apparatus for voltage measurement.

(*a*) HIGH-VOLTAGE TRANSFORMERS. Such transformers, for testing purposes, require careful design, owing to the fact that they are subjected to transient voltages and surges during their normal operation when the insulation under test breaks down. To withstand the internal stresses set up by such disturbances the transformer insulation must be carefully proportioned. The transformers are usually single-phase, core type, and are oil-immersed, Bakelite cylinders being used for insulation between high- and low-tension windings. Transformers for cable testing, and certain other purposes, may also be required to give a considerable current, and the question of regulation and cooling must therefore be carefully considered. In insulation testing the current taken from the transformer, when the test specimen breaks down, is limited by the insertion of water resistances in the test circuit, so that for such purposes the transformer need not have a large kVA capacity. The following table gives the order of the capacities required for various purposes, and also the approximate values of maximum voltage used.

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* See also Refs. (8), (9), (10), (11), (12).

TABLE XII

Purpose of Transformer	Approximate Capacity (kVA)	Maximum Voltage (kV)
Routine pressure tests upon motors and switchgear	Small	2 or 3
Insulation testing	10 to 20	50
Routine testing of cables	50	10 to 30
High-voltage transformer and porcelain insulator testing and research . .	20 to 50	100 to 200
Research and testing of strings of insulators.	½ to 1 kVA per kilovolt	500 to 2,000
High-voltage cable testing . . .	100 to 500	100 to 500

Equipments giving a maximum high-tension voltage up to 500 kV have usually a single high-voltage transformer. For the higher voltages—1,000 kV and upwards—two or more transformers are generally used, connected *in cascade*—a method of connection due to Prof. Dessauer. This arrangement is found to be more convenient than the single transformer for very high voltages, owing to the very large size and high cost of a single transformer for the purpose. A common method of connecting transformers in cascade is shown in Fig. 268. The low-tension supply is connected to the primary winding of transformer I, the tank of which is earthed. One end of the h.t. secondary winding of this transformer is connected to the earthed tank. From the other end of the secondary winding, a lead passes through a high-voltage bushing, which provides insulation for the full secondary voltage between this lead and the tank. Through this bushing also runs a second lead from a tapping point on the secondary winding, the voltage between this tapping point and the high-voltage end of the secondary winding being that required for the primary winding of transformer II. One end of the secondary winding of transformer II is connected to its tank, which is insulated from earth for the full secondary voltage of transformer I. The other end of the secondary winding provides the high voltage terminal of the equipment, the total voltage—to earth—obtainable being approximately the sum of the secondary voltages of the two transformers.

(*b*) Voltage Regulation. In order to avoid surges on the high-tension side of the transformer, and also for accuracy of voltage measurement, it is essential that the regulation of voltage shall be smooth. Sudden variations of the testing voltage *must* be avoided in most tests. Another requirement of voltage-regulating apparatus is that it shall not cause distortion of the voltage waveform.

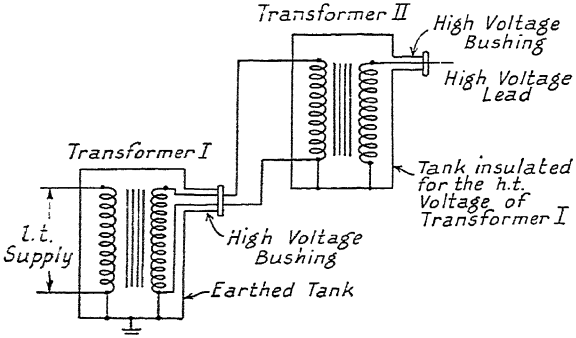

Fig. 268. Cascade Connection of Transformers

The transformer secondary voltage is regulated by variation of the voltage applied to its primary winding. This may be done either—

(i) By variation of the alternator field current, or

(ii) By insertion of either resistance or inductance in the supply circuit from the alternator, or

(iii) By means of an induction regulator, or

(iv) By means of a tapped transformer.

(i) *Variation of the Alternator Field Current.* This method can only be used, of course, when a separate alternator is used for the supply to the testing plant. Except in the case of comparatively small plants for routine testing, a separate alternator is generally used. The alternator should have a voltage waveform which is as nearly as possible sinusoidal on no load, and the distortion under load conditions should be small. This is achieved by making the air gap of the alternator large; by special design of the armature windings; and by suitable regulation of the number, and shape, of the slots.

This method of voltage regulation has the advantages of smooth control of the voltage from zero to the full voltage, absence of impedance for regulation purposes in the transformer primary circuit (which may produce waveform distortion), good waveform, convenience, simplicity, and freedom from appreciable disturbance by frequent short-circuiting of the transformer during testing.

The field current of the alternator is varied directly in the case of fairly small machines, but when a large alternator is used the field current of the exciter may be varied, thus varying the alternator field current indirectly. Fig. 269 shows the connections for direct variation of the alternator field current. A potential divider, connected across a steady d.c. supply, is used, and the connections are arranged so that a small field current in a reverse direction to the normal one, may be obtained. By this arrangement zero voltage may be obtained by neutralization of the residual magnetism in the field. Smooth and gradual voltage regulation is obtained by a special design of the potential divider, which should have a large number of turns and some provision for very gradual and steady movement of the sliding contact.

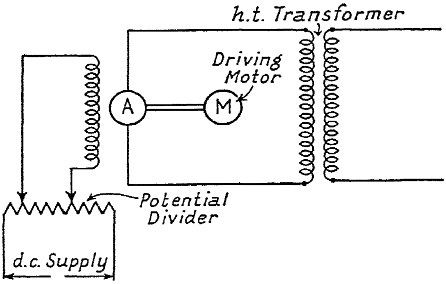

Fig. 269. Alternator Field Current Method of Voltage Control

(ii) *Resistance, or Inductance, Control.* When a separate alternator is not used—in the case of small equipments—resistance or inductance must be inserted in the a.c. supply circuit for voltage regulation.

If resistance is used, it is better to make the connections as in Fig. 270, using the resistance as a potential divider, than to insert it in series with the transformer primary winding. Zero voltage can be obtained in this way. A slider resistance should be used for smooth voltage regulation.

The disadvantages of the resistance method are the loss of power in the resistance and the impracticability of its use for high powers, owing to the very large size and great cost of resistances for the purpose. For small equipments (2 or 3 kVA) it has the advantages of cheapness, convenience, smooth voltage variation, and small distortion of the voltage waveform.

A choke coil, connected in series with the transformer primary, may be used instead of resistance. Voltage regulation is then obtained by the withdrawal or insertion of the iron core of the coil. This method has the advantage of greater efficiency than the resistance method, on account of its low power factor, but has several serious disadvantages. These are—

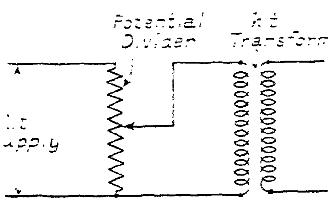

FIG. 270. POTENTIAL DIVIDER CONTROL

(a) the large size of coil required if the power to be dealt with is high;

(b) the distortion of waveform owing to the iron core;

(c) the fact that increase of its inductance will *increase* the primary voltage of the transformer instead of decreasing it if the power factor of the load on the secondary side of the testing transformer is leading, as is often the case.

(iii) *Induction Regulator Method.* This method of voltage regulation has the advantage of smooth regulation from zero to full voltage, and may be used for all loads and power factors. It is, therefore, much to be preferred to the resistance and inductance methods described above.

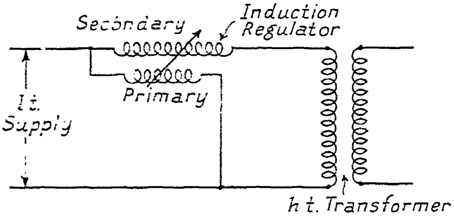

FIG. 271. INDUCTION-REGULATOR CONTROL

An induction regulator is, essentially, a transformer, or mutual inductance, the secondary voltage of which can be varied by the rotation of the primary through any required angle up to 180°. The connections are shown in Fig. 271. If the primary and secondary have equal numbers of turns, by rotation of the primary winding the voltage induced in the secondary may be varied from $-E$ to $+E$ (where E is the low tension supply voltage). Thus the voltage applied to the h.t. transformer primary may be varied from zero to $2E$.

Careful design, with distributed windings on the rotor, are necessary to prevent waveform distortion. This method of regulation is often used in cable-testing equipments, since its gradual voltage variation at loads of any magnitude is advantageous for such work.

(iv) *Voltage Variation by Means of a Tapped Transformer.* Fig. 272 gives the connections of this method of regulation. An intermediate regulating transformer is used, having its primary supplied from the l.t. supply. The secondary winding of the transformer has a large number of tappings whereby the voltage applied to the primary of the h.t. transformer may be varied. In order to avoid surges, due to the opening of the secondary circuit of the regulating transformer when the tapping switch is moved, two contact brushes are used, making contact with adjacent studs, and with a "buffer" resistance or reactance coil between them to prevent short-circuit of a section of the transformer winding. An auto-transformer may be used instead of the double-wound transformer shown in the diagram. The operation is similar to that in the case of the double-wound transformer.

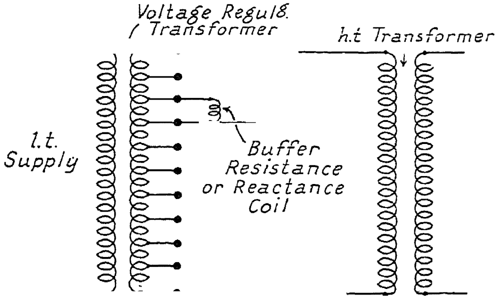
Voltage Regul^g. Transformer
h.t Transformer
l.t. Supply
Buffer Resistance or Reactance Coil

FIG. 272. TAPPED-TRANSFORMER REGULATION

For gradual regulation a number of coarse tappings are used together with fine tappings. The method has the advantages of high efficiency and small waveform distortion, but the regulation is not smooth unless a very large number of tappings is used and, when the power of the equipment is large, the switchgear must be large and is expensive.

(*c*) CONTROL GEAR AND CONNECTIONS. On the low tension side —i.e. in the primary circuit of the h.t. transformer, there should be—

A *main switch*, to isolate the testing apparatus from the supply.

Fuses.

A *circuit breaker*, with arrangements for tripping the breaker over a wide current range, and having a no-volt coil in order to protect apparatus from damage in the event of failure of the supply.

An *over-voltage relay*, which short-circuits the no-volt coil of the circuit breaker if the supply voltage exceeds a predetermined value.

Interlocks. These are arrangements which guard against the supply to the transformer being switched on except at a low value, or which make it impossible for the operator to pass inside the screen-work surrounding the high-tension testing area while the supply is on. The latter takes the form of a switch on the gate of the screen.

Earth connections. Both for safety and for the protection of apparatus it is necessary that all metal parts, forming part of the testing equipment, which should be at earth potential, should be definitely earthed. This applies to the framework of switchgear and to the guard screen, transformer tanks (except where a cascade arrangement is used), and to one end of each of the transformer windings.

On the high-tension side care must be taken in deciding the size and shape of the leads and connections, in order to avoid "corona" loss, which may distort the voltage waveform and may influence the breakdown or flash-over voltages measured, on account of the surges in the circuits, and because of ionization of the air in the test room, which it may produce. "Corona" is the name given to the luminous glow which surrounds a high-tension conductor when the potential gradient at its surface exceeds the disruptive strength of air. Breakdown of the air at the conductor surface occurs first of all, and this spreads outwards, ionizing the air immediately surrounding the conductor, and producing both a glow and a hissing sound. The result of such corona is a loss of power, with the possible effects mentioned above. The voltage at which corona occurs depends upon the diameter of the high-tension conductor, the voltage increasing with increasing diameter. Thus, to avoid such effects the conductors in the high-voltage circuit should be made much larger than is necessitated by current-carrying considerations.

E. T. Norris and F. W. Taylor (Ref. (13)) recommend for the diameters of high-tension conductors "1 in. for 100 kV and about 12 in. for 1,000 kV, if the atmospheric conditions are normal and the high-tension clearances are ample." High-tension conductors are almost always bare, the surrounding air forming the principal insulation. The supports for these conductors are usually Bakelite tubes or rods, or porcelain insulators. Since the current to be carried is usually very small, good contact at joints is not often essential, mere touch being sufficient. Sharp edges, which produce concentration of electrostatic field, must be avoided throughout the circuit. Terminals are usually fitted with spherical metal caps, and sharp bends in conductors are fitted with spherical elbow pieces to distribute the electric stress.

As a protection from surges which may be produced when a sphere-gap, or the apparatus under test, breaks down, a high resistance (about ½ ohm per volt), or choke coil, is inserted in the main high-voltage circuit. A sphere-gap is also connected across the high-voltage winding as a protection in the event of excessive voltage being applied to the circuit. This may be the same gap as is used for voltage measurement. A water resistance, consisting of a glass tube or tubes containing water, with some common salt added, is connected in the sphere-gap circuit to prevent pitting of the sphere surfaces by limiting the current flowing at breakdown. These resistances should be about 1 ohm per volt to limit the current to 1 amp.

Fig. 273 gives a simplified diagram of connections for a high-voltage testing equipment having a separate supply alternator.

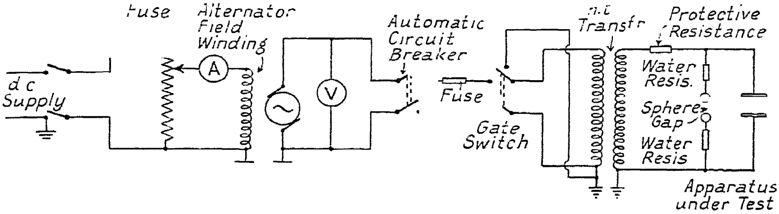

FIG. 273. CONNECTIONS OF TESTING EQUIPMENT

(*d*) APPARATUS FOR VOLTAGE MEASUREMENTS. Owing to the difficulty of designing electrostatic voltmeters, for the measurement of extra high voltages, which will be free from errors due to corona effects within the instrument, and to external electrostatic fields, a number of special methods of measurement have been devised for the purpose. Some method must be available, also, for calibrating such voltmeters when constructed. Work by the British Electrical and Allied Industries Research Association (Ref. (14)), and the American Institute of Electrical Engineers, led to the standardization of the sphere-gap method as the most reliable one.

(i) *Sphere-gap.* Two metal spheres are used, separated by an air gap. The potential difference between the spheres is raised until a spark passes between them. The value of the potential difference required for spark-over depends upon the dielectric strength of air, the size of the spheres, their distance apart, and upon a number of other factors. As a result of the research work already mentioned, the British Standards Institution has drawn up a list of "British Standard Rules for the Measurement of Voltage with Sphere-gaps" (No. 358-1939). Some years ago the American Institute of Electrical Engineers drew up a list of rules for the measurement of voltages up to 50 kV, using needle points for a spark-gap, instead of spheres as at present used. The disadvantages of this form of gap were the errors caused by variation in sharpness of the needles; by variation of the humidity of the atmosphere; and by the corona which formed at the points before the gap actually sparked over.

In B.S. 358-1939, twelve sizes of spheres are to be used to cover a range of voltage from 2 to 2,500 kV, namely, of diameters 2, 5, 6·25, 10, 12·5, 15. 25, 50, 75, 100, 150 and 200 cm. The following are extracts from these rules—

1. "A spark-gap may be used for the determination of the peak value of a voltage-wave, and for the checking and calibration of voltmeters and volt-measuring devices for high-voltage tests.

2. "Peak voltages may be measured from about 2 kilovolts to about 2,500 kilovolts by means of sphere-gaps used in conjunction with the calibrations provided herewith."

3. "The appropriate Tables may also be used for the measurement of impulse voltages of either polarity and of the standard wave forms, having fronts of at least one microsecond and times-to-half-value of at least 5 microseconds."

4. "The length of any diameter shall not differ from the correct value by more than 1 per cent for spheres of diameter up to 100 cm., or by more than 2 per cent for larger spheres."

5. "One sphere may be earthed or both may be insulated, the voltages being symmetrical with respect to earth in the latter case."

Subject to certain specified conditions,
"no conductor or body having a conducting surface (except the supporting shanks) shall be nearer the sparking point of the high-voltage sphere than the distance given by the expression $\left(0{\cdot}25 + \frac{U}{300}\right)$ metres, where U is the voltage in kV (peak) which is to be measured."

6. "It is recommended that spheres of brass, bronze, steel, copper, aluminium or light alloys be used, and that the surface of the spheres should be cleaned immediately before use. A high degree of polish should be avoided."

The rules provide that the current at spark-over shall be limited to about 1 amp. and that the interval between consecutive discharges shall be great enough to prevent appreciable heating of the spheres.

The density of the air affects the spark-over voltage for a given gap-setting. The "relative air density," is given by

$$d = \frac{p}{760} \cdot \frac{273 + 20}{273 + t} = 0{\cdot}386 \left(\frac{p}{273 + t}\right) \quad . \quad . \quad (230)$$

where p = barometric pressure in millimetres of mercury

t = temperature in degrees Centigrade

A correction factor, by which the spark-over voltage for a given gap-setting, under the standard conditions (760 mm. pressure and 20° C.) must be multiplied, in order to obtain the actual spark-over voltage, may be obtained from a table given in the B.S.I. rules. This factor is approximately equal to d for values of d above 0·9.

With regard to humidity, the rules state that over the sparking distances covered by the calibration tables, the breakdown voltage of the sphere-gap is independent of the humidity of the atmosphere but deposited dew on the surface of the spheres lowers the breakdown voltage and invalidates the calibrations.

For the calibration of a spark-gap from a standard sphere-gap the two gaps should not be connected in parallel. "Equivalent spacings should be determined by comparing each gap in turn with a suitable indicating instrument."

F. W. Peek (Ref. (15)), Russell (Ref. (16)), and others have obtained formulae by which the spark-over voltage for a given gap-setting and sphere-diameter may be calculated, but the use of the calibrations given in the B.S.I. rules is to be recommended instead of the use of such formulae, the application of which is limited.

The spark-over voltages given in the rules are in terms of r.m.s. (sine-wave) voltage. To obtain the corresponding peak values—which are important since it is the peak value of the voltage, rather than the r.m.s. value, which determines the flash-over, or breakdown of insulation—the r.m.s. values given must be multiplied by $\sqrt{2}$.

A disadvantage of the sphere-gap method of measurement is that it cannot be used to give a continuous record of voltage. It is generally used to calibrate some indicating instrument, or other apparatus, which does give such a continuous record. In the following methods of measurement, wherever calibration is referred to, it is to be understood that the sphere-gap method is implied unless otherwise stated.*

(ii) *Transformer Ratio Methods of Measurement.* In this method the primary voltage of the high-tension transformer is measured by a calibrated voltmeter, and this is multiplied by the turns ratio of the transformer to obtain the secondary voltage. The assumption is made that the transformer ratio is unaffected by the transformer impedance and by variations in the load. The method also gives r.m.s. values of voltage, and the secondary voltage waveform must be determined in order that peak values of voltage may be obtained from the "crest factor" of the wave, i.e. the ratio

$$\frac{\text{Peak voltage}}{\text{r.m.s. voltage}}$$

Some high-tension transformers carry a separate voltmeter-coil having a number of turns which is a definite fraction of the number of turns on the secondary winding. The voltage induced in this coil (measured by a low-voltage voltmeter), when multiplied by the turns-ratio, gives the secondary voltage of the transformer. This method cannot be used with the cascade arrangement of transformers and may only be relied upon to within 1 or 2 per cent.

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* The calibration of the sphere-gap itself is described by S. Whitehead and A. P. Castellain (Ref. (14)), who used a number of different methods for the purpose, and obtained consistency between them.

(iii) *Potential Divider Methods.* In such methods, either a high resistance, or air capacitor, must be used, connected across the high-voltage winding, a definite fraction of the total secondary voltage being measured by means of a low-voltage electrostatic voltmeter.

High resistances for this purpose have the disadvantage of residual phase-angle errors due to the effects of distributed capacitance, unless special precautions are taken in their design (Ref. (17)). Such resistances have been constructed, however, and are used in several high-voltage equipments.

The advantages of capacitors as compared with resistors are: their comparatively simple construction, their freedom from heating, and the fact that they are much more easily shielded from extraneous capacitance effects than are high resistances. B. G. Churcher and C. Dannatt (Refs. (18), (19)) have described the construction and use of such capacitors.

In the capacitor method an electrostatic voltmeter, whose capacitance is large compared with that of the high-voltage standard capacitor, is connected in series with the latter, and is shunted by a capacitance of much greater value than its own. The capacitance of the standard capacitor must be accurately known, and it is important that it shall be free from dielectric loss. For the latter reason air capacitors are always used for this purpose, their capacitance usually being from 50 to 100 $\mu\mu f$.*

If C is the capacitance of the standard capacitor, and C_1 that of the capacitor which shunts the electrostatic voltmeter (see Fig. 274), then the voltmeter reading v is given by

$$v = \frac{i}{\omega C_1}$$

where i is the current passing through the capacitors and $\omega = 2\pi \times$ frequency.

Neglecting the capacitance of the voltmeter, the effective capacitance of C and C_1, in series, is $\frac{CC_1}{C + C_1}$, and the voltage V of the high-tension circuit, is

$$V = \frac{i}{\omega\left(\frac{CC_1}{C + C_1}\right)}$$

Thus,

$$V = \frac{C + C_1}{C} \cdot v \quad . \quad . \quad . \quad . \quad (231)$$

This method measures r.m.s. voltage values.

Other methods utilizing a standard impedance involve the measurement of the current which flows through the impedance—either a standard air capacitor or high resistance—when it is connected across the high-tension winding of the transformer. The current may be measured by means of a thermo-junction and galvanometer; by measuring the voltage drop across a known resistance, in series with the standard high impedance, using an electrometer; or by a valve-voltmeter method described by S. Whitehead and A. P. Castellain (*loc. cit.*).

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* The effect of humidity of the atmosphere and of dust deposits, upon the purity of capacitors is given by B. G. Churcher and C. Dannatt (*loc. cit.*).

A *thermo-junction* consists of a small "heater" of fine wire, through which the current to be measured is passed. To the centre of this heater is attached a thermo-junction of two dissimilar metals, the attachment being by means of a small bead of a material which, while being electrically insulating, is a good thermal conductor.

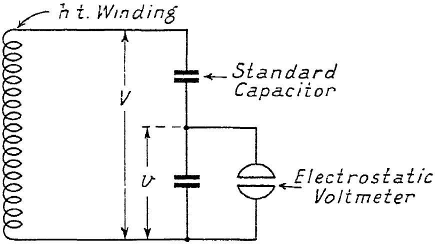


FIG. 274. CAPACITOR POTENTIAL DIVIDER

An e.m.f. is set up, in the circuit containing the thermo-junction, when a current is passed through the heater. This e.m.f. depends upon the magnitude of the heating effect of the current, and thus the r.m.s. value of a small alternating current may be measured by comparison with a known direct current. The thermo-junction, which may be of the vacuum type—in which the heater and junction are enclosed in a small evacuated glass bulb with terminals brought out—or air type, is used in conjunction with some form of galvanometer. The arrangement is calibrated with direct current, and is then used for the measurement of alternating currents of the same order of magnitude as the direct, calibrating current. Such thermo-junctions may be obtained for either 2·5 mA, 5 mA, 10 mA, and so on, up to 10 amp. Below 1 amp. they are vacuum type, and above 1 amp. air type.

From the measured value of the current flowing through a standard impedance, the r.m.s. values of the high-tension voltage may be obtained by multiplication by the value of the impedance.

(iv) *Measurement of Peak Voltage.* Owing to the fact that the *maximum* or *peak* value of the applied voltage is that which produces the actual breakdown stress in the material under test, it is important that this value should be known in most cases. If methods of voltage measurement which give r.m.s. voltage values are used, the peak voltage may be obtained from the crest-factor of the voltage wave. It is often more satisfactory, however, to use some method of voltage measurement which gives the peak value of the voltage directly. From the fact that the breakdown of air is involved in the sphere-gap method, this is obviously one method which depends upon the peak values, and the calibration curves for sphere-gaps, already referred to, give such peak values.

Other methods of measurement of peak voltage have been devised and are as follows—

1. *Rectified Capacitor Charging-current Method.* This method depends upon the fact that the peak voltage value is proportional to the average charging-current of a standard air capacitor. The charging-current is therefore rectified, either by a mechanical rectifier, by specially designed thermionic valves or metal rectifiers, and its average value is then measured by a d.c. moving-coil, permanent-magnet milliammeter, or by a d'Arsonval galvanometer.

The valves used depend for their action upon the fact that if two electrodes are enclosed within a highly evacuated tube, one of these electrodes—the plate—being comparatively cold, whilst the other—the filament—is heated, current is passed only in the direction plate to filament. The amount of current which it is possible to rectify by this means is dependent upon the filament temperature.

One-half of each wave must be entirely suppressed by such a valve. In order to prevent ripples in the output current wave, two valves are used with a suitable proportioning of inductance and capacitance in the circuit.

The capacitor used may be either of the type already mentioned, the capacitance being calculated from the dimensions of the capacitor—when the method is an absolute one—or it may take the form of a capacitor bushing, fitted to the transformer, in which case calibration is needed.

Theory. If v is the voltage across the capacitor, of capacitance C, at any instant, and Q is the quantity of electricity in the capacitor at that instant, we have

$$\int i\,dt = Q = Cv$$

where i = charging current and t = time.

Now, during one whole period, the voltage rises to a positive maximum, falls to zero, rises to a negative maximum, and falls again to zero, thus giving a total change in voltage, during the cycle, of $4V_{max}$, where V_{max} is the peak, or maximum, value. Meanwhile, the quantity of electricity supplied to the capacitor during this period is $\int_0^T i\,dt$, T being the periodic time.

Thus
$$\int_0^T i\,dt = C\,.\,4V_{max}$$

or
$$\frac{1}{T}\int_0^T i\,dt = \frac{C\,.\,4V_{max}}{T}$$

Now, $\frac{1}{T}\int_0^T i\,dt$ is the average value of the charging-current, i_{av}. Hence,

$$i_{av} = \frac{4CV_{max}}{T} = 4Cf\,.\,V_{max}$$

where f is the frequency.

Thus $$V_{max} = \frac{i_{av}}{4Cf} \quad . \quad . \quad . \quad . \quad . \quad (232)$$

which gives the peak voltage value if the capacitance of the capacitor and the frequency (which must be kept constant throughout the test) are known.

Fig. 275 shows the connections of an arrangement for peak voltage measurement using full-wave rectification (see Ref. (20)).

This method of peak voltage measurement was first described by Chubb and Fortescue, and its accuracy is considered by R. Davis, G. W. Bowdler, and W. G. Standring (Ref. (21)), who state that, for a sine wave of voltage, the method is accurate provided that the impedance of the rectifier is negligibly small compared with that of the capacitor; that the rectifier is efficient in entirely suppressing one-half of the alternating wave; and that the milliammeter, or galvanometer, indicates the mean current correctly.

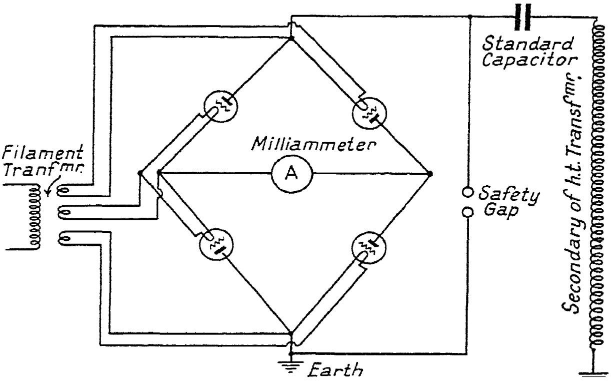


FIG. 275. CIRCUIT FOR PEAK VOLTAGE MEASUREMENT BY RECTIFIED CAPACITOR CHARGING CURRENT

With regard to wave-form, the same authors state that: "the method . . . is satisfactory for all wave-shapes with the exception of (1) wave-shapes with different positive and negative maxima, and (2) wave-shapes with more than one peak in each half cycle."

2. *Ryall Crest Voltmeter.* L. E. Ryall (Ref. (22)) has developed a simple form of crest voltmeter which is independent of frequency. It consists of a neon lamp, used in conjunction with a capacitor potential-divider. Two capacitors, one—a variable one—of much greater capacitance than the other, are connected in series across the voltage whose peak value is to be measured. A neon lamp of a specially chosen type, is connected across the variable capacitor (see Fig. 276). Neon lamps have the characteristic that they only

commence to glow when the voltage applied to them reaches a certain definite value, called the "striking" voltage.

In this method of peak voltage measurement the variable air capacitor, across which the lamp is connected, is adjusted until the lamp "strikes." Then the peak value of the h.t. voltage is given by

$$V_{max} = V_{a.c.s.}\left(1 + \frac{C_1}{C_2}\right) \quad . \quad . \quad . \quad (233)$$

where $V_{a.c.s.}$ is the a.c. striking voltage of the lamp (which is a constant quantity if the type of lamp is suitably chosen), and C_1 and C_2 are the capacitances of the variable and fixed capacitors respectively.

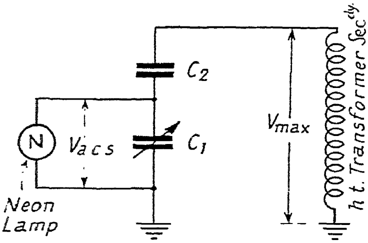


FIG. 276. CONNECTIONS OF RYALL CREST VOLTMETER

Ryall found that the a.c. striking voltage for a neon lamp "is constant for all frequencies above 25 cycles per second, and can be measured to within ± ½ per cent if about 5 min. is allowed to elapse between successive flashes of the lamp, so that the lamp can return to its original ionized condition.

"The a.c. extinguishing voltage, $V_{a.c.e.}$, is also constant for all frequencies above 25 cycles per second, and can be measured to within ± ¼ per cent."

The accuracy of the method obviously depends upon the accuracy with which $V_{a.c.s.}$ may be determined, and upon the accuracy with which the capacitances C_1 and C_2 are known.

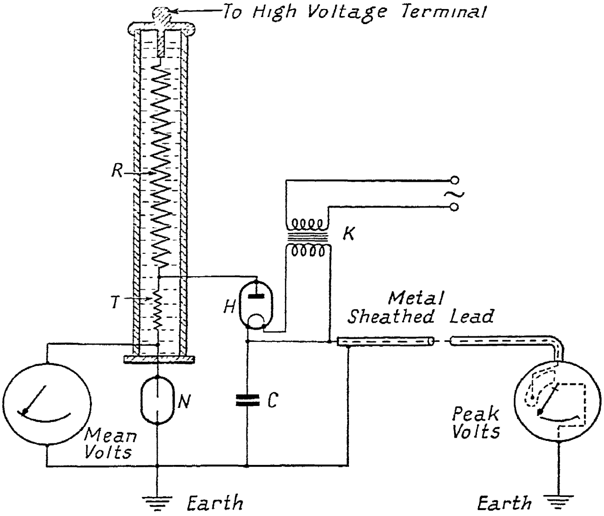
Both the lamp and capacitor C_1 are enclosed within metal boxes for screening purposes, and the lamp, in addition, is screened by a wrapping of lead foil, which is earthed, as the lamp glows when placed in a strong electrostatic field. A design of such a crest voltmeter for 150 kV (max.) is given by Ryall (*loc. cit.*).

Fig. 277 shows the principle of the Everett-Edgcumbe peak voltmeter for high-voltage measurements up to 500 kV (peak). This instrument gives a direct and continuous indication of the peak voltage under all conditions of loading and its readings are independent of waveform.

A resistor *R* is connected between the high-voltage terminal and earth. Across a section *T* of this resistor is joined a high-voltage rectifying valve *H* in series with a capacitor *C* and an electrostatic voltmeter reading direct in peak volts. For safety, the connecting leads are enclosed in an earthed flexible metal sheathing.

In the diagram of connections a mean voltmeter is also included.

(v) *High-voltage Voltmeters.* For the measurement of voltages up to about 200 kV, several forms of voltmeter have been designed which may be connected across the high-voltage circuit directly, without any potential-dividing device. Most of these are of the "attracted-disc" type, based on Lord Kelvin's volt balance. The latter is an absolute instrument. It consists of two flat discs, one fixed, and the other suspended from one arm of a balance. The fixed disc is insulated from the case of the instrument and is charged to a high potential. The pull between the two discs is given by



(*Everett Edgcumbe & Co., Ltd.*)

FIG. 277. EVERETT EDGCUMBE PEAK KILOVOLTMETER

$$F = \frac{\kappa . A . V^2}{8\pi d^2} \quad . \quad . \quad . \quad . \quad . \quad (234)$$

where *A* is the area of the moving disc, *V* the voltage between the discs, *d* the distance between them, and κ the permittivity of the medium. This force is balanced by weights on the balance arm, the magnitude of the weights giving the force F, and hence *V*.

Fig. 278 shows the construction of the Siemens and Halske instrument, based on the above principle, while Fig. 279 shows the Abraham voltmeter, the latter being commonly used in high-voltage testing equipments.

In the former instrument the high-tension plate is in the form of a hollow metal spheroid, suspended, by means of an aluminium rod and metal filament, in oil. The oil tends to prevent the formation of an arc between the high-tension plate and the earthed plate, and by enabling the two plates to be brought nearer together, increases the working forces of the instrument. The earthed plate is in the form of a shallow metal pan, which also screens the instrument from external electrostatic effects. Another guard screen is provided in the oil just above the high-tension plate, as shown.

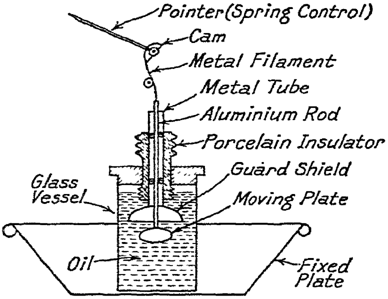


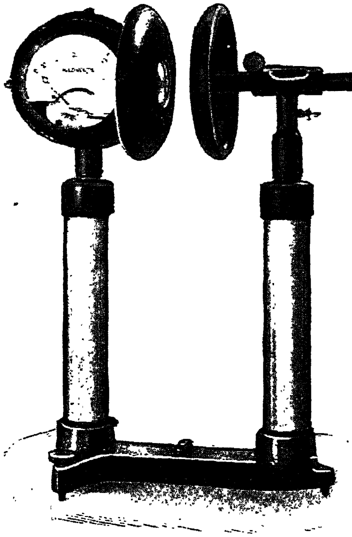
FIG. 278. SIEMENS AND HALSKE HIGH-TENSION VOLTMETER

Control of the instrument is by a spiral spring on the pointer spindle, and damping is by fluid friction of the high-tension plate in the oil. The cam on the pointer spindle gives a scale which is almost uniform over the upper 70 per cent of its range.

In the Abraham instrument—manufactured by Messrs. Everett-Edgcumbe—there are two hollow metal mushroom-shaped discs arranged as shown. The right-hand disc forms the high-tension plate, while the centre portion of the left-hand disc is cut away and encloses a small disc which is movable and is geared to the pointer of the instrument. The two mushroom-shaped discs are fixed in position except that the right-hand one may be set to different distances from the other, to alter the range of the instrument. A scale is provided, for this setting, on the right-hand support. The two large discs form adequate protection for the working parts of the instrument against external electrostatic disturbances. This instrument is made in six sizes covering the range of voltage 3 kV to 500 kV.

Most of the high-voltage electrostatic voltmeters which are absolute instruments depend upon a direct measurement of the forces brought into play by the voltage applied. But there is another class of absolute instrument for this purpose which operates through the indirect action of these forces, i.e. through their influence upon the period of oscillation of a conducting body. The best known example of such an instrument is the ellipsoid voltmeter (see Ref. (66)) but E. Bradshaw, S. A. Husain, N. Kesavamurthy and K. B. Menon (Ref. (87)) have discussed their general theory and have considered their scope and limitations.

Ionic Wind Voltmeter. When a highly-charged point is situated in air or other gas, a movement of the air surrounding the point is observed. This is referred to as the "electric wind," and is brought about by the repulsion of ions from the surface of the point by the intense electrostatic field. These ions, colliding with uncharged molecules of air in the neighbourhood of the point, carry the latter with them and set up the "electric wind." If an earthed electrode is placed near to the highly-charged one, an intense electric field exists, of course, between the two, and a similar wind is observed also at the earth electrode.



(Everett-Edgcumbe & Co., Ltd.)

FIG. 279. ABRAHAM-VILLARD VOLTMETER

This phenomenon has been investigated by W. M. Thornton and by M. Waters and W. G. Thompson, and has been put to a useful purpose in their Ionic Wind Voltmeter (Ref. (23)).

Fig. 280 shows the principle of the instrument. A hot wire, of platinum-gold alloy, included in one arm of a Wheatstone bridge network, is used as the earthed electrode of a high-tension gap. Before the high voltage is applied to the gap the bridge network is balanced—i.e. the galvanometer *G* is set to zero. When the voltage across the gap exceeds a certain value—called the "threshold voltage"—the electric wind cools the hot wire and hence reduces its resistance. A reduction of 25 per cent in resistance is obtainable under suitable conditions, and this is sufficient to cause an appreciable out-of-balance voltage in the bridge network, as indicated by the galvanometer deflection. The cooling, and hence the galvanometer deflection, depends upon the potential gradient (and hence upon the applied voltage), as well as upon the temperature of the electrode, the nature and pressure of the gas in which the electrostatic field exists, and upon the frequency.

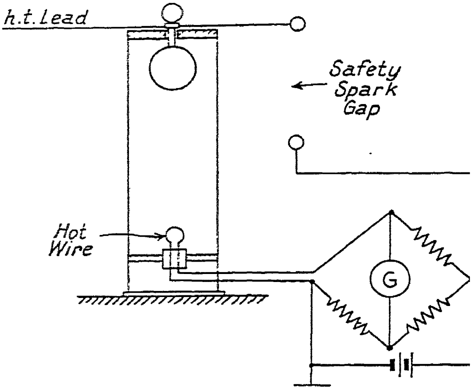


FIG. 280. IONIC WIND VOLTMETER

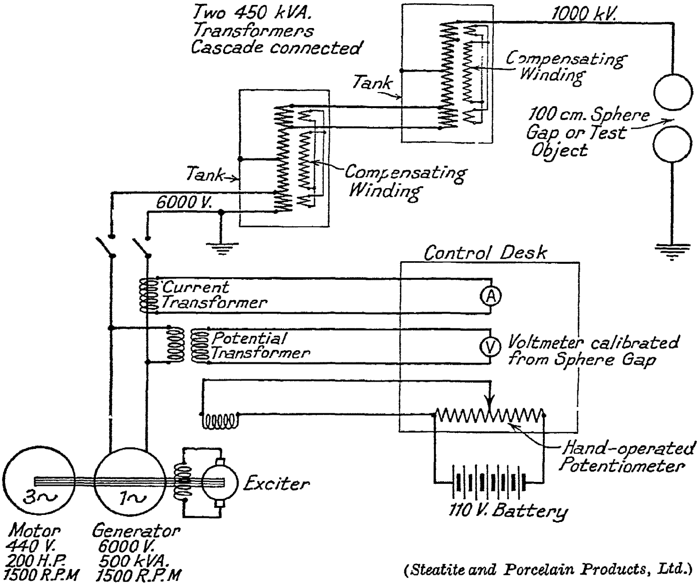
The "threshold voltage" is the voltage which is required before the potential gradient is sufficient for ionization to commence. When the applied voltage is alternating, therefore, only that portion of the voltage-wave which lies above the threshold value is effective from the point of view of cooling of the hot wire. The area of the effective portion of the voltage-wave—and hence the cooling—varies for different wave shapes, even though the waves have the same r.m.s. value. Voltage waveform, therefore, influences the instrument readings, but the authors have shown how this influence may be predetermined. Calibration of the instrument is on a sine waveform.

The voltmeter can be used to indicate either crest or r.m.s.

alternating voltages, or direct voltages. Several forms of the instrument have been constructed, both for out-door and laboratory use, and for voltages up to 300 kV.

The principle advantages are that the high voltage may be measured by an observer at some distance from the charged conductors, and the robust construction, and freedom from disturbance by weather and temperature conditions which make it suitable for outdoor use.

A description of the vacuum-enclosed electrostatic voltmeters recently developed by Metropolitan-Vickers Elec. Co., Ltd. is given in Chapter XVIII, p. 661.



(*Steatite and Porcelain Products, Ltd.*)

FIG. 281. LOW-FREQUENCY H.T. TESTING EQUIPMENT

Special Apparatus for Tests other than Low-frequency Alternating Current Tests. The apparatus described in the foregoing pages is, to a great extent, common to all methods of high-voltage testing. Except for a few special pieces of apparatus which may be required in individual cases, according to the purpose for which the equipment is to be used, no further apparatus is required for sustained low-frequency high-voltage tests. A complete diagram of connections for an equipment of this type is shown in Fig. 281.

Apparatus, in addition to that already described, is required however, for the other types of test—namely, constant direct current tests, high-frequency tests, and surge tests.

CONSTANT DIRECT CURRENT TESTS. As previously mentioned, high-voltage direct current is chiefly used for the testing of high-voltage cables after installation. For the generation of high-tension direct current a number of comparative low-voltage generators can be connected in series. A specially designed machine, called a "transverter" (Ref. (24)), can also be used to obtain d.c. from a three-phase a.c. supply. Both of these methods are, however, only of use when a large amount of power is to be dealt with, and cannot be adapted to a small power-testing set, which, for the purpose of cable testing, should be portable.

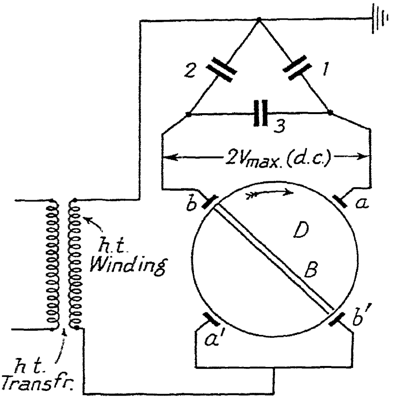


FIG. 282. DELON MECHANICAL RECTIFIER

The methods of obtaining the direct current now used is by the rectification of high-voltage alternating current, the latter being obtained from a high-voltage transformer.

The rectification may be done either by a mechanical rectifier, by a thermionic valve rectifier or by selenium or copper oxide rectifiers. Metal rectifiers are now in common use.

Delon Mechanical Rectifier. Fig. 282 illustrates the principle of this apparatus. An insulated disc D carries a conducting bar B, the ends of which come, in turn, opposite to one pair of the collecting brushes aa' and bb'. This disc is driven in synchronism with the alternating current supply, so that the rotating bar B is opposite to collecting brushes bb' (say) at the instant when the high voltage wave is at its positive maximum point, and opposite aa', when the voltage is negative maximum. Sparks pass between the collecting brushes and the connecting bar at these instants, the result being that the capacitor 2 is charged to a voltage of $+ E_{max}$ (above earth) and capacitor 1 to a voltage of $- E_{max}$ (below earth). Thus the potential across capacitor 3 is $2E_{max}$ and this voltage is *direct* not alternating. The high-tension d.c. is thus obtained from capacitor 3, and since the charging is taking place continuously, direct current can be supplied to a circuit under test.

In tests upon cables the internal capacitance between the cable cores, and from cores to ground, form the capacitors 1, 2, and 3 but the spark-over which takes place at the collecting brushes in the mechanical rectifier tends to set up surges and oscillations which, by weakening the cable dielectric, may cause breakdown at a lower indicated voltage than would otherwise be obtained.

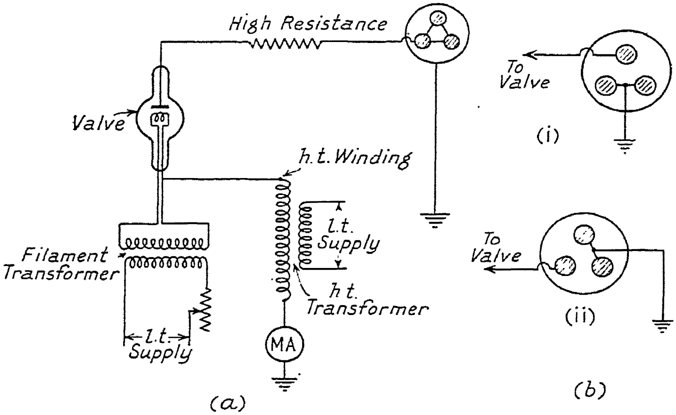


FIG. 283. CONNECTIONS FOR D.C. HIGH-VOLTAGE CABLE TESTS

For this reason the mechanical rectifier has been largely displaced, by thermionic valves or metal rectifiers.

Tests with Thermionic or Metal Rectifiers. Fig. 283 shows the connections for a high-voltage d.c. test upon a three-core cable—cores to ground.

Current passes through the valve, or rectifier which now commonly replaces it, only during one half-wave of the voltage cycle. During this time the capacitor formed by the cable cores and the sheath is charged to a potential equal to that of the maximum value, V_{max}, of the high-tension transformer secondary voltage. When this voltage reverses—during the next half-wave—the potential of this cable-capacitor remains the same, while the potential of the valve filament rises to a maximum potential of V_{max} in the opposite direction. Thus the valve must be able to withstand a voltage of $2V_{max}$ between its plate and filament. The filament-transformer windings must, also, be insulated for a voltage of V_{max} between windings. The milliammeter *MA* reads the charging and leakage current through the capacitor formed by the cable.

The voltage for the valve filaments is usually 8 volts, and care must be taken that the correct voltage is used, since departure from it, by a comparatively small percentage, causes a considerable fall in the output current of the valves if the filament voltage is low. Resistances of about $\frac{1}{4}$ megohm are connected in the plate circuits of these valves for protection against surge effects.

Tests are usually made upon three-core cables with connections as shown in Fig. 283(*b*) as well as with the connections of Fig. 283(*a*).

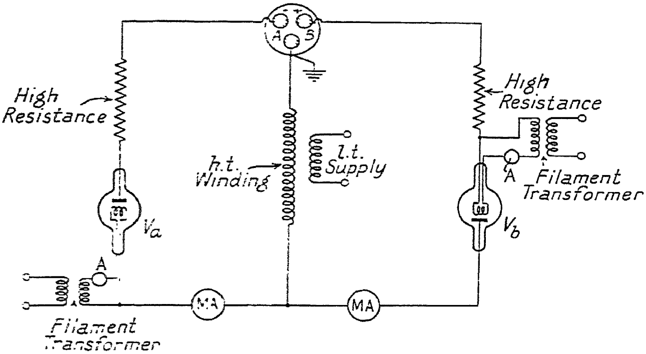


FIG. 284. CONNECTIONS FOR HIGH-VOLTAGE D.C. TEST UP TO 200 kV

Two valves are used for test voltages up to 200 kV. The connections for such a test are shown in Fig. 284, the test being one between cores *A* and *B* of a three-core cable. Valve V_a allows current to flow only in the direction plate to filament. Thus, during one half-cycle current flows through it in this direction, and core *A* of the cable is charged to a potential V_{max} below earth, V_{max} being the maximum value of the voltage wave of the h.t. transformer secondary winding. Meanwhile, valve V_b is allowing no current to flow through it. During the next half-cycle the process is reversed, current flowing through valve V_b and charging core *B* to a potential V_{max} above earth. Thus the d.c. potential between cores is $2V_{max}$, and from each core to earth V_{max}.

In the case of a single-core cable, if the same pressure ($2V_{max}$) is to be applied between core and sheath as between cores in the three-core cable, two auxiliary capacitors must be used, connected, as shown in Fig. 285.

In some portable high-voltage d.c. testing sets a single-phase low-voltage alternator, driven by a petrol engine, is used for the supply. The primary of the high-voltage transformer is supplied through an auto-transformer with a variable inductance choke-coil for fine adjustment. Voltage measurement is made by an electrostatic voltmeter of the Abraham Villard type, directly connected across the cable cores under test, although a sphere-gap is often included. In order to avoid trouble from surges, which may cause breakdown of apparatus, it is advisable to increase the testing pressure to its full value gradually, and also to discharge the cable through a high resistance. Owing to dielectric absorption it is necessary, also, to maintain the discharging connection for a considerable length of time, in order to avoid a subsequent dangerous rise of voltage.

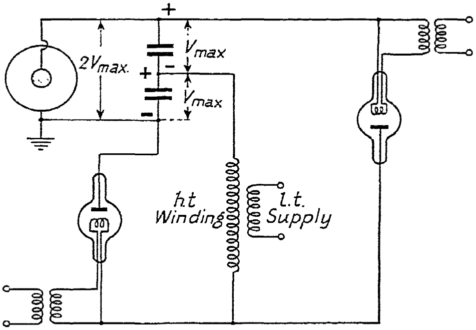


Fig. 285. Connections in the Case of a Single-core Cable

Other connections for high-voltage d.c. cable-testing are given by J. Urmston (Ref. (25)).

Equivalence of D.C. and A.C. Test Voltages. Owing to the "electric osmosis" effect, any moisture which may exist within the cable dielectric tends to move towards the negatively-charged electrode (the sheath or one of the cores, depending upon the test), when a d.c. voltage is applied. Although the amount of such moisture is usually small, it may yet be sufficient to cause breakdown due to its concentration near the negative electrode. If the applied voltage is alternating, no such movement of moisture occurs, the moisture remaining uniformly distributed. Again, breakdown may occur when testing cable-samples due to surges which are produced by spark-discharge and corona effects at the cable ends. These effects are more severe with a.c. than with d.c.

For these reasons it is obvious that there are other considerations beyond mere equivalence of potential gradient, which must determine to what alternating voltage a given direct voltage is equivalent from the point of view of insulation breakdown.

N. A. Allen, to whom the author is indebted for much of his information upon cable-testing with direct currents, gives (Ref. (24)) a table showing the ratios of d.c. to a.c. test voltage, quoted by various authorities as giving an equivalent breakdown-test upon cables and dielectrics. For paper-insulated cables, although a d.c./a.c. ratio of 2·5 has been used, Allen suggests that a ratio of 1·5 to 2·0 would be more satisfactory. Owing to the fact, also, that this ratio increases with increasing insulation thickness, he suggests the adoption of a ratio 1·5 for cables up to 33,000, and a ratio of about 2·0 for voltages above this. The following table—taken from the same paper—gives the d.c./a.c. ratios usually adopted by English cable makers.

TABLE

Working Voltage	Standard Test-voltage		Ratio D.C./A.C.	Average Thickness of Dielectric
	A.C.	D.C.		
11,000	20,000	30,000	1·50	0·30 in. (7·62 mm.)
22,000	44,000	75,000	1·71	0·40 in. (10·2 mm.)
33,000	66,000	100,000	1·52	0·50 in. (12·7 mm.)

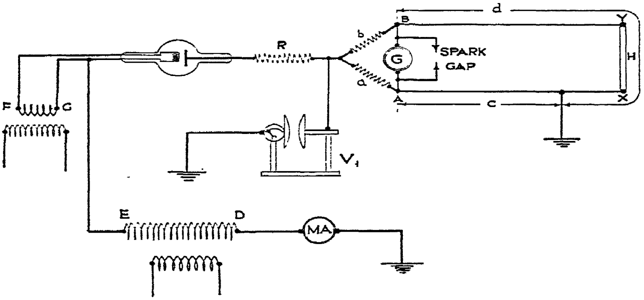
Localization of Faults in High-voltage Cables. Another application of the rectifying valve, is in the localization of faults in long lengths of high-voltage cable. Such faults are often of very high resistance and cannot easily be located, with accuracy, by the ordinary methods of fault localization given in the following chapter.

In high-voltage cables the large amount of oil present in the cables and joints causes faults in the cable (which may be obvious—owing to their low resistances—when the high testing voltage is applied) to seal up rapidly when this voltage is withdrawn, resulting in their having, again, a high resistance. One method of dealing with such a fault is to reduce its resistance by the continued application of a high testing voltage which "burns out" the fault, and enables location to be made by ordinary methods.

The method which has been largely adopted, as being more satisfactory than this, is to use a valve rectifier high-tension d.c. testing set for the supply of current for a "Murray loop" test. This and similar h.t. tests are described by Allen (Ref. (24)) and by Urmston (Ref. (25)).

Fig. 286 gives the connections for such a test, using a single-valve d.c. high-voltage testing set whereby a testing voltage of about 60 kV—which is usually sufficient for the purpose—can be obtained. A two-valve set may be used, if necessary, for higher testing voltages.

A loop is made by connecting together (by means of a short-circuiting strap) the two distant ends of a sound cable and the faulty one. A highly insulated slide-wire, having a sliding contact which can be operated by a long insulating handle, is connected across the near ends of the loop, and a galvanometer and spark-gap —the latter for protection of the galvanometer—are also connected across these ends as shown. The d.c. testing voltage is applied to an intermediate point on the slide-wire, and an Abraham voltmeter is used to indicate the applied voltage. A high resistance and milliammeter are connected in the supply circuit for the limitation and measurement, respectively, of the supply current—i.e. of the fault current.



(*Watson, London*)

FIG. 286. CONNECTIONS OF AN EQUIPMENT FOR LOCALIZATION OF FAULTS IN HIGH VOLTAGE CABLES

The procedure, as given by Urmston (Ref. (25)) is as follows: "Pressure is then applied, and when the breakdown occurs the fault current is limited to 3 or 4 mA., and a preliminary balance made. The current is then increased to 40 or 50 mA., and the balance readjusted."

The cable voltage is usually low when the fault is fully broken down, but the voltage may rise very suddenly if the fault clears, and this necessitates the precautions regarding insulation of the slide-wire in order to safeguard the operator. The spark-gap, which is set to about 0·01 in. by micrometer, is for the protection of the galvanometer and slide-wire when the fault resistance falls, at breakdown, resulting in the discharge of the cable. The inductance of the galvanometer and bridge circuit cause this discharge current to pass across the spark-gap rather than through them.

If a and b are the resistances of the slide-wire ratio arms at balance (see Fig. 286), and c and d are the resistances of the cable lengths to the fault, then

$$\frac{b}{a}=\frac{d}{c}$$

or
$$\frac{a+b}{a}=\frac{c+d}{c}$$
$$\therefore c=\frac{a(c+d)}{a+b}$$

Thus, assuming the resistance of the cable-loop, per unit length, to be uniform, and calling the total length of the loop L, we have—

$$\text{Distance of fault from point } A=\frac{aL}{a+b}$$

The resistance and length of the short-circuiting strap may be assumed negligible.

HIGH-FREQUENCY TESTS. It has been stated already that the behaviour of insulating materials at high frequencies is very different from that at ordinary commercial frequencies. This is largely due to the very much greater dielectric power loss, within the material at the high frequency. The heat produced by this power loss tends to produce breakdown of the insulation at voltages which are smaller than those at which breakdown occurs when the frequency is low. Such tests are useful, also, in the detection of lack of homogeneity in compound-filled porcelain insulators.

Two kinds of high-frequency high-voltage tests are carried out—

(*a*) Tests with apparatus which produces undamped high-frequency oscillations.

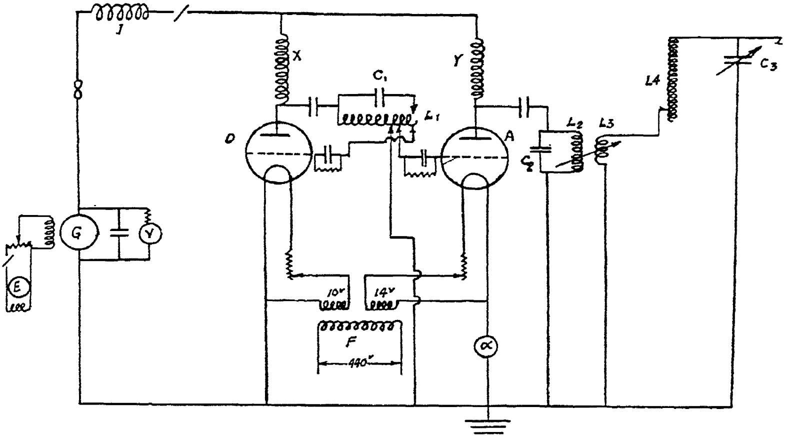
(*b*) Tests with apparatus producing damped high-frequency oscillations.

(*a*) *Undamped oscillations* do not occur in power systems, but are useful for insulation testing purposes, especially for insulation to be used in radio work.

High-frequency alternators have been used for frequencies up to 30,000 cycles per second, and high-frequency arc generators have also been used, but these have the disadvantage that smooth voltage variation is difficult.

Fig. 287 shows a valve circuit used by the Metropolitan-Vickers Electrical Co., Ltd., for such tests. Voltage variation is effected by variation of the coupling between the anode circuit of the main valve, and the secondary circuit, by means of the variometer, or by variation of the anode voltage of the valves. The voltage obtainable is 150 kV, and the frequency 100,000 cycles per second.

(*b*) *Damped high-frequency oscillations* are obtained by the use of a Tesla coil, together with a circuit containing a quenched spark-gap, as shown in Fig. 288. The Tesla coil constitutes the high-voltage transformer. It consists of two air-cored coils which are placed concentrically. The high-voltage secondary coil has a large number of turns, and is wound on a frame of insulating material, the insulation between turns being air, or in some cases, oil. If the Tesla coil is oil-immersed, the spacing between turns can be made smaller than when air is the insulation. The primary winding has only a few turns, wound on an insulating frame. The supply is usually 50 cycle a.c. to the primary of a high-voltage testing transformer, although a valve rectifier may be used on the secondary side of this transformer to give a d.c. supply to the primary side of the Tesla transformer. Two capacitors, C_p and C_s are connected in the primary and secondary circuits respectively, of the Tesla transformer. C_p is an air capacitor. C_s is usually made up of a sphere-gap, for voltage measurement purposes, the internal capacitance of the secondary winding of the Tesla coil, and the capacitance of the apparatus under test, the latter being usually small in comparison with the other two. The primary circuit of this transformer also contains a trigger spark-gap.



(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 287. HIGH-FREQUENCY TESTING EQUIPMENT FOR 150 kV

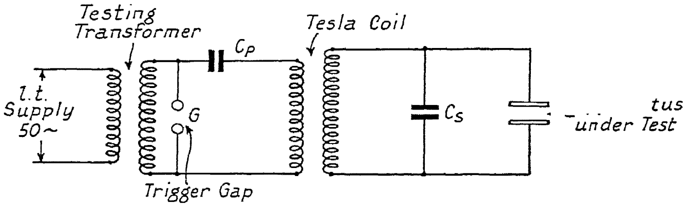


FIG. 288. CIRCUIT FOR HIGH-VOLTAGE TESTS WITH DAMPED HIGH-FREQUENCY OSCILLATIONS

Assuming the supply to the primary of the Tesla transformer to be alternating—as it usually is—the capacitor C_p is charged to some maximum voltage, which depends upon the voltage on the secondary side of the supply transformer, and upon the setting of the "trigger-gap." At this voltage value the trigger-gap breaks down, the capacitor C_p discharges, and a train of damped oscillations, of high frequency, is produced in the circuit containing C_p, the spark-gap, and the primary winding of the Tesla transformer. During the time taken for this train of oscillations to die away (this time being a very small fraction of a second), the spark-gap is conducting, due to the formation of an arc across it.

This charge and discharge of capacitor C_p takes place twice in one voltage cycle—i.e. in $\frac{1}{50}$ sec. for 50 cycle supply. Thus there will be 100 of these trains of damped oscillations per second.

The frequency of the oscillations themselves is very high—100,000 cycles per second being a usual value—its actual value depending upon the inductance and capacitance of the oscillatory circuit.

B. L. Goodlet (Ref. (26)) states, in connection with the test frequency to be adopted, that "to the author's knowledge the highest frequency oscillation from which any serious trouble has been experienced on any transmission system is 80,000 cycles." He also gives results of tests which show that the spark-over voltage of an insulator is very little higher at about 600,000 cycles than it is at 100,000 cycles, and suggests that 100,000 cycles per second is therefore the maximum required for testing purposes.

The frequency is given approximately by the expression

$$f = \frac{1}{2\pi\sqrt{L_pC}} \quad . \quad . \quad . \quad . \quad (235)$$

where L_p is the inductance of the primary circuit of the Tesla transformer.

Oscillations are induced in the secondary circuit of the Tesla transformer by oscillatory current in the primary, and these will be of the same frequency as those in the primary circuit if the secondary inductance and capacitance are adjusted so that the two circuits are in tune—i.e. if $L_pC_p = L_sC_s$—L_s and C_s being the inductance and capacitance of the secondary oscillatory circuit. In this way a series of trains of damped oscillations are applied to the apparatus under test, connected as shown.

To prevent the energy of the oscillatory discharge from surging backwards and forwards between the primary and secondary circuits of the Tesla transformer, the trigger spark-gap must be quenched by air-blast cooling. This is helped by using a rotating spark-gap.

A sphere-gap is used for voltage measurement, and if the waveform of the Tesla secondary voltage is required a cathode-ray oscillograph must be used. Accurate measurements of the high-frequency voltage are not always required, as such tests are often carried out by allowing high-frequency discharge to take place across the surface of an insulator for a certain time, and observing its effect upon the insulator. The frequency of the discharge is obtained from the constants of the oscillatory circuits.

Voltage and Frequency Values in the Oscillatory Circuits. The voltage relationships in the primary and secondary circuits of the Tesla transformer can be determined approximately as follows (Ref. (26))—

Let V_p = maximum voltage to which the primary capacitor C_p is charged.
„ V_s = maximum voltage to which the secondary capacitor C_s is charged.

Then,

$$\text{Energy in primary capacitor at breakdown of trigger-gap} = \frac{1}{2} C_p V_p^2$$

This energy is passed on to the secondary circuit, with some loss due to resistance and dielectric losses, and to the fact that the electromagnetic coupling between the two circuits is not perfect.

Energy given to secondary circuit $= \frac{1}{2} C_s V_s^2$. If ε is the efficiency of the energy transfer, then

$$\frac{1}{2} C_s V_s^2 = \varepsilon \cdot \frac{1}{2} \cdot C_p V_p^2$$

or

$$\frac{V_s^2}{V_p^2} = \varepsilon \frac{C_p}{C_s}$$

Thus,

$$\frac{V_s}{V_p} = \sqrt{\varepsilon \cdot \frac{C_p}{C_s}}$$

To determine the oscillation frequencies in the two circuits, let M be the mutual inductance between the two windings of the Tesla transformer. Let I_p and I_s be the currents in the primary and secondary oscillatory circuits, respectively. Then neglecting the resistances of the two circuits, since these are usually small compared with the other impedances in the circuits, we have—

For the primary circuit,

$$I_p \left(j\omega L_p - \frac{j}{\omega C_p} \right) - j\omega M I_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

(ω being $2\pi \times$ frequency).

In the secondary circuit,

$$I_s \left(j\omega L_s - \frac{j}{\omega C_s} \right) - j\omega M I_p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (237)$$

Then

$$\frac{I_p}{I_s} = \frac{j\omega M}{j\omega L_p - \dfrac{j}{\omega C_p}} = \frac{j\omega L_s - \dfrac{j}{\omega C_s}}{j\omega M}$$

Hence,

$$\omega^2 M^2 - \omega^2 L_p L_s - \frac{1}{\omega^2 C_p C_s} + \frac{L_s}{C_p} + \frac{L_p}{C_s} = 0$$

or

$$\omega^2 (M^2 - L_p L_s) + \frac{L_s}{C_p} + \frac{L_p}{C_s} - \frac{1}{\omega^2 C_p C_s} = 0$$

Let $\dfrac{M^2}{L_p L_s} = K^2$ or $K = \dfrac{M}{\sqrt{L_p L_s}}$

(K is the "coefficient of coupling" of the circuits.)

Then $\omega^2 (K^2L_pL_s - L_pL_s) + \frac{L_s}{C_p} + \frac{L_p}{C_s} - \frac{1}{\omega^2C_pC_s} = 0$

or $$\omega^2(1-K^2) - \frac{1}{L_pC_p} - \frac{1}{L_sC_s} + \frac{1}{\omega^2L_pL_sC_pC_s} = 0$$

Factorizing, we have,

$$\left[\omega(1-K) - \frac{1}{\omega L_pC_p}\right]\left[\omega(1+K) - \frac{1}{\omega L_sC_s}\right] = 0$$

since $L_pC_p = L_sC_s$ when the two circuits are tuned.

Thus, $\omega^2 = \frac{1}{L_pC_p(1-K)}$

or $f_1 = \frac{1}{2\pi\sqrt{L_pC_p(1-K)}}$ (f_1 being one value of the frequency)

or $\omega^2 = \frac{1}{L_sC_s(1+K)}$

or $f_2 = \frac{1}{2\pi\sqrt{L_sC_s(1+K)}}$ (f_2 being another value of the frequency)

Now, K is usually small compared with L_pC_p and L_sC_s, so that as previously stated, the frequency is given, approximately, by

$$f = \frac{1}{2\pi\sqrt{L_pC_p}} = \frac{1}{2\pi\sqrt{L_sC_s}} \qquad . \quad . \quad . \quad (238)$$

With regard to the closeness of the coupling between the two circuits of the Tesla transformer, Goodlet (*loc. cit.*) points out that if this is very close an impulse effect will be obtained rather than an oscillatory one, and, if the coupling is very loose, the conditions existing with undamped oscillations will be obtained.

SURGE OR IMPULSE TESTS. In surge tests it is required to apply to the circuit or apparatus under test, a high direct voltage whose value rises from zero to maximum in a very short time and dies away again comparatively slowly—i.e. a voltage having a very steep wave-front and a flat tail is to be applied. This is done by means of a circuit due to Prof. E. Marx (Refs. (88), (89), and (90)) and called the Marx circuit after him. To illustrate the principles of the method the connections of a multi-stage surge generator are given in Fig. 289. Many modifications of the circuit have been used.

C_1, C_2, C_3, and C_4 are capacitors which are connected in parallel with high resistances, R_1, R_2, etc., between them. S_1, S_2, and S_3 are trigger spark-gaps.

The capacitors are charged, in parallel, from a high-tension transformer, through a rectifying valve. At a certain voltage, depending upon their setting, the trigger-gaps break down and connect the capacitors in series instantaneously. Thus the voltage between the point T and earth, at this instant, is the sum of the voltages to which the capacitors were charged. The discharge takes place in so short a time that the energy loss in the resistors is negligible. Thus, the whole of the energy which is stored in the capacitors during charging, is suddenly discharged through the test circuit when the trigger-gaps break down. The gaps are set so that S_1 breaks down at a slightly lower voltage than S_2, S_2 at a slightly lower voltage than S_3, and so on. Then S_1 breaks down first and S_2 and S_3 follow instantly.

Although only three gaps are shown in the figure, the number of gaps and capacitors may be increased to give any desired multiple of the charging voltage and it has been found feasible in practice to operate a 50-stage impulse generator. The number which can be used successfully is limited to some extent, however, by the fact that the high resistance between the supply and the distant capacitors, when a large number are used, may prevent them from receiving a full charge. A large number of gaps in series also reduces the impulse voltage obtainable.

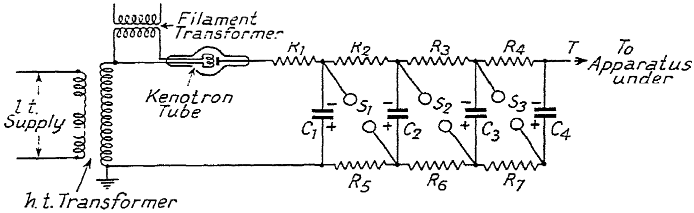


FIG. 289. CONNECTIONS OF MULTI-STAGE SURGE GENERATOR

F. S. Edwards, A. S. Husbands and F. R. Perry (Ref. (87)) have reviewed subsequent developments of the Marx circuit and described the construction of impulse generators as well as their practical application. In this paper the simplified circuit shown in Fig. 290 (*a*) is given for a single-stage impulse generator. C_1 = discharge capacitance of the generator, C_2 = capacitance of the load, L_1 = internal inductance of generator, L_2 = external inductance of load and connections, R_1 or R'_1 = resistance for control of wave-tail (there are two alternative positions for this resistance), R_2 = resistance for control of wave-front. These authors state, however, that if the tail of the wave is long compared with its front—as it is for a standard wave—very little error is caused by ignoring the wave-tail resistance during the calculation of the wave-front duration. The circuit can then be further simplified to that of Fig. 290 (*b*).

The critical resistance of the circuit (see Chapter XVI) is

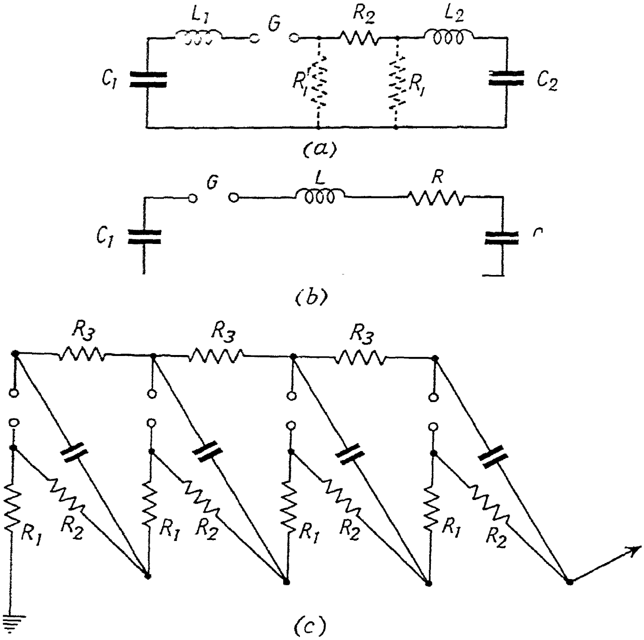
$$R = \sqrt{\frac{4L}{C}}, \text{ where } \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

The voltage across the load C_2, for a charging voltage E across C_1, is then given by

$$V = \frac{CE}{C_2}\left[1 - \left(1 + \frac{2t}{CR}\right)\varepsilon^{-\frac{2t}{CR}}\right] \quad . \quad . \quad (239)$$

or, if L is reduced to zero, R remaining the same,

$$V = \frac{CE}{C_2}\left(1 - \varepsilon^{-\frac{t}{CR}}\right) \quad . \quad . \quad . \quad . \quad (240)$$



(a)

(b)

(c)

FIG. 290

V approaches $\frac{CE}{C_2}$ as t approaches infinity.

The "nominal wave-front" defined in B.S. 923 : 1940 is equal to $2{\cdot}75CR$ when L is zero, and to $2{\cdot}1CR$ when the circuit is critically damped. In calculating the resistances the value of R_2 needed to make the circuit of Fig. 290 (*a*) non-oscillatory with R_1 absent is obtained. Then the value of R_2 (if $> \sqrt{\frac{4L}{C}}$) required to give the desired wave-front is computed, and finally the value of R_1 to give the required wave-tail when R_1 is on the generator side of R_2.

The efficiency of the generator can then be approximately estimated as $\frac{C_1}{+ C}$ multiplied by a factor of about 0·95 for a 1/50-microsec. wave.

Wave-front-control resistances can be connected either outside or within the generator, or partly in and partly outside. The best arrangement is probably to have about half of the resistance outside the generator. Wave-tail-control resistors can be used as the charging resistors within the generator circuit.

Fig. 290 (*c*) shows the connections of a widely used multi-stage circuit which combines high efficiency with distributed wave-front resistors. The value R_3 is made large compared with R_1 but R_2 is as small as is necessary to give the required length of wave-tail. Since the current through R_2 does not flow through R_1 it does not reduce the initial generator-output voltage, however small the value of R_2 or however large R_1 may be.

A sphere-gap is used for voltage measurement, and measures the peak value of the impulse wave. Owing to the fact that the sphere-gap and apparatus under test are in parallel, care must be taken in using the former for voltage measurement. The conditions of its use for this purpose are specified in B.S. 368 : 1939. This states that the accuracy of measurement is within ± 3 per cent provided that the duration of the impulse voltage is not less than that of a 1/5-microsec. wave.

A single-beam cathode-ray oscillograph is used, with a simple capacitance-type potential divider, to obtain either a visual or photographic record of the impulse wave shape. A. K. Nuttall (Ref. (91)) has described the construction of an oscillograph for the measurement of moderately high impulse voltages directly.

Impulse Ratio. The "impulse ratio" of an insulator is the ratio

$$\frac{\text{Minimum spark-over voltage when tested with an impulse voltage}}{\text{Spark-over voltage when tested at power frequency}}$$

This is not constant for any given insulator, but is always greater than unity. It depends upon—

(*a*) The polarity of the impulse voltage. The spark-over voltage is usually less, by some 10 per cent, when the insulator pin (in the case of a porcelain insulator) is positive, than when it is negative.

(*b*) The steepness of the impulse wave-front and the time of decay of the impulse voltage. The highest spark-over voltage is obtained with an impulse voltage which rises most rapidly to its crest value and also falls away again rapidly. Goodlet (*loc. cit.*) gives the figures 1·3 to 1·5 for the mean impulse ratio in the case of pin-type porcelain insulators, and 1·2 to 1·4 for suspension insulators.

Notes on the Testing of Insulators, Insulating Materials, and Cable Lengths. While impulse and high-frequency tests are carried out, as above described, for research purposes, and by manufacturers, in order to ensure that their finished products will give satisfactory performance in service, the most general tests upon insulating materials are carried out at power frequencies. Such tests may be carried out in accordance with the purchaser's specifications, and their exact nature then depends upon individual requirements. So many factors, such as barometric pressure, temperature, time of application of the testing voltage, and so on, influence the results of these tests that the British Standards Institution, and similar authorities in other countries, have drawn up standard specifications which state standard test conditions for various types of manufactured apparatus and materials.

Testing of Porcelain Insulators. Such insulators are designed so that spark-over occurs at a lower voltage than puncture, thus safeguarding the insulator, in service, against destruction in the case of line disturbances. Flash-over tests are thus very important in this case. Flash-over, or surface breakdown, is really due to a breakdown of the air at the insulator surface, and the voltage at which it occurs for a given insulator depends upon—

(*a*) The barometric pressure.
(*b*) The temperature.
(*c*) The shape of the electrostatic field.
(*d*) The humidity.
(*e*) The nature of the contact between the insulator and the electrodes.

Of these factors, the first two can be taken into account by assuming that the flash-over voltage is directly proportional to the "air density correction factor" $\left(\frac{0{\cdot}386p}{273 + t}\right.$, p being the pressure in millimetres of mercury and t the temperature in degrees Centigrade), provided this is not very different from unity.

If the electrostatic field has no component in a direction normal to the surface of the insulator, the breakdown voltage is simply that for the air alone, and is practically independent of the material of the insulator. Usually, however, the electrostatic field lies partly in the surrounding air and partly in the insulator, and the shape of this field becomes very important. It is thus necessary to take into account the disposition of any metal parts, used for mounting purposes, when considering results.

The flash-over voltage decreases with increasing humidity and correction factors are given in B.S. 137 : 1941. Dirt of any kind upon the surface has a similar effect to that of deposited moisture in lowering the flash-over voltage, and thus for consistent results upon a given insulator it should be both clean and dry.*

Unless the contacts between the electrodes and insulator are good, ionization of the air near the contacts will take place, and the breakdown voltage will be reduced.

The time of application of the voltage may also affect the results.

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* See also Ref (84).

If the applied voltage is high, and is maintained for some time at a value a little below the normal flash-over voltage, breakdown often occurs.

In addition to the dry flash-over test, rain tests, and tests in a misty or smoke-laden atmosphere are often carried out, the rain test being specified in British Standard Specification No. 137 : 1941, for "Porcelain and Toughened Glass Insulators for Overhead Power Lines" (3·3 kV and upwards).

The specification gives a table of standard insulator rating numbers and test voltages, the latter being separated into "Minimum dry flash-over voltage," "Minimum wet flash-over voltage," "One-minute dry test voltage," "One-minute rain test voltage" and "Puncture test voltages."

The various tests are then specified and the following notes give an indication of their nature (the specification should be referred to for full details).

POWER FREQUENCY TESTING VOLTAGE. This shall be of frequency between 25 and 100 c/s and approximately sine-wave form. The peak value, determined by sphere gap, oscillograph or other approved method, shall not exceed 1·45 × r.m.s. value.

FLASH-OVER TESTS. These are (i) 50 per cent dry impulse flash-over test, using an impulse generator delivering a positive 1/50-microsecond impulse wave. "The voltage shall then be increased to the 50 per cent impulse flash-over voltage, i.e. the voltage at which approximately half of the impulses applied cause flash-over of the insulator." (ii) Dry flash-over and dry one-minute test in which the test voltage given in the table (mentioned above) is applied. "The voltage shall be raised gradually to this value in approximately ten seconds and shall be maintained for one minute. The voltage shall then be increased gradually until flash-over occurs." (iii) Wet flash-over and one-minute rain test in which the insulator is "sprayed throughout the test with artificial rain drawn from a source of supply at a temperature within 10° C. of the ambient temperature in the neighbourhood of the insulator." The resistivity of the water is to be between 9,000 and 11,000 ohm cm.

SAMPLE TESTS. Specifications are given for temperature-cycle tests, mechanical tests, electro-mechanical tests, puncture tests and porosity tests.

ROUTINE TESTS. These are to be applied to all insulators and "shall be commenced at a low voltage and shall be increased rapidly until a flash-over occurs every few seconds. The voltage shall be maintained at this value for a minimum of five minutes or, if failures occur, for five minutes after the last punctured piece has been removed. At the conclusion of the test the voltage shall be reduced to about one-third of the test voltage before switching off."

POTENTIAL DISTRIBUTION ALONG A SUSPENSION INSULATOR STRING. Owing to the earth capacitances between the metal fittings (caps and pins) of an insulator string and the supporting tower or pole, the potential distribution across the various component units of the string is by no means uniform. The potentials across the units adjacent to the line conductor are much greater than those across units nearer to the supporting cross-arm. If all the units are identical, the consequence is that the "string efficiency," i.e. the ratio $\frac{\text{actual breakdown voltage of the complete string}}{\text{breakdown voltage per unit} \times \text{number of units}}$ is low.*

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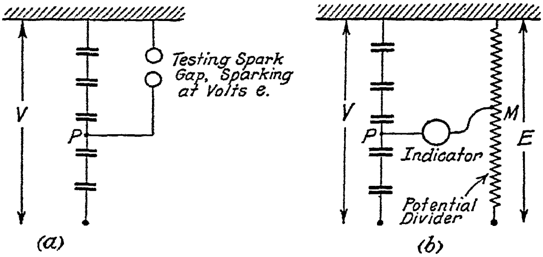
* For a fuller discussion of the question, see Ref. (78); also F. W. Peek, *Trans. A.I.E.E.*, Vol. XXXI, p. 907; or H. Cotton, *The Transmission and Distribution of Electrical Energy.*

Measurement of the potential distribution along such a string may be carried out in the laboratory by either (*a*) direct measurement or (*b*) a potential divider or null method. These methods may be understood from the simplified diagrams of Fig. 291, in which (*a*) shows the direct method and (*b*) the potential-divider method.

In (*a*) a small test spark-gap is employed, set to spark over at a voltage e across it. The potential V across the string is brought up to such a value as to cause the gap to spark. The connection P is moved from one unit to another down the string, and for each position the required value of V for gap spark-over is determined.



G. 291

The results give the percentage voltage drops e/V for the various portions of the string.*

Although this method is very simple and involves only a capacitance current, which does not cause pitting of the spheres, it is rather rough, and the presence of the exploring wire, unless this is correctly placed, may influence the electrostatic field distribution. Again, the capacitance of the gap itself may be comparable to that of the insulator units. In Method (*b*) the contact M is moved until the indicator shows zero, when the potentials of M and P are obviously the same, so that that of P is known, for a given applied voltage V, from the calibration of the potential divider.

The potential divider may be a resistance or capacitor, or it may be replaced by a separate source of supply. This source, either from an induction regulator† or a second alternator driven in synchronism with that supplying the insulator string, must be in the same phase and of the same waveform as the supply volts V.

The indicator‡ may be: (i) a spark-gap; (ii) a neon tube of special construction; (iii) an electrometer across a high resistance;

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* See R. H. Marvin, *Trans. A.I.E.E.*, January and June, 1916.

† Drewnowsky, *Archiv. für Elektrotechnik*, Vol. XXVII, p. 229.

‡ See also A. Schwaiger, "Voltage Distribution for Insulator Suspension Chains," *E. and M.*, No. 50, 1919; and "Theory of the High-voltage Insulator," *E.T.Z.*, No. 43, 1920; and *E. and M.*, No. 38, 1920; also Schering and Raske, *E.T.Z.*, Vol. LVI, p. 75, 1935.

(iv) a vibration galvanometer in series with a high resistance; or
(v) a triode.

The use of an electrometer for the measurement of high alternating voltages has been described by P. W. Baguley and H. Cotton.* They used a Lindemann electrometer, which is a very useful instrument for the measurement of very small currents. The instrument, described by F. A. Lindemann and T. C. Keeley,† is manufactured by the Cambridge Instrument Co.

TESTING OF INSULATING MATERIALS. In the case of such materials it is not the voltage which produces spark-over breakdown which is important, but rather the voltage for puncture of a given thickness (i.e. the dielectric strength). The measurements made upon insulating materials are usually, therefore, those of dielectric strength and of dielectric loss and power factor, the latter being intimately connected with the dielectric strength of the material.

The breakdown of solid insulating materials is a complex process, and does not occur simply when the applied potential gradient, or field strength, exceeds a certain critical value, regardless of other conditions.

It is found that the dielectric strength of a given material depends, apart from the chemical and physical properties of the material itself, upon many factors including—

(*a*) The thickness of the sample tested.

(*b*) The shape of the sample.

(*c*) The previous electric and thermal treatment of the sample.

(*d*) The shape, size, material, and arrangement of the electrodes.

(*e*) The nature of the contact which the electrodes make with the sample.

(*f*) The waveform and frequency of the applied voltage (if alternating).

(*g*) The rate of application of the testing voltage and the time during which it is maintained at a constant value.

(*h*) The temperature and humidity when the test is carried out.

(*i*) The moisture content of the sample.

It is obviously very necessary, therefore, that tests shall be carried out under standard conditions, and with standard sizes and shapes of both sample and electrodes if the results are to have any real significance.

The Nature of Dielectric Breakdown. The theory of breakdown which was most generally adopted in the past was the *Thermal* theory. K. W. Wagner first attempted to give a definite mathematical theory of thermal breakdown, and Prof. Miles Walker, during a discussion on E. H. Rayner's paper on "High-voltage Tests and Energy Losses in Insulating Materials" (*Jour. I.E.E.*, Vol. XLIX p. 3) stated the essentials of the theory, which are as follows—

Dielectric losses occur in insulating materials, when an electrostatic field is applied to them. These losses result in the formation of heat within the material. Most insulating materials are bad thermal conductors, so that, even though the heat so produced is small, it is not rapidly carried away by the material. Now, the conductivity of such materials increases considerably with increase of temperature, and the dielectric losses, therefore, rise and produce more heat, the temperature thus building up from the small initial temperature rise. If the rate of increase of heat dissipated, with rise of temperature, is greater than the rate of increase of dielectric loss with temperature rise, a stable condition (thermal balance) will be reached. If, however, the latter rate of increase is greater than the former, the insulation will break down owing to the excessive heat production, which burns the material.

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* *World Power*, Vol. XIX, No. CXI, March, 1933.

† *Phil. Mag.*, 1924, Vol. XLVII, p. 577.

Now, the dielectric losses per cubic centimetre in a given material and at a given temperature, are directly proportional to the frequency of the electric field and to the square of the field strength. Hence the decrease in breakdown voltage with increasing time of application and increasing temperature, and also the dependence of this voltage upon the shape, size, and material of the electrodes and upon the form of the electric field.

Much research has been done, particularly by the British Electrical and Allied Industries Research Association, during recent years on this question of the dielectric breakdown of insulating materials. S. Whitehead (Ref. (93)) has reviewed the subject comprehensively and this work together with those mentioned in Refs. (2), (94), and (95), should be consulted by readers wishing to study the subject. H. F. Church and C. G. Garton (Ref. (95)) list five well-established causes of failure in solid insulation and state that two others are now under investigation.

Importance of Dielectric Loss Measurements. From the above it will be realized that the measurements of dielectric loss in insulating materials are very important, and give a fair indication as to comparative dielectric strengths of such materials. In the case of cables, dielectric loss measurements are now generally recognized as the most reliable guide to the quality and condition of the cable.

The measurement of dielectric losses was dealt with in Chapter IV.

Directions for the Testing of Solid Specimens. A number of reports issued by the British Electrical and Allied Industries Research Association* give directions for the study of insulating materials of all classes, including solid dielectrics, papers, fabrics, varnishes, and oils.

The following extracts from the B.S.I. Standard Specification (No. 234 (1942)) for "Ebonite for Electrical Purposes," will serve to indicate what precautions are to be taken and what methods adopted in the testing of ebonite, which may be taken as a typical solid dielectric.

"ELECTRIC STRENGTH. The ebonite shall withstand for one minute without breakdown the following r.m.s. test-voltages when applied in accordance with the method described in Appendix D—

For Grade I—2,000 volts per mil (80 kV per mm).
For Grade II—1,000 volts per mil (40 kV per mm).
For Grade III—750 volts per mil (30 kV per mm)."

Appendix D gives the following method—

"A sheet or disc of the material, not less than 4 in. (101·6 mm) in diameter, shall be taken and recessed on both sides so as to accommodate the spherical electrodes described below, with a wall or partition of the material between them 20 mils (0·508 mm) thick, as shown in Fig. 4.

"The test shall be carried out at a temperature of from 15° C. to 25° C.

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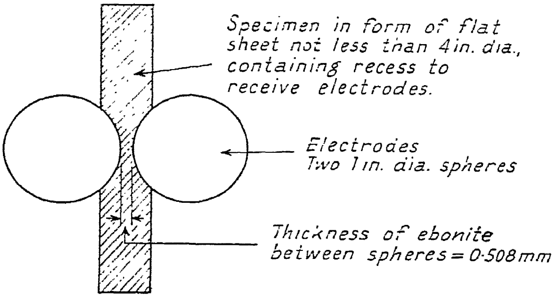
* See references at end of chapter.

"The electrical stress shall be applied to the specimen by means of two 1 in. diameter spheres fitting into the recesses without leaving any clearance, especially at the centre."

"Fig. 4," referred to in the extract, is reproduced in Fig. 292. It is further specified that the specimen shall be conditioned, before testing, by being "maintained in an ordinary room atmosphere for at least 24 hours immediately before test." The applied voltage is to be of approximately 50 cycles frequency and of sinusoidal wave-form. This voltage must be commenced at about one third the full value and increased rapidly to the full testing voltage.

It is laid down also that the power-loss factor (defined as the "product of the power factor and the permittivity which shall be determined at a frequency from 800 to 1,600 cycles per second by the method described in Appendix E") shall not exceed


Specimen in form of flat sheet not less than 4in. dia., containing recess to receive electrodes.
Electrodes Two 1in. dia. spheres
Thickness of ebonite between spheres = 0·508mm

(*By courtesy of the B.S.I.*)

FIG. 292. ARRANGEMENT OF ELECTRODES FOR THE TESTING OF EBONITE

0·018 for Grade I

0·030 for Grade II

0·080 for Grade III

(Appendix E describes a Schering bridge method the operation of which is specified more fully in B.S. 903 : 1950.)

THE TESTING OF INSULATING OILS. In British Standard specification (No. 148 : 1951) for "Insulating Oil," the method of applying the testing voltage (which must be alternating, or approximately sine wave-form, of frequency between 25 and 100 cycles per second and with a peak factor of $\sqrt{2} \pm 5$ per cent) which is recommended, is shown in Fig. 293, taken from the specification.

It is specified that—

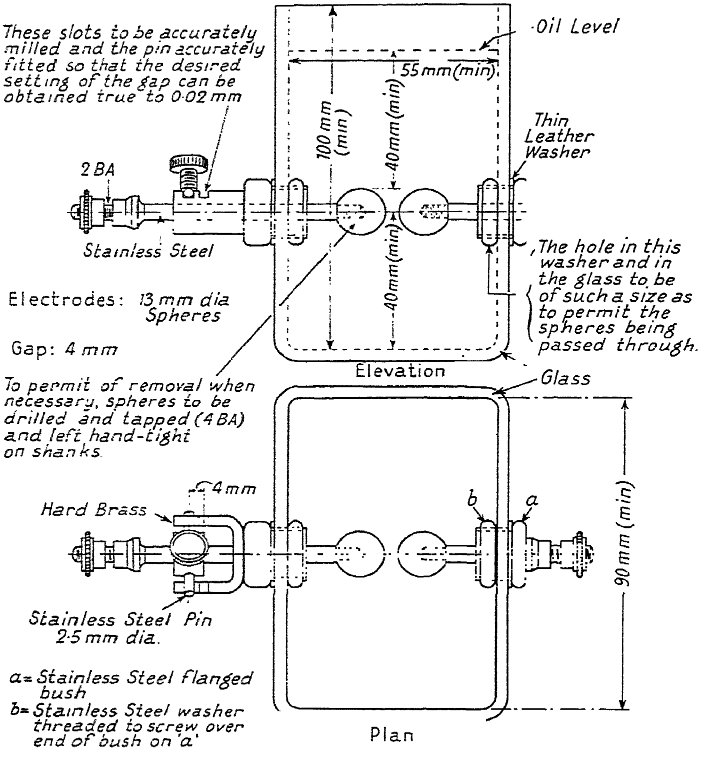
"The minimum internal dimensions of the test-cell shall be 55 mm. × 90 mm. × 100 mm. high.

"The electrodes shall be polished spheres of 12·7–13 mm. diameter, preferably of brass, arranged horizontally with their axis not less than 40 mm. above the bottom of the cell. For the test, the distance between the spheres shall be 4 ± 0·02 mm. A suitable gauge shall be used for checking the gap.

"Spheres shall be replaced as soon as pitting caused by discharges is observed."

On account of the serious effect of suspended solids and moisture in the oil, upon the electric strength, great care must be taken in preparing the oil sample for test and in cleaning the test-cell. These matters are dealt with in the specification.

Cable Testing. The cable tests which have already been described in this chapter have been tests upon cables already installed.


These slots to be accurately milled and the pin accurately fitted so that the desired setting of the gap can be obtained true to 0·02 mm
Oil Level
55 mm (min)
100 mm (min)
40 mm (min)
40 mm (min)
Thin Leather Washer
2 BA
Stainless Steel
The hole in this washer and in the glass to be of such a size as to permit the spheres being passed through.
Electrodes: 13 mm dia Spheres
Gap: 4 mm
Elevation
Glass
To permit of removal when necessary, spheres to be drilled and tapped (4 BA) and left hand-tight on shanks.
4 mm
Hard Brass
b
a
90 mm (min)
Stainless Steel Pin 2·5 mm dia.
a= Stainless Steel flanged bush
b= Stainless Steel washer threaded to screw over end of bush on 'a'
Plan

(*By courtesy of the B.S.I.*)

Fig. 293. Apparatus for the Measurement of the Dielectric Strength of Oil

Acceptance tests, which are tests called for by purchasers before accepting cables from the manufacturers, are also of importance. The results of such tests must be a reliable guide as to the probability of the cables being satisfactory in service.

At one time breakdown tests upon short lengths of cable with alternating voltages of commercial frequency, rapidly applied, were called for, but it was later realized that such tests gave little or no reliable information regarding the cable, since the breakdown value, when the voltage is applied for a considerable time, is usually considerably smaller than that obtained with rapid application of voltage. Thus, tests with a voltage applied for 15 or 30 min. were carried out in addition to the rapid tests. As a development of these tests, time-voltage curves for short lengths of cable are now carried out by some manufacturers (Ref. (29), (30)). These are obtained by first determining the breakdown voltage for one length with rapid application of voltage, and then, with other lengths, finding the length of time required before breakdown occurs, with applied voltages of gradually decreasing magnitudes for each test length. For example, if 100 kV produces breakdown of sample 1, when rapidly applied, and it is found that it takes 5 min. for 90 kV to produce breakdown of sample 2, 12 min. for 85 kV to produce breakdown of sample 3, and so on, a time-voltage curve for the cable can be plotted. This method is suggested by Dunsheath (Ref. (6)). E. A. Beavis (Ref. (30)) has investigated the question of the preparation of the ends of the cable lengths for such tests, in order to avoid breakdown by flash-over instead of by puncture.

Dielectric loss and power-factor tests are regarded as giving the most reliable information as to the quality of the cable, and such tests are of greater importance than breakdown tests upon short lengths.

It is found that the dielectric losses in high-voltage cables do not increase with the square of the voltage, as is the case with most simple dielectrics, nor does the power factor remain constant. The losses increase in proportion to some power of the voltage greater than the square, and the power factor rises with voltage, these effects being due to "ionization" caused by air which is entrapped within the cable insulation. Measurements of power factor with different voltages applied to the cable are therefore made, and the power factor–voltage curve plotted. This should be practically flat if the cable under test is such as will be satisfactory in service. The variation of power factor with temperature is also of importance. In a good cable the power factor should increase very little with increase of temperature. To investigate this power factor, measurements are made at different dielectric temperatures, obtained by heating the cable by the passage of current through its cores. The power factor–voltage curve is also obtained when the cable has cooled after a heat run. This should not be appreciably different from the curve obtained before the heat run.

LIVE CONDUCTOR DETECTION. While this scarcely comes under the heading of a high-voltage measurement it is perhaps worthy of mention here.

It is very important to be able to determine whether a high voltage conductor is live or dead prior to earthing it and a difficulty arises from the fact that, if the detecting device used is defective, it may indicate that the conductor under test is dead when, in fact it is not.

The requirements for a satisfactory potential indicator have been discussed by G. F. Shotter and E. E. Hutchings (Ref. (97)). They are—

"(i) To indicate reliably if a conductor is not at earth potential.

"(ii) To discriminate whether the voltage on a conductor is due to conductive connection with a source of supply, or to electrostatic charge, or electrostatic or inductive coupling with a live circuit. This involves a quantitative indication.

"(iii) To be of a form suitable for use by a linesman either indoors or outside under all weather conditions.

"(iv) To be such that it can be tested before use to ensure that it is operating satisfactorily."

This report made recommendations for the construction of a potential indicator of the resistance type and such an instrument is now made by Everett Edgcumbe and Co., Ltd. It has a series resistor comprising a number of ceramic resistance units in a tube of Bakelized material and a robust moving-coil indicator having high sensitivity. A permanently attached flexible lead is used for earthing. The indicator reads direct in kilovolts to earth and has a scale which is closed up at higher readings. This increases the effective range and enables the detector to be checked, before use, on a medium-voltage circuit.

BIBLIOGRAPHY AND REFERENCES

(1) *Electrical Measurements*, F. A. Laws.

(2) *Dielectric Phenomena*, S. Whitehead.

(3) *Electrical Power Transmission and Interconnection*," C. Dannatt and J. W. Dalgleish.

(4) *Hochspannung Technik*, A. Roth.

(5) *Insulated Cables*, C. J. Beaver.

(6) *High-voltage Cables*, P. Dunsheath.

(7) *Electric Power Transmission*, H. Waddicor.

(8) "Surges and Over-voltage Phenomena on Transmission Lines, due to Lightning," H. Norinder, *Jour. I.E.E.*, Vol. LXVIII, p. 525.

(9) "Lightning," G. C. Simpson. *Jour. I.E.E.*, Vol. LXVII, p. 1269.

(10) "Measurement of Surge Voltages on Transmission Lines due to Lightning," E. S. Lee and C. M. Foust, *General Electric Review*, March, 1927, p. 135.

(11) "Testing Porcelain Insulators," W. Bucksath, *Electrician*, 24th April, 1925.

(12) "Lightning Arresters," H. Cotton, *B.E.A.M.A. Journal*, June, July, and August, 1922.

(13) "High-voltage Testing Equipments," E. T. Norris and F. W. Taylor, *Jour. I.E.E.*, Vol. LXIX, p. 673.

(14) "Sphere-gap Calibration," S. Whitehead and A. P. Castellain, *Jour. I.E.E.*, Vol. LXIX, p. 898.

(15) "The Sphere-gap as a Means of Measuring High-voltage," F. W. Peek, Jun., *General Electric Review*, Vol. XVI (1913), p. 286.

(16) "The Dielectric Strength of Air," A. Russell, *Proc. Phys. Soc.* (London), Vol. XX (1905–07), p. 49.

(17) "The Design and Construction of a Shielded Resistor for High Voltages," R. Davis, *Jour. I.E.E.*, Vol. LXIX, p. 1028.

(18) "The Use of Air Condensers as High-voltage, Standards," B G. Churcher and C. Dannatt, *Jour. I.E.E.*, Vol. LXIX p. 1019.

(19) "The Use of the Schering Bridge at 150 kV," B. G. Churcher and C. Dannatt, *World Power*, Vol. V, No. XXIX, p. 238.

(20) "Notes on the Testing of Static Transformers," J. L. Thompson and H. Walmsley, *Jour. I.E.E.*, Vol. LXIV, p. 505.

(21) "The Measurement of High Voltages, with special reference to the Measurement of Peak Voltages," R. Davis, G. W. Bowdler, and W. G. Standring," *Jour. I.E.E.*, Vol. LXVIII, p. 1222.

(22) "The Construction and Operation of a Simple Neon-tube High-tension Crest Voltmeter," L. E. Ryall, *Jour. I.E.E.*, Vol. LXIX, p. 891.

(23) "The Ionic Wind Voltmeter and Thermo-electrostatic Relay," W. M. Thornton, M. Waters, and W. G. Thompson, *Jour. I.E.E.*, Vol. LXIX, p. 533.

(24) "Testing High-voltage Cables with D.C. after Installation," N. A. Allen, *Electrical Communication*, Vol. IV (January, 1926), p. 184.

(25) "The Electrical High-pressure Testing of Cables and the Localization of Faults," J. Urmston, *Jour. I.E.E.*, Vol. LXIX, p. 983.

(26) "The Testing of Porcelain Insulators," B. L. Goodlet, *Jour. I.E.E.*, Vol. LXVII, p. 1177.

(27) "Dielectric Phenomena at High Voltages," B. L. Goodlet, F. S. Edwards, and F. R. Perry, *Jour. I.E.E.*, Vol. LXIX, p. 695.

(28) "The Nature of Breakdown in Solid Insulating Materials," R. Liljeblad, *A.S.E.A. Journal*, Vol. III (1926), p. 163.

(29) "A Modern High-voltage Cable-testing Plant," T. N. Riley, *Electrical Communication* (March, 1927), p. 284.

(30) "Some Developments in the Routine Pressure-testing of High-voltage Cables," E. A. Beavis, *Jour. I.E.E.*, Vol. LXVI, p. 1086.

(31) "A High-voltage Testing Laboratory," *English Elec. Jour.*, Vol. V, No. 1.

(32) "High-voltage Laboratories," A. P. M. Fleming, *World Power*, Vol. VI, No. XXXI, p. 16.

(33) "The Laws of Dielectrics," W. S. Flight, *Jour. I.E.E.*, Vol. LII, p. 764.

(34) "The Effect of Heat on the Electric Strength of Some Commercial Insulating Materials," W. S. Flight, *Jour. I.E.E.*, Vol. LX, p. 218.

(35) "High Voltage Testing," *Electrician* (6th February, 1925).

(36) "Comparative Measurements with Three Different Connections for Surge-tests on Transformers," *The Brown-Boveri Review* (March, 1931), p. 126.

(37) "Le problème des isolants. Quelques rèsultats d'essais et l'application a la construction des transformateurs a très haute tension," *A.C.E.C. Journal*, No. 128 (1930), p. 125.

(38) "Insulation for 132 kV Electrical Apparatus," C. C. Garrard, *G.E.C. Journal* (February, 1931), p. 161.

(39) "Dielectric Problems in High-voltage Cables," P. Dunsheath, *Jour. I.E.E.*, Vol. LXIV, No. 349 (1926).

(40) "The Permittivity and Power Factor of Micas," C. Dannatt and S. E. Goodall, *Jour. I.E.E.*, Vol. LXIX, p. 490.

(41) "Dielectric Loss-angle Measurement of Multi-core High-tension Cables, with special reference to the Schering Bridge," L. G. Brazier, *Jour. I.E.E.*, Vol. LXIX, p. 757.

(42) "Insulating Oils for High-voltage Cables," T. N. Riley and T. R. Scott, *Jour. I.E.E.*, Vol. LXVI, p. 805.

(43) "Research on Insulating Oils," *The Electrical Review* (18th November, 1921).

(44) "High-voltage Precision Measurements," W. M. Thornton, *Jour. I.E.E.*, Vol. LXIX, p. 1273.

(45) "Directions for the Study of Varnish-paper and Varnish-fabric Boards and Tubes," B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LXII, p. 160.

(46) "Tentative Directions for the Determination of the Electric Strength of Solid Dielectrics," B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LXIV, p. 165.

(47) "Directions for the Study of Electrical Insulating Papers (Unvarnished) for Purposes Other than the Manufacture of Cables," B.E.A.I.R.A. Report, *Journal I.E.E.*, Vol. LXI, p. 982.

(48) "Directions for the Study of Vulcanized Fibre for Electrical Purposes," B.E.A.I.R.A. Report, *Jour.I.E.E.*, Vol. LXI, p. 964.

(49) "Directions for the Study of Hard Composite Dielectrics (Insulating Materials)," B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LX, p. 565.

(50) "Measuring Small Alternating Currents," E. H. W. Banner, *Electrician*, 20th August, 1926.

(51) *The High-voltage Research Laboratory*, Metropolitan-Vickers Electrical Co., special publication 7000/1.

(52) "The Testing of Materials used in the Manufacture of Electrical Equipment," C. Dawson, *Jour. I.E.E.*, Vol. LXI, p. 59.

(53) "Directions for the Study of Micanite," B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LXVII, p. 1243.

(54) "Directions for the Study of Pressboard for Electrical Insulating Purposes," B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LXI, p. 486.

(55) "Directions for the Study of Varnished Cotton Cloth (excluding Adhesive Tape), B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LXII, p. 173.

(56) "Low Power-factor Measurements at High Voltages," E. H. Rayner, W. G. Standring, R. Davis, and G. W. Bowdler, *Jour. I.E.E.*, Vol. LXVIII, p. 1132.

(57) "The Effect of Added Impurities on the Breakdown Voltage of Insulating Oils," B.E.A.I.R.A., Report, *Jour. I.E.E.*, Vol. LXVII, p. 750.

(58) "Directions for Determining the Electric Strength of Fibrous Insulating Materials," B.E.A.I.R.A. Report, *Jour. I.E.E.*, Vol. LX, p. 794.

(59) "Contact Effects between Electrodes and Dielectrics," B. G. Churcher, C. Dannatt, and J. W. Dalgleish, B.E.A.I.R.A., Report, *Jour. I.E.E.*, Vol. LXVII, p. 271.

(60) "Properties and Characteristics of Insulating Materials," R. T. Fleming, *Jour. I.E.E.*, Vol. LX, p. 58.

(61) "The Prevention of Ionization in Impregnated Paper Dielectrics," S. G. Brown and P. A. Sporing, *Jour. I.E.E.*, Vol. LXVII, p. 968.

(62) "The Testing of Underground Cables with Continuous Current," O. L. Record, *Jour. I.E.E.*, Vol. LIV, p. 608.

(63) "Calibration of the Sphere Gap," J. R. Meador, *Trans. A.I.E.E.*, Vol. 53, p. 942.

(64) "Dielectric Strength of Mineral Oils," F. M. Clark, *Trans. A.I.E.E.*, Vol. 54, p. 50.

(65) "The new ASEA Oil-testing Apparatus," L. Mörtsell, *ASEA Jour.*, Oct., 1934.

(66) "The Absolute Measurement of High Electrical Pressures," W. M. Thornton and W. G. Thompson, *Jour. I.E.E.*, Vol. LXXI, p. 1.

(67) "Cathode-Ray Oscillographic Studies of Surge Phenomena," T. E. Allibone, W. G. Hawley and F. R. Perry, *Jour. I.E.E.*, Vol. LXXV, p. 670.

(68) "A Resistor for High-Voltage Measurements," G. W. Bowdler, *Jour. I.E.E.*, Vol. LXXIII, p. 65.

(69) "ASEA's Short Circuit and High Tension Laboratory at Ludvika," R. Kempe, *ASEA Jour.*, Sept., 1934.

(70) "Conduction through Transformer Oil at High Field Strengths," J. F. Gillies, *Jour. I.E.E.*, Vol. LXXVI, p. 647.

(71) "Directions for the Measurement of the Power Factor of Cables at Power Frequencies and at Voltages up to 200 kV" (E.R.A. Report), *Jour. I.E.E.*, Vol. LXXVII, p. 845.

(72) "Standardization of Impulse-voltage Testing," T. E. Allibone and F. R. Perry, *Jour. I.E.E.*, Vol. LXXVIII, p. 257.

(73) "The Measurement of Discharges in Dielectrics," A. N. Arman and A. T. Starr, *Jour. I.E.E.*, Vol. LXXIX, p. 67.

(74) "Routine Over-Voltage Testing of High-Voltage Cables," C. Kibblewhite, *Jour. I.E.E.*, Vol. LXXIX, p. 82.

(75) "Some New High-Tension Measuring Instruments," W. M. Thornton, *Jour. I.E.E.*, Vol. LXXIX, p. 483.

(76) "The Effects of Impulse Voltages on Transformer Windings," T. E. Allibone, D. B. McKenzie, and F. R. Perry, *Jour. I.E.E.*, Vol. LXXX, p. 117.

(77) "Sixty-cycle Calibration of the 50-centimetre Ungrounded Sphere Gap," C. S. Sprague and G. Gold, *Trans. A.I.E.E.*, Vol. LVI, p. 594.

(78) "Electrical Characteristics of Suspension Insulators," C. L. Dawes and R. Reiter, *Trans. A.I.E.E.*, Vol. LVI, p. 59.

(79) "The Calibration of Sphere Gaps with Impulse Voltages," R. Davis and G. W. Bowdler, *Jour. I.E.E.*, Vol. LXXXII, p. 645.

(80) "The Calibration of the Sphere Spark Gap for Voltage Measurement up to One Million Volts (Effective) at 50 Cycles," F. S. Edwards and J. F. Smee, *Jour. I.E.E.*, Vol. LXXXII, p. 655.

(81) "Dielectric Strength of Porcelain," P. L. Bellaschi and M. L. Manning, *Electrical Engineering*, Dec., 1939.

(82) "The Electric Strength of Air at High Pressure," H. H. Skilling, *Electrical Engineering*, April, 1939.

(83) "A Note on Standard Calibrations for Sphere Gaps," S. Whitehead, *Jour. I.E.E.*, Vol. LXXXIV, p. 408.

(84) "Testing of Transmission-line Insulators under Deposit Conditions," W. J. John and C. H. W. Clark, *Jour. I.E.E.*, Vol. LXXXV, p. 590.

(85) "High Voltage Rectifier Valves for X-ray Diagnostics," J. H. Van der Tuuk, *Phillips Tech. Rev.*, Vol. 8, No. 7, p. 199.

(86) "Surge Phenomena," *E.R.A. Report Ref. S/T*35.

(87) "The Development and Design of High-voltage Impulse Generators," F. S. Edwards, A. S. Husbands and F. R. Perry, *Jour. I.E.E.*, Vol. 98, Pt. I, p. 155.

(88) "Absolute Measurement of High Voltages by Oscillating Electrode Systems," E. Bradshaw, S. A. Husain, N. Kesavamurthy and K. B. Menon, *Jour. I.E.E.*, Vol. 95, Pt. II, p. 636.

(89) *Deutsches Reichspatent, No.* 455933, E. Marx, 1923.

(90) "Versuche über die Prüfung von Isolatoren mit Spannungsstossen," E. Marx, *E.T.Z.*, 1924, Vol. 45, p. 652.

(91) *Hochspannungs Praktikum*, E. Marx, (Julius Springer, Berlin, 1941).

(92) "A Cathode-ray Oscillograph for the Direct Measurement of High, voltage Transients," A. K. Nuttall, *Jour. I.E.E.*, 1936, Vol. 78, p. 229.

(93) *Dielectric Breakdown of Solids*, S. Whitehead, Oxford University Press, 1951.

(94) "The Deterioration and Breakdown of Dielectrics Resulting from Internal Discharges," J. H. Mason, *Jour. I.E.E.*, Vol. 98, Pt. I, p. 44.

(95) "Some Mechanisms of Insulation Failure," H. F. Church and C. G. Garton, *Proc. I.E.E.*, Vol. 100, Pt. IIA, pp. 111–20.

(96) "The Electrical Breakdown of Air between Insulators," W. L. Harries, *Proc. I.E.E.*, Vol. 100. Pt. IIA, pp. 132–7.

(97) "Potential Indicators for Live Conductor Detection," G. F. Shotter and E. E. Hutchings, *E.R.A.*, *F/T* 133.

CHAPTER XII

LOCALIZATION OF CABLE FAULTS

THE routine testing of cables with high voltages and the localization of faults in high-tension cables, using high-voltage d.c., have already been dealt with in the previous chapter. Only the localization of faults in cables which are in service, and for use with the lower distribution voltages, will be considered here.

The faults which are most likely to occur are: (*a*) a breakdown of the insulation of the cable which allows current to flow from the core to earth or to the cable sheath—called a "ground" fault; (*b*) a "cross" or short-circuit fault, in which case the insulation between two cables, or between two cores of a multi-core cable, is faulty; and (*c*) an open-circuit fault where the conductor becomes broken or a joint pulls out. The methods used for locating an open-circuit fault differ from those used in the other two cases.

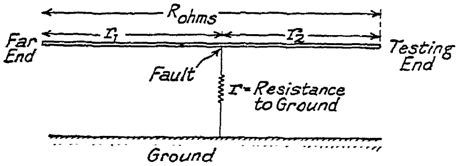


FIG. 294. GROUND FAULT ON A SINGLE CABLE

The causes of such faults are numerous and need not concern us. It is important that their exact position shall be determined, however, in order that repairs may be undertaken without loss of time and effort.

In the case of multi-core cables it is advisable, first of all, to measure the insulation resistance of each core to ground and also between cores. If the fault is a "ground" this will enable the faulty core to be discovered; and if a short-circuit, the cores which are involved can be determined.

Blavier and Earth Overlap Tests. These tests enable one to find the position of a ground on a single cable—i.e. when no other cables run along with the faulty one.

The case is illustrated by Fig. 294. The total resistance of the cable, before the occurrence of the fault, is assumed to be known. Let this be R ohms. Suppose also that r is the resistance of the fault to ground, and that r_1 and r_2 are the resistances of the lengths of cable, "far end" to fault, and "testing end" to fault, respectively.

In the Blavier test the resistance between line and ground is measured, first with the "far end" disconnected from earth, and then with the "far end" earthed.

These measurements may be made with the aid of a low-tension supply, and either an ammeter and voltmeter, or a bridge network.

Theory. Let the two measured values of resistance be M_1 and M_2.

Then
$$M_1 = r_2 + r \qquad \text{(i)}$$
$$M_2 = r_2 + \frac{rr_1}{r + r_1} \qquad \text{(ii)}$$
and
$$R = r_1 + r_2 \qquad \text{(iii)}$$
From (i),
$$r = M_1 - r_2$$
Hence, in (ii),
$$M_2 = r_2 + \frac{(M_1 - r_2)\,r_1}{(M_1 - r_2) + r_1}$$
from which
$$(M_2 - r_2)(M_1 - r_2 + r_1) = (M_1 - r_2)\,r_1$$
$$\therefore\ M_1M_2 - M_1r_2 - M_2r_2 + r_2^2 + r_1M_2 = M_1r_1$$
Substituting from (iii), we have, after simplifying,
$$M_1M_2 - 2M_2r_2 + r_2^2 + RM_2 = M_1R$$
or
$$r_2^2 - 2M_2r + M_2^2 - M_2^2 + M_1M_2 = R\,(M_1 - M_2)$$
$$(r_2 - M_2)^2 = (M_1 - M_2)(R - M_2)$$
Thus
$$r_2 = M_2 \pm \sqrt{(M_1 - M_2)(R - M_2)} \qquad (241)$$

Since r_2 is obviously less than M_2, the negative sign in this expression is the one to be taken, i.e.
$$r_2 = M_2 - \sqrt{(M_1 - M_2)(R - M_2)} \qquad (242)$$

In the "earth overlap" test, the two measurements made are: first, the resistance M_1 between line and ground, measured from the "testing end," with the "far end" earthed; and then the resistance M_2, line to ground, measured at the "far-end," with the "testing end" earthed.

Then
$$M_1 = r_2 + \frac{rr_1}{r + r_1}$$
$$M_2 = r_1 + \frac{rr_2}{r + r_2}$$
and, as before, $R = r_1 + r_2$.

By elimination, as in the Blavier test, we have
$$r_2 = M_1\left(\frac{R - M_2}{M_1 - M_2}\right)\left\{1 - \sqrt{\frac{M_2(R - M_1)}{M_1(R - M_2)}}\right\} \qquad (243)$$
and also
$$r_1 = M_2\left(\frac{R - M_1}{M_2 - M_1}\right)\left\{1 - \sqrt{\frac{M_1(R - M_2)}{M_2(R - M_1)}}\right\}. \qquad (244)$$

In each of the above tests, the distance of the fault from either end is obtained from the resistance between the fault and the end by using the known value of the resistance, per unit length, of the cable. The accuracy of these two methods is not high, mainly because it is often difficult to make connections to earth, at the ends of the cable, having a low enough resistance for this to be considered negligible—as it is in the theory of the tests (see Ref. (8)).

Voltage-drop Tests. These tests can be used when a second cable, free from faults, runs along with the faulty cable, the sound cable being used, either as part of the current circuit as in Fig. 295, or as a potential lead to the voltmeter as in Fig. 296.

Referring to Fig. 295, a large *steady* current is passed through the loop formed by the sound and faulty cables, joined together at the distant end, as shown, by a low resistance connection. The current is from a number of accumulators, and is regulated and measured by the resistance R and ammeter A. By means of the throw-over switch S the voltmeter V, one terminal of which is earthed, is connected first across the section *ed* of the loop, and then across the section *abcd*. Let the two readings obtained be V_1 and V_2. Since the same current flows through both of these loop-sections (neglecting the voltmeter current) we have

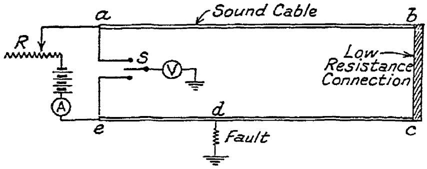


FIG. 295. SOUND CABLE USED IN CONJUNCTION WITH THE FAULTY ONE

$$\frac{V_1}{V_2} = \frac{\text{Resistance of section } ed}{\text{Resistance of section } abcd}$$

or

$$\frac{V_1}{V_1 + V_2} = \frac{\text{Resistance of section } ed}{\text{Resistance of the whole loop } abcde}$$

Now, if the cross-section of the cable is the same throughout the length, we have

$$\frac{\text{Distance of fault from } e}{\text{Length of the whole loop}} = \frac{V_1}{V_1 + V_2}$$

from which the position of the fault can be found. If the cable cross-section is not uniform, a correction must be applied to allow for the fact.

The resistance of the voltmeter should be large compared with the resistance of the fault, since the latter forms part of the voltmeter circuit. This fact constitutes an objection to the method. Another reason for the high-resistance voltmeter is that the instrument would otherwise take an appreciable current and so introduce errors in measurement.

In the circuit shown in Fig. 296, the fault resistance does not enter into the resistance of the voltmeter circuit. In this case the voltmeter measures the voltage drop across the length *ed* of the cable, and the resistance of this length is obtained by dividing by the current through it (as indicated by the ammeter *A*).

The voltmeter resistance should be high, as compared with the resistance of both the length *ed*, and of length *abcd*, of the cable, if corrections for voltmeter current are to be avoided. If the voltmeter is electrostatic, no corrections are, of course, necessary.

The length *ed* is calculated, as before, from the known resistance per unit length. Care must be taken that the current passed through the length *ed* of the cable is not sufficient to produce appreciable heating as the resistance per unit length will then be different from that used in the calculation of length.

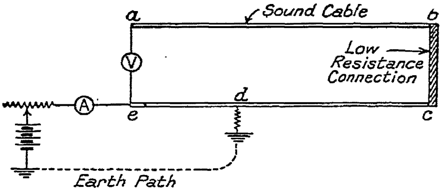


FIG. 296

The same precaution must be taken in the previous method if the cable cross-section is not uniform. If the cross-section is the same throughout, however, the proportionality renders the precaution less necessary, although it is not advisable to pass such a current as will produce appreciable heating, even in this case.

Loop Tests. These tests can be carried out for the location of either a ground or a short-circuit fault, provided that a sound cable runs along with the grounded cable or with the two cables (or cores in a multi-core cable) which are short-circuited. Such tests have the advantage that the resistance of the fault does not affect the results obtained, provided this resistance is not very high. If it is high it may adversely affect the sensitivity.

MURRAY LOOP TEST. The connections for this test are shown in Fig. 297. Connections (*a*) are for a test for a ground fault, and (*b*) for a short-circuit fault. Both circuits are essentially Wheatstone bridge networks, *G* being a galvanometer and *P* and *Q* resistances, or a slide-wire, forming two ratio arms. Referring to Fig. 297(*a*), the bridge is balanced by adjustment of *P* and *Q* until *G* indicates zero deflection.

Then,
$$\frac{P}{Q} = \frac{R}{X}$$

or
$$\frac{P+Q}{Q} = \frac{R+X}{X}$$

$$\therefore \frac{Q}{P+Q} = \frac{X}{R+X} = \frac{X}{2r} \quad . \quad . \quad . \quad (245)$$

where r is the resistance of one of the cables when free from faults. The value of r may be obtained from the lengths, cross-sections and temperatures, of the two cables all of which are assumed to be the same for each. The distance of the fault from the lower end of resistance Q may then be obtained from the value of X obtained as above.

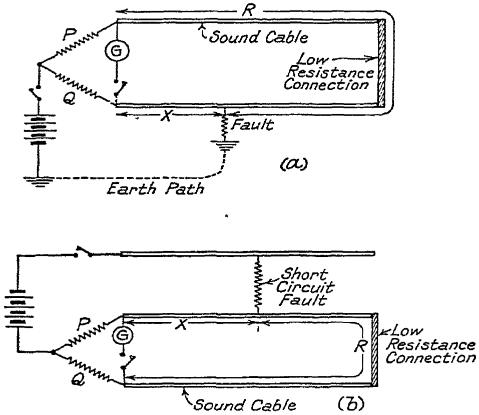


FIG. 297. MURRAY LOOP TEST

It should be noted that the resistance of the fault enters only in the battery supply circuit and, provided it is not sufficiently large to reduce the sensitivity, will not affect the results.

In Fig. 297(*b*), the connections are practically the same as in the ground test, except that a portion of one of the short-circuiting cables is substituted for an earth path in the battery circuit. Balance is obtained as before, by the adjustment of P and Q and, at balance,

$$\frac{Q}{P} = \frac{R}{X}$$

or
$$\frac{P}{P+Q} = \frac{X}{R+X} \quad . \quad . \quad . \quad . \quad . \quad (246)$$

The total resistance $(R + X)$ of the loop is assumed to be known, so that X, and hence the distance of the fault from the upper end of P, may be calculated. Again, the fault resistance enters only into the battery circuit.

VARLEY LOOP TEST. This test makes provision for the measurement of the total loop resistance instead of obtaining it from the known lengths of cable and their resistance per unit length. The connections are shown in Figs. 298, 299 for both the ground and short-circuit tests respectively. In this loop test the ratio arms P and Q are fixed, balance being obtained by adjustment of a variable resistance S, placed in series with the section of the loop having the smaller resistance. When balance is obtained with the throw-over switch, in the battery circuit, on contact c, then, in either test, the magnitude of the resistance X may be obtained from the setting of S for balance, together with the values of P and Q and of the resistance $R + X$ (i.e. the total resistance of the loop).

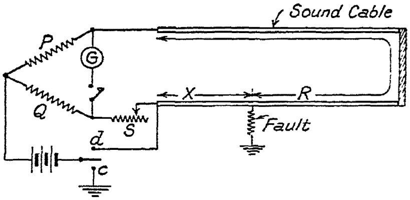


FIG. 298. VARLEY LOOP TEST FOR GROUND FAULT

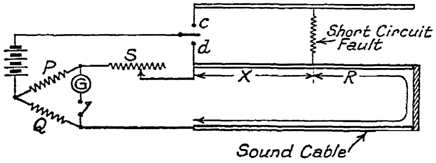


FIG. 299. VARLEY LOOP TEST FOR SHORT CIRCUIT FAULT

At balance, in the ground test,

$$\frac{P}{Q} = \frac{R}{X + S}$$

or

$$\frac{P + Q}{Q} = \frac{R + X + S}{X + S}$$

From which,

$$X = \frac{Q\,(R + X) - SP}{P + Q} \qquad . \quad (247)$$

In the short-circuit test,

$$\frac{Q}{} = \frac{R}{X + S}$$

Now, P, Q, and S are known. $R + X$ may be measured by throwing over to contact d and obtaining a balance by adjustment of S as in the ordinary Wheatstone bridge network. In the ground test, as connected in Fig. 298, at balance,

$$\frac{P}{Q} = \frac{R + X}{S_1}$$

where S_1 is the new setting of S. Thus $R + X$ can be found. In Fig. 299, at balance,

$$\frac{P}{Q} = \frac{S_2}{R + X}$$

where S_2 again is the required setting of S for balance. The measured value of $R + X$ is then used in the calculation of X, from whose value the position of the fault is obtained as before. The use of the Evershed and Vignoles "Bridge-Megger Testers" for the location of cable faults was mentioned on p. 307.

FISHER LOOP TEST. In this test, developed by H. W. Fisher, two sound conductors, running from the testing end to the far end of the faulty cable, must be available. The lengths and resistances of these two conductors need not be known, but the length and resistance of the faulty cable must be known.

Two balances of the bridge network are necessary, the two connections for these being as shown in Fig. 300 (a) and (b). In the first test, one of the sound cables—of resistance R_1—is left disconnected, and in the second test it is merely used as a lead from the battery to the far end of the faulty cable. R_2 is the resistance of the other sound cable, while x and r are the resistances between the fault and the testing end, and between fault and the far end respectively.

Theory. Let P_1 and Q_1 be the balance values of P and Q in the first test, and P_2 and Q_2 their values in the second test.

Then,
$$\frac{P_1}{Q_1} = \frac{R_2 + r}{x}$$
and
$$\frac{P_2}{Q_2} = \frac{R_2}{x + r}$$

R_2 is eliminated as follows—

$$\frac{P_1}{Q_1} + 1 = \frac{R_2 + r}{x} + \frac{x}{x} = \frac{R_2 + r + x}{x}$$

$$\frac{P_2}{Q_2} + 1 = \frac{R_2}{x + r} + \frac{x + r}{x + r} = \frac{R_2 + r + x}{x + r}$$

$$\therefore \frac{\frac{P_2}{Q_2} + 1}{\frac{P_1}{Q_1} + 1} = \frac{x}{x + r}$$

$$\therefore x = (x + r)\frac{\left(\frac{P_2}{Q_2} + 1\right)}{\left(\frac{P_1}{Q_1} + 1\right)} \quad . \quad . \quad . \quad . \quad . \quad (248)$$

If the resistance per unit length of the faulty cable is uniform, we have

Distance of fault from testing end

$$= \left(\frac{\frac{P_2}{Q_2} + 1}{\frac{P_1}{Q_1} + 1} \right) \times \text{Total length of faulty cable}$$

Fault Localizing Bridges. Several forms of fault-localizing bridge, which are portable and are arranged for determination of the distance of a fault from the testing end directly (i.e. without calculation), are made up by different manufacturers. Fig. 301 shows the connections and lay-out of the Raphael bridge, developed by F. C. Raphael, and manufactured by Messrs. Muirhead & Co., Ltd. It consists of a double slide-wire, with a scale, and two movable contacts S and P. The former contact connects one end of the cable loop to any point on the bridge wire, while P is the sliding contact for balance adjustment. The galvanometer and battery are connected to the cable loop, as in the Murray loop test (Fig. 297(a)).

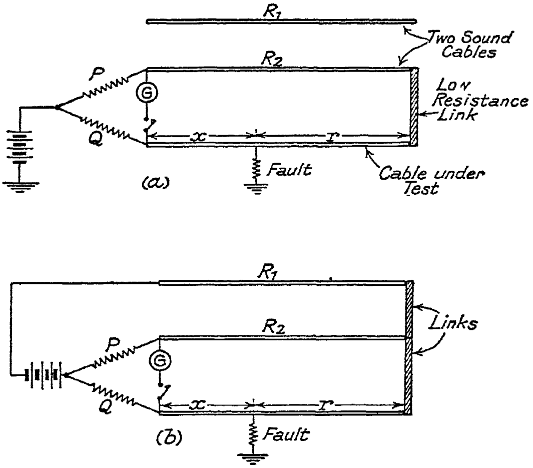


FIG. 300. FISHER LOOP TEST

In carrying out a test for a ground fault location, the contact S is placed in such a position on the slide-wire that the number of scale divisions of the working portion of this wire is a convenient multiple of, or is the same as, the length of the loop in yards. P is then moved until balance is obtained, when the scale reading opposite P gives the distance of the fault from the testing end, in yards, directly.

The relationship at balance is the same as in the Murray loop test, namely, $\frac{Q}{P+Q} = \frac{x}{2r}$

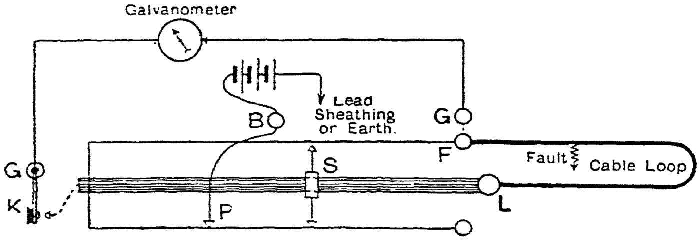
Corrections to be Applied in Loop Tests. If a ratio such as

$$\frac{\text{Resistance to fault}}{\text{Resistance of the whole loop}}$$

is, in one of these tests, determined in terms of resistances in the bridge ratio arms, it is obviously equal to

$$\frac{\text{Distance to fault}}{\text{Length of the whole loop}}$$

only if the cable section (and also the temperature) is uniform throughout the loop. If this is not so, corrections must be applied.



(*Muirhead & Co., Ltd.*)

FIG. 301 RAPHAEL FAULT LOCALIZING BRIDGE

Correction when the Cross-section and Length of both Faulty and Sound Cables are Known.

Let L_f = length of faulty cable.
" a_f = cross-section of faulty cable.
" L_s = length of sound cable.
" a_s = cross-section of sound cable.

Then, equivalent length of the whole loop is $L_f + L_s \cdot \frac{a_f}{a_s}$ and this length must be used in calculating the distance to the fault, instead of the actual length of loop. If resistances of the two cables, per unit length r_f and r_s, are used instead of cross-sections, the equivalent length is $L_f + L_s \frac{r_s}{r_f}$. Temperature corrections are applied in a similar way if the temperatures of the two cables are known or can be estimated with reasonable accuracy.

Correction when the Cross-section of the Faulty Cable is Not Uniform. Suppose the faulty cable consists of a number of sections, in series, these sections having different resistances per unit length. Let L_1, L_2, L_3, etc., be the lengths of these sections, and r_1, r_2, r_3, etc., be their resistances per unit length. Then, the first section has resistance L_1r_1, the second L_2r_2, and so on. If x is the resistance—obtained by measurement—of the cable from testing end to the fault, it is first necessary to determine in which section the fault exists. Thus, if x is greater than $L_1r_1 + L_2r_2$, but less than $L_1r_1 + L_2r_2 + L_3r_3$, the fault is in the third section, and its distance L along this section from the point of junction with section 2 is given by

As mentioned above, temperature corrections must also be applied if any appreciable difference of temperature exists between the various sections. It may be necessary, also, to correct for the resistance of joints if these are numerous.

Although other cases requiring corrections exist, enough has been said to indicate the method of applying such corrections in any particular case.

Tests for an Open-circuit Fault. If a complete disconnection occurs in a cable—either by a fault burning clear without causing a ground fault, or on account of the cable being pulled out at a joint—its position may be found by a capacitance test. The capacitance of a cable, to ground or to another parallel conductor, is proportional to the length of the cable. If the capacitance C of the whole length of the cable, when sound, is known, and the capacitance C_x of the length between one end and the fault is measured, the distance of the fault from the testing end is $\frac{C_x}{C} \times$ length of the whole cable. If the capacitance C is not known, tests must be carried out from each end of the cable, the sum of the two capacitances so measured giving the capacitance C. The distance of the fault from either end can then be obtained as above.

For the measurement of capacitance, either a ballistic galvanometer or an alternating current bridge method, using a high-frequency generator and telephone detector, may be employed.

Connections for the measurement of the capacitance of the cable up to the fault are shown in Fig. 302, two alternatives being given. The first method of measurement is by direct deflection, using a ballistic galvanometer. C is a standard capacitor. The galvanometer BG is shunted by an Ayrton shunt. The galvanometer throw is first observed when the capacitance represented by the cable-length is charged from the battery, the standard capacitor being then out of circuit. This capacitor is then substituted for the cable by means of the switches shown, and the galvanometer "throw" produced, when it is charged from the same battery, is observed. If D_c and D_s are these two throws—corrected if necessary for variations of the shunting powers in the two cases—we have

$$\frac{\text{Capacitance of cable-length to the fault}}{\text{Capacitance of standard capacitor}} = \frac{D_c}{D_s}$$

In the second method a capacitor bridge is employed, the supply being a high-frequency generator, and the detector T a telephone. The cable capacitance is then measured in terms of the standard capacitor C, the non-inductive resistors P and Q being adjusted to give a balance of the bridge, when

$$\frac{\text{Capacitance of cable-length to the fault}}{\text{Capacitance of standard capacitor}} = \frac{Q}{P}$$

The distance to the fault is then obtained, as described above, from this measured value of the capacitance, up to the fault, compared with the capacitance of the whole cable.

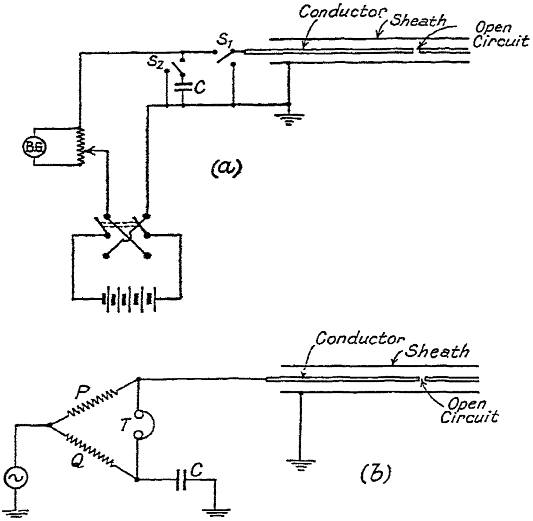


FIG. 302. TEST FOR AN OPEN CIRCUIT FAULT

An open-circuit fault locator operating on this principle is manufactured by Messrs. H. W. Sullivan, Ltd. A Kelvin-Varley slide wire is used for the ratio arms P and Q.

Induction Method of Testing. This method, which is used for the localization of ground faults in cables, can only be successfully employed when the cable has no metallic sheath or armouring. It is, however, useful for the localization of such faults in vulcanized bitumen cables, and has the merit of discovering the fault directly without reliance having to be placed upon calculations, or upon assumptions as to cable resistance.

A battery and interrupter are used to send an interrupted direct current through the faulty cable. An exploring, or search coil, to the terminals of which a telephone receiver is connected, is then placed with its plane parallel to the direction of the cable as shown in Fig. 303. This coil consists of about two hundred turns of 26 or 30 S.W.G. wire, wound on an equilateral triangle former of about 3 ft. side.* The intermittent current in the cable induces an e.m.f. in the coil, and a note is thus heard in the telephone receiver. If the search coil is moved along the cable in the direction of the fault, this note will cease immediately the fault is passed, since the testing current there passes to earth, and from thence passes to the earthed terminal of the testing battery. The position of the fault is thus directly determined from the cessation of the telephone note.

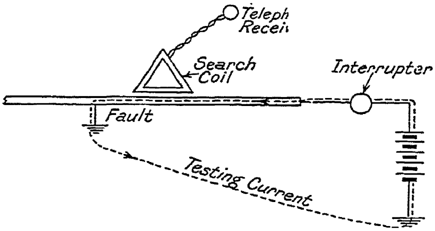


FIG. 303. INDUCTION METHOD OF LOCATING A GROUND FAULT

If the cable is lead-covered or is armoured, the method is rendered uncertain, owing to the currents flowing in the sheath (which is connected to the cable at the fault) and to the magnetic shielding effect of the steel armouring in the latter case.

A test set on this principle, containing an impulse generator as well as two different forms of search coil, is the "P & B" Cable Tracer and Fault Locator, made by Price and Belsham, Ltd. From the point of view of tracing the run of an underground cable the magnetic shielding effect of the armouring is negligible.

BIBLIOGRAPHY AND REFERENCES

(1) *The Localization of Faults in Electric Light Mains,* F. C. Raphael.

(2) *Underground Cable Systems,* G. W. Stubbings.

(3) *Electrical Measurements,* F. A. Laws.

(4) *Methods of Measuring Electrical Resistance,* E. F. Northrup.

(5) *The Theory of Electric Cables and Networks,* A. Russell.

(6) "The Electrical High-pressure Testing of Cables and the Localization of Faults," J. Urmston, *Jour. I.E.E.*, Vol. LXIX,, p. 983.

(7) "Methods of Locating Breaks in Cable Conductors," J. H. Savage, *Distribution of Electricity,* Dec., 1933.

(8) "Cable Fault Localization: Sources of Errors and their Magnitudes," E. W. Golding, *Electrical Times,* 22nd July, 1937.

(9) "Testing Cables with D.C. Pressures and Localizing High Resistance Cable Faults," C. Grover, *Distribution of Electricity,* May, 1937, *et seq.*

(10) "Localization of Flashing Faults in Cables," A. T. Starr and H. T. Gooding, *Jour. I.E.E.*, Vol. LXXXIV, p. 699.

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* These dimensions are taken from Raphael's *Localization of Faults.*

CHAPTER XIII

ELECTRICAL METHODS OF MEASURING TEMPERATURE

General. The most important methods of measuring temperature are—

(*a*) By mercury thermometers.
(*b*) By gas thermometers.
(*c*) By pyrometers.

Mercury thermometers are generally used for temperatures up to about 300° C., mercury remaining liquid over the range − 39° C. to 357° C. If the mercury is under pressure of an inert gas such as nitrogen or carbon dioxide, the boiling point is increased so that temperatures up to 500° C. may be measured. Borosilicate glass is used for high-range thermometers. To avoid trouble from softening of the glass, quartz tubes containing mercury, under gas pressure, can be used up to 700° C. Tin, which melts at 232° C., has been used instead of mercury for temperatures up to 1,000° C.

Gas thermometers depend for their operation upon the fact that for certain gases, the change in volume, at constant pressure, or the change in pressure, at constant volume, as their temperature varies, obeys a regular law. They are therefore used as standards for the measurement of temperature and, although such thermometers are not suitable for commercial and ordinary laboratory measurements, they have been used to determine the melting points of a number of metals. These melting points, once established, may be used in the calibration of other—and more convenient—instruments for temperature measurement. Gas thermometers have been used with hydrogen up to about 400° C. and with nitrogen up to 1,600° C. Their probable error has been given by A. L. Day* as ± 2° C. He also gives the melting points of a number of metals, including that of pure platinum, at 1,755° C. (although the temperature now accepted is 1,773·5° C.).

Gas thermometers have been used to measure very low temperatures and one using helium is applicable down to a temperature of about one degree above absolute zero.

Kelvin showed that it was possible to formulate a temperature scale to be independent of the properties of any particular substance. This scale would be the same as that for a "perfect" gas. The departures of the gases, used in gas thermometers, from perfection, in this sense, are small and can be calculated.

An International Temperature Scale was adopted at the Seventh General Conference of Weights and Measures at Sèvres in 1927.

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* *Transactions of the Faraday Society*, November, 1911. p. 142.

This scale is defined in "The Units and Standards of Measurement Employed by the National Physical Laboratory" (H.M. Stationery Office) which states—

> "The thermodynamic Centigrade scale, on which the temperature of melting ice and the temperature of condensing water vapour, both under the pressure of one standard atmosphere, are numbered 0° and 100° respectively, is recognised as the fundamental scale to which all temperature measurements should ultimately be referable."

The scale is based on a number of fixed and reproducible equilibrium temperatures ranging from—

> "The temperature of equilibrium between liquid and gaseous oxygen at the pressure of one standard atmosphere . . . – 182·97° C,"

to

> "the temperature of equilibrium between solid gold and liquid gold at normal atmospheric pressure . . . 1,063° C."

Pyrometers are instruments, or pieces of apparatus, which are generally used for the measurement of high temperatures and for the indication and recording of temperatures under circumstances which could not easily be met by mercury thermometers. There are four types of pyrometers, namely, platinum resistance, thermo-electric, radiation, and optical pyrometers. Of these, the first three are electrical instruments, and it is with these that we are most concerned in this chapter, although in the commonest form of optical pyrometer electrical measurements are involved.

The change in electrical resistance, corresponding to any given temperature change, is found to be a perfectly constant quantity for any given piece of metal. The thermo-electric e.m.f., set up in a given thermo-junction of two dissimilar metals, for any given change in temperature, is also quite definite. Since both electrical resistance and e.m.f. can be conveniently measured with great precision, electrical pyrometers, depending upon these effects, form very convenient and sensitive methods of measuring temperature.

The measurement of temperature by all the methods dealt with in this chapter as well as by methods outside the present scope is discussed comprehensively by Dr. Ezer Griffiths (Ref. (19)).

Electrical Resistance Pyrometers. These pyrometers may be used to cover a range from – 200° C. to 1,200° C., readings to 1° C. being possible when the resistance material is platinum, as it usually is. The law of resistance variation with temperature has been fully investigated by Callendar and Griffiths, using an air thermometer as standard. Copper and nickel have also been used for low temperature ranges—up to about 100° C. The chief requirement of the materials used in such thermometers is that their resistance at a certain temperature must be the same after subsequent heating as it was before the heating. Again, the change in resistance, per degree alteration in temperature, should be as large as possible; but this change is not uniform over any large temperature range for any of the resistance metals in use. Provided, however, the law of variation is known for any particular specimen, this can be used as a standard for the calibration of other resistance thermometers. If the material is suitably chosen, and is properly treated before being used, constancy of resistance at any given temperature can be obtained. Platinum has been found very satisfactory in this respect. The resistance thermometer in which such a material is used will compare favourably, in constancy, with the best mercury thermometers, and forms one of the most accurate methods of measuring temperatures within their range.

Laws of Resistance Variation. The law of increase of resistance of platinum with increase of temperature has been found to be

$$R_t = R_0 (1 + at - \beta t^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (250)$$

where R_t = the resistance at temperature t° C.

R_0 = ,, ,, ,, 0° C.

and a and β are constants. Callendar found that for pure platinum $a = 0{\cdot}0037$ and $\beta = 0{\cdot}00000057$.

For simplicity a single constant k has been introduced, called the "fundamental coefficient," which can be used to express the law of variation of resistance in the form

$$R_t = R_0 (1 + kt) \quad . \quad . \quad . \quad . \quad . \quad . \quad (251)$$

with sufficient accuracy for small temperature changes. This constant is determined by testing a platinum resistance pyrometer with melting ice and boiling water—i.e. at 0° C. and 100° C.

Then, if R_{100} and R_0 are its resistance at 100° C. and 0° C. respectively, we have

$$R_{100} = R_0 (1 + 100k)$$

from which

$$k = \frac{R_{100} - R_0}{100R_0}$$

Then, for the resistance R at any temperature t° C. we have

$$R = R_0 (1 + k \, . \, T_p)$$
$$= R_0 \left(1 + \frac{(R_{100} - R_0)}{100R_0} T_p\right)$$

or

$$R = R_0 + \frac{R_{100} - R_0}{100} \, . \, T_p \quad . \quad . \quad . \quad (252)$$

T_p is called the "platinum temperature," and a correction must be applied in order to determine the temperature in degrees centigrade.

From the above expression, we have

$$T_p = \frac{R - R_0}{R_{100} - R_0} \times 100$$

$R_{100} - R_0$ is the "fundamental interval," and is constant for any particular pyrometer. The correction to be applied to convert platinum temperatures to centigrade temperatures is rendered necessary, when higher temperatures than 100° C. are to be determined, on account of the importance of the constant β at these temperatures.

If R is the resistance at an actual temperature of t° C., this temperature is obtained from the platinum temperature T_p, from the following expression, due to Callendar,

$$t - T_p = \delta \left[\left(\frac{t}{100}\right)^2 - \left(\frac{t}{100}\right) \right] \quad . \quad . \quad . \quad . \quad (253)$$

This "difference formula" is correct to within $\frac{1}{10}$°C. up to 500° C., and to within $\frac{1}{2}$° C. up to 1,000° C. The constant δ depends upon the purity of the platinum, its value being 1·5 if the platinum is very pure. This constant is determined from measurements of the resistance of the pyrometer at three known temperatures, namely, those of melting ice, boiling water, and boiling sulphur, which are 0° C., 100° C., and 444·6° C. respectively.

A calibration curve for the particular pyrometer is used, in practice, to obtain the centigrade temperatures, instead of using the formula.

C. F. Marvin* found that the resistance of pure nickel varies with temperature, according to the law

$$\log_e R = a + bt \qquad (254)$$

where R = resistance in ohms

t = temperature in degrees centigrade

and a and b are constants of the order of 1 and 0·0017 respectively, depending upon the purity. For a temperature range of − 25° C. to 350° C. the assumption that this law is true will not introduce a greater error than 1° C. If the range of temperature is small, the error will be under 0·1° C.

Construction of Platinum Resistance Thermometers. A bare platinum wire (usually about 0·2 mm. in diameter) is wound on a mica frame, and is enclosed within a porcelain, or fused silica, tube. This tube may also be enclosed within a steel tube for protection, its use being dependent upon the temperature to be measured and upon the situation of the instrument when in use. The bridge, or measuring circuit, is often situated at a considerable distance from the body whose temperature is to be measured, and for this reason resistance changes with temperature in the long leads required, may introduce errors of measurement, unless compensating leads are used. The latter are in the form of a loop of wire, of the same material as the leads to the thermometer, running along with these leads and ending in a loop of platinum wire, inside the thermometer tube, but not connected to the resistance coil. In use, these compensating leads are connected in the opposite side of the bridge to that in which the thermometer itself is connected. Any temperature changes which affect the leads to the thermometer also affect the compensating leads, equally, and thus the balance of the bridge network is not affected by them. Copper or steel tubes are sometimes used, instead of porcelain or silica ones, for temperatures up to about 700° C. It is essential that the platinum should be protected from fumes which may corrode it and cause change of resistance. Joints within the thermometer itself should be welded, since metallic solderings give off fumes, when hot, which cause deterioration of the platinum. Also, the platinum wire is subjected to a special annealing treatment, to avoid subsequent changes of resistance due to strain when in use.

These pyrometers may be used for continuous work up to about 900° C., and, intermittently, for temperatures up to 1,200° C. Above

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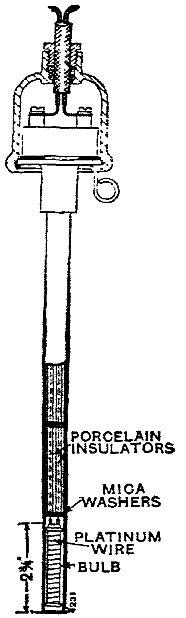
* *Physical Review*, April, 1910, p. 522.

900° C. trouble may be experienced due to deterioration of both the platinum and of the mica insulation. The accuracy obtainable is of the order of 0·01° C. up to 100° C., 0·5° C. up to 500° C., and 3° C. up to 1,000° C.

The construction of a platinum resistance thermometer manufactured by the Cambridge Instrument Co., Ltd., is shown in Fig. 304.



(Cambridge Instrument Co., Ltd.)
FIG. 304. PLATINUM RESISTANCE THERMOMETER

Method of Use. These pyrometers may be used either in a Wheatstone bridge network of the ordinary type, or in conjunction with an indicating instrument, such as the Whipple Indicator, which gives the temperature directly. They may be used, also, with a recording instrument such as the Callendar Recorder, which records temperature variations on a revolving drum. A Kelvin double bridge is used instead of a Wheatstone bridge when the thermometer resistance is low.

Fig. 305 shows the connections for the measurement of the change in resistance of the thermometer with temperature by a Wheatstone bridge network. P and Q are equal ratio arms. In a third arm are a variable resistance R and the compensating leads. The thermometer coil is contained in the fourth arm, with a slide-wire in between these last two arms, as shown.

A special bridge, designed for the purpose, is often used, and the slide-wire is graduated in degrees, the position of the slide-wire contact for balance of the bridge giving the temperature directly.

Temperature Indicators and Recorders. The Whipple indicator consists of a bridge of this type, the slide-wire being wound spirally on a drum. The indicator is calibrated so that the position of the drum, for balance, gives the temperature directly in degrees.

Other forms of temperature indicators are in the form of a bridge network, but do not require adjustment of resistance after the initial balance. If the bridge is balanced at the lowest temperature of the range of the thermometer variation of temperature, producing resistance variation, will cause a galvanometer current to flow. This current will depend upon the resistance variation—i.e. upon the temperature—so that the galvanometer can be calibrated to read temperature directly. Errors due to changes of battery voltage in this form of instrument are avoided by the provision of a standard resistance, which can be substituted for the thermometer resistance, and the bridge adjusted to give a standard indication upon the indicator scale with this standard resistance in circuit.

In the Callendar Recorder, the bridge is kept balanced automatically by sensitive relays which are operated by the current in the galvanometer branch of a Wheatstone network. A sliding contact is moved along a slide-wire by these relays. A drum, whose axis is parallel to the slide-wire, revolves slowly and carries a chart upon which a pen, carried by the sliding contact arm, traces the movements of the sliding contact. These movements are proportional so the temperature variations, and thus a record of the temperature it made upon the chart.

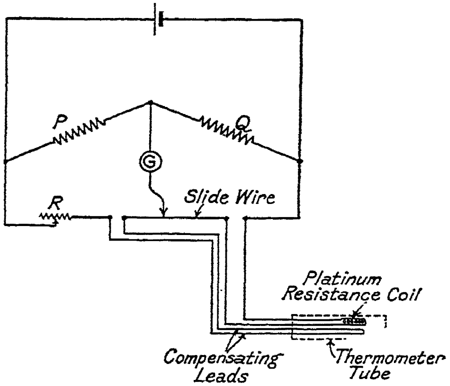
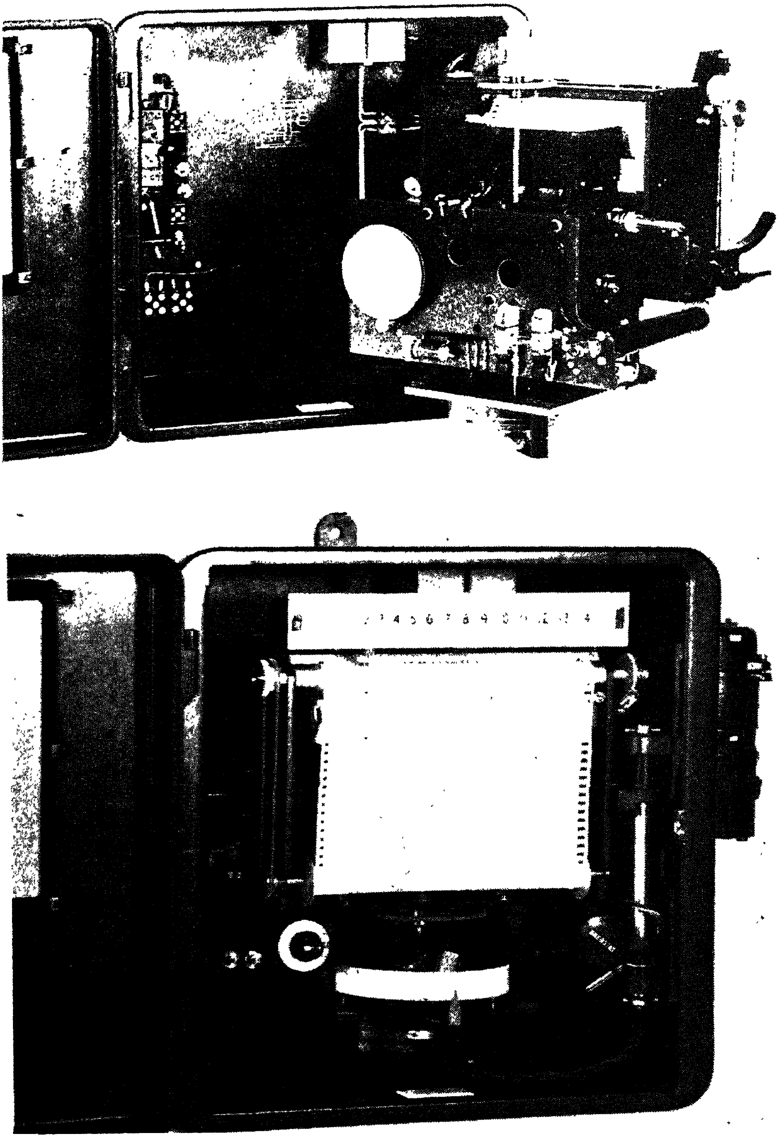


FIG. 305 BRIDGE NETWORK FOR TEMPERATURE MEASUREMENTS

Fig. 306 shows the construction of a modern form of Callendar Recorder made by the Cambridge Instrument Co., Ltd. It consists essentially of a Wheatstone Bridge in which balance is automatically maintained. Up to six simultaneous records can be taken on a monthly chart.

Thermo-electric Pyrometers. These pyrometers, which can measure temperatures up to 1,400° C., are generally used for less precise work than that for which resistance thermometers are used, and have the advantage of being cheaper than the latter. They also have the advantage of following temperature changes with very little time-lag, and are thus suitable for recording comparatively rapid changes. They are very convenient, also, for measuring the temperature at one particular point in a piece of apparatus.

If two wires, of different metals, are joined together at each end so as to form a complete electrical circuit, it is found that a current flows in the circuit if one of the junctions is at a higher temperature



(*Cambridge Instrument Co., Ltd.*)

FIG. 306. CALLENDAR RECORDER

than the other.* This current is the result of an e.m.f. which is set up in the circuit and which is a function of the temperature of the "hot" junction if the other—the "cold" junction—is maintained at a constant temperature. If, instead of the two wires being directly joined together at the "cold" junction, two leads are taken from these ends of the wires to an indicating instrument, the thermoelectric effect is the same, provided the ends are maintained at the same "cold" temperature.

In thermo-electric pyrometers the e.m.f. set up is either measured by a potentiometer or is allowed to send a current through a galvanometer, connected in the circuit as just described, the galvanometer deflection then being proportional to the thermo-electric e.m.f. if the resistance of the circuit is constant. Since the thermoelectric e.m.f. is really dependent upon the *difference* of temperature between the two junctions, it is necessary to keep the temperature of the "cold" junction constant, if the temperature of the "hot" junction is to be accurately measured.

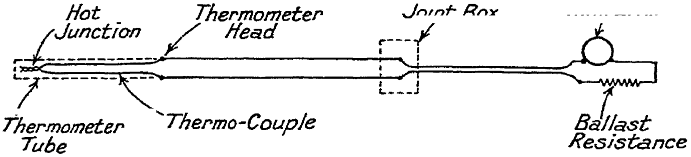


FIG. 307. CONNECTIONS OF THERMO-ELECTRIC PYROMETER

Compensating Leads. The tube containing the thermo-couple itself is necessarily fairly short. This means that the temperature of the two open ends of the couple may be very far from constant. For this reason, "compensating leads" are used to connect these ends to the indicating or measuring instrument. The materials of these leads are chosen so that the thermoelectric e.m.f's set up at their junctions with the open ends of the thermocouple are equal and opposite, and so neutralize one another. The effect obtained is therefore that of removing the cold junction from the thermometer head to the terminals of the indicating instrument, or to some point where the temperature is reasonably constant and can be controlled. In the latter case, flexible copper leads are taken from the constant-temperature point to the indicating instrument. A "ballast " resistance is also included in the circuit. This is a resistor of material having a negligibly small temperature coefficient, and its value is made large compared with the resistance of the rest of the circuit so as to make changes of resistance of the latter with temperature of no importance.

The complete circuit is as shown in Fig. 307. If a high-resistance galvanometer is used, the ballast resistance may be unnecessary. This method of use is most suitable when the temperature to be measured is very high. The absolute constancy of the temperature of the cold junction is not then of such importance as when the temperature to be measured is lower. For measurements of lower temperatures the cold junction may be contained within a vacuum flask, and maintained at a known low temperature, the connections then being as in Fig. 308. Crushed ice may be used in the cold junction flask. Alternatively the reference junction may be maintained at a steady—though higher—temperature in a small, electrically heated chamber.

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* The "Seebeck" effect.

Thermo-electric E.M.F.s. The law of e.m.f. produced with variation of the temperature difference between hot and cold junctions may be written

$$e = \alpha (T - T_0) + \beta (T^2 - T_0^2) \quad . \quad . \quad . \quad . \quad (255)$$

where e = thermo-electric e.m.f. in volts

T and T_0 = the *absolute** temperatures of the hot and cold junctions respectively

and α and β are constants which depend upon the metals forming the couple.

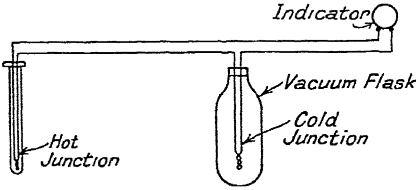


Fig. 308

This law, obtained experimentally, holds for most thermo-couples. It may be extended as follows—

$$e = (T - T_0)[\alpha + \beta (T + T_0)]$$

or

$$e = (T - T_0)\left[\alpha + 2\beta \frac{(T + T_0)}{2}\right]$$

Expressing the temperatures in degrees centigrade (on the ordinary scale instead of as absolute values), we have,

$$e = (t - t_0)\left[\alpha + 2\beta \frac{(t + 273 + t_0 + 273)}{2}\right]$$

or

$$e = (t - t_0)[\alpha + 2\beta \times 273 + 2\beta t_{av}] \quad . \quad . \quad . \quad (256)$$

where t_{av} is the average temperature, in degrees centigrade, of the hot and cold junctions. Writing γ for $(\alpha + 2\beta \times 273)$, we have for the e.m.f. per degree centigrade temperature difference between the junctions,

$$\frac{e}{t - t_0} = \gamma + 2\beta t_{av} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

Wedmore and Onslow (Ref. (2)) give the thermo-electric e.m.f's of a number of metals when used as a thermo-junction with lead—which has no thermo-electric effect—as the other metal of the junction. The e.m.f's in micro-volts per degree centigrade, are given in the form of the above equation, taking lead as zero.

For example, compared with lead, the e.m.f's of iron, and copper, in micro-volts per degree centigrade, are given as

Iron . . .	$+ 17{\cdot}34 - 0{\cdot}0487t_{av}$
Copper . .	$+ 1{\cdot}36 + 0{\cdot}0095t_{av}$

---

* Absolute temperatures on the centigrade scale are obtained by adding 273° to the centigrade temperature.

Thus, an iron-copper couple would give $15\cdot98 - 0\cdot0582t_{av}$ microvolts per degree centigrade.

It can be seen from this expression that when

$$t_{av} = \frac{15\cdot98}{0\cdot0582} \quad 275^\circ \text{ C.}$$

there will be no thermo-electric e.m.f.

Below 275° C. current flows, therefore, from the copper to the iron, at the hot junction, and above 275° C. from iron to copper.

The mean temperature at which the thermo-electric e.m.f. per degree C. is zero is called the "neutral" temperature.

The law

$$\log_{10} e = A \log_{10} T + B \quad . \quad . \quad . \quad (258)$$

where e = e.m.f. in microvolts

T = temperature of the test junction in degrees Centigrade (cold-junction temperature being 0° C.)

and A and B = constants depending on the materials used in the couples, is used for thermo-electric work. An accurate formula for Copper-constantan couples over the range 0° to 350° C. is $e = 74\cdot672\ T - 13{,}892(1 - \varepsilon^{-0\cdot00261T})$ where e = e.m.f. in microvolts and ε is the base of natural logarithms.

The e.m.f. produced for any given temperature difference between hot and cold junctions is constant, at a given hot-junction temperature, for any pair of metals, regardless of the size of the two wires and of the areas in contact at the junctions.

Metals used for Thermo-junctions. Two classes of metals are used in thermo-junctions—base metals and rare metals. The advantages of the base metal couples are that such metals are comparatively cheap and the couples, therefore, may be made more robust than when expensive metals are used. They have also a high thermo-electric e.m.f. The rare metals—usually platinum and platinum alloys—are more durable for a given size, and can withstand higher temperatures, than the base metals. Their thermo-electric e.m.f's are somewhat small (about one-fifth as great as those with base metals) but are more constant than those obtained with the base metals.

Amongst the latter are couples of copper-constantan, silver-constantan and iron-constantan. These can be used for temperature measurements up to 500° C., 700° C., and 900° C., respectively, and their thermo-electric e.m.f's in millivolts at 500° C. (the cold junction being at 0° C.) are approximately 27·8, 27·6, and 26·7 respectively. The Hoskins couple—a nickel wire with a wire made of an alloy of 90 per cent nickel and 10 per cent chromium—is also used up to 1,100° C., its thermo-electric e.m.f. at 500° C. being approximately 10·0 millivolts when the cold junction is at 0° C.

Platinum with platinum-rhodium (alloy), and platinum with platinum-iridium, are rare-metal couples which are commonly used for temperatures (continuous) up to 1,400° C. and 1,000° C. respectively, their thermo-electric e.m.f's at 500° C. being 4·4 millivolts and 7·4 millivolts respectively.

The laws for these last two couples are—

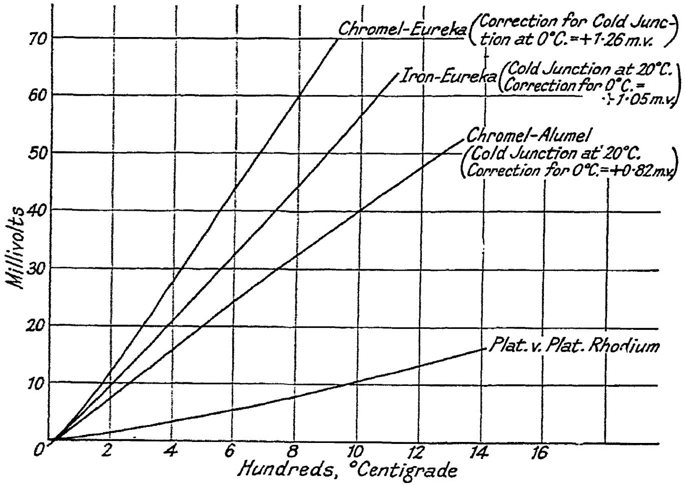
For platinum—platinum-iridium 10 per cent,

$$\operatorname{Log}_{10} e = 1{\cdot}10 \log_{10} T + 0{\cdot}89 \quad . \quad . \quad . \quad (259)$$

For platinum—platinum-rhodium 10 per cent,

$$\operatorname{Log}_{10} e = 1{\cdot}19 \log_{10} T + 0{\cdot}52 \quad . \quad . \quad . \quad (260)$$

e being in microvolts and T in degrees centigrade. The temperature of the cold junction is assumed to be 0° C.



(Foster Instrument Co.)

FIG. 309. VOLTAGE-TEMPERATURE CURVES FOR THERMO-COUPLES

For temperature measurements up to about 80° C. a bismuth-antimony couple is often used.

A graph, showing the results of tests upon various thermo-couples by the Foster Instrument Co., is given in Fig. 309.

Construction and Use of Thermo-electric Pyrometers. The construction of the thermo-electric pyrometer is illustrated in Fig. 310, which shows a pyrometer manufactured by the Cambridge Instrument Co. for high temperature measurements (up to 900° C.) in which the couple is iron-constantan.

To protect the thermo-couple, a sheath, surrounding the hot junction, is almost always used. This may be of steel, or nickel-chrome, for base-metal couples employed at the lower temperatures, but is usually of porcelain or quartz, in the case of the rare-metal pyrometers for use at the highest temperatures. Alundum (fused alumina and fireclay) sheaths are sometimes used also at the very high temperatures. In rare-metal thermo-couples the wires forming the couple may be about 0·02 in. in diameter, but in the base metal couples the wire diameter is usually about 0·1 in. The junction itself may be formed by either twisting or fusing the two wires together.

Fig. 311 shows a thermo-electric pyrometer manufactured by the Cambridge Instrument Co. for the measurement of the temperatures of surfaces. The thermo-couple takes the form of a thin strip (about 0·01 in. thick and 0·25 in. wide), consisting of pieces of two dissimilar metals welded together end to end. This strip is flexible and, if pressed lightly upon a hot surface such as a heated metal roller, it conforms to the shape of the latter and makes intimate contact with it.

The e.m.f. set up in the strip is measured on an indicator whose scale is graduated directly in degrees centigrade.

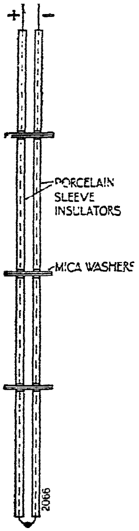
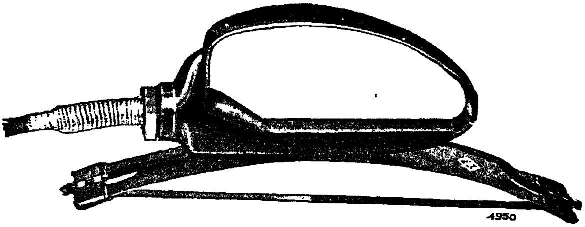


FIG. 310
THERMO-ELECTRIC PYROMETER



Cambridge Instrument Co., Ltd.)

FIG. 311. SURFACE PYROMETER

As previously stated, thermo-electric e.m.f's may be measured either directly by a galvanometer, or by a potentiometer. The former is the more convenient for most practical purposes, since the galvanometer may be calibrated—for a given temperature of the cold junction—to give temperatures directly. If the galvanometer has a high resistance, changes of resistance of the thermo-couple circuit with temperature will introduce no appreciable errors. Calibration of the galvanometer may be by comparison with another instrument (already calibrated), or may be carried out by observing the deflection for a number of known temperatures, such as the boiling points or melting points of various substances.*

This method of using the thermo-couple may be applied either to indication, or to the continuous recording, of temperatures.

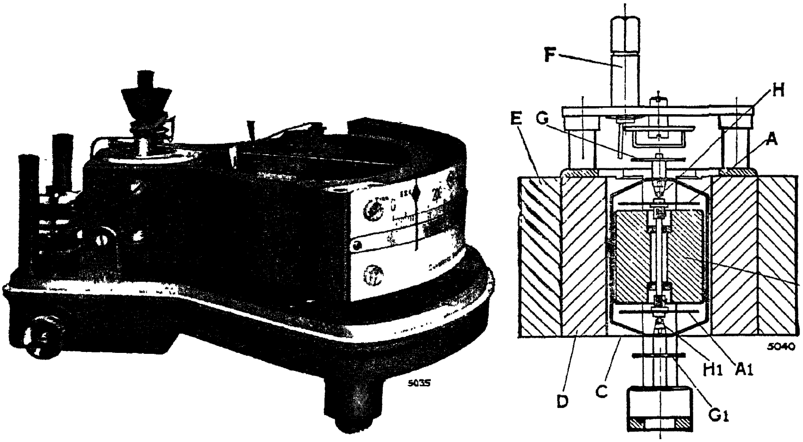
---

* For tables of such temperatures, see Ref. (2).

The potentiometer method, in which the thermo-electric e.m.f. is measured by comparison with the e.m.f. of a standard cell, is, however, more precise than the above method.

Special potentiometers for such purposes are manufactured by various scientific instrument manufacturers, and have already been mentioned in Chapter VIII.

Fig. 312 shows the construction of an indicator manufactured by the Cambridge Instrument Co. for use with their thermo-electric pyrometers. The moving coil, *C*, of the indicator is suspended magnetically in the field of the magnet of which *D* is the pole block. *A* and A_1 are iron discs, *B* is an iron core, *G* and G_1 are control springs, and *H* and H_1 pivots. The magnetic suspension ensures resilience and protects the pivots against the effects of vibration.



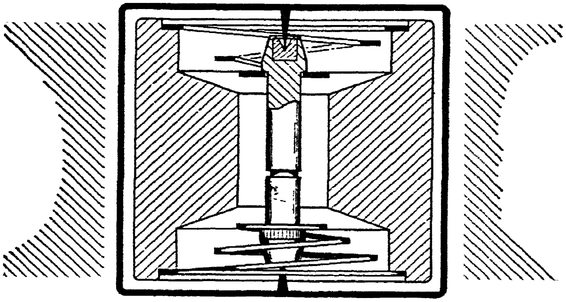
(*Cambridge Instrument Co., Ltd.*)

FIG. 312. TEMPERATURE INDICATOR

An indicator for the same purpose, manufactured by the Foster Instrument Co., is illustrated in Fig. 313. The "Resilia" patent design of the moving system protects the latter against vibration by ensuring that the pivots of the coil and the jewels shall not be separated by such vibration. The pivots are turned inwards from the coil and rest in conical jewels carried by a very light staff. This staff is held in position, relative to the core, by light springs, so that when the instrument is subjected to mechanical vibration the coil, pivots, jewels, and staff move together and the pivots are uninjured.

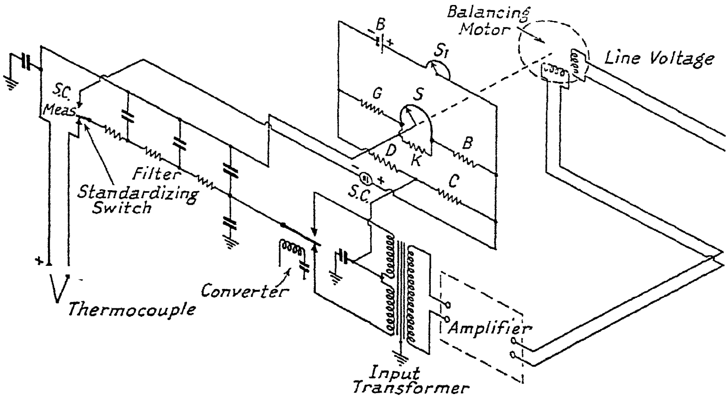
For the recording of temperature variations, a recorder, similar in construction to that already described, in dealing with resistance thermometers (see page 502), is used. Instead of the moving contact, which carries the pen, moving over the slide-wire of a Wheatstone bridge network, it now moves over a slide-wire in a potentiometer circuit, the latter being included in the recorder itself.

Fig. 314 shows the principle of operation of a quick-response recorder which employs an electrical balancing system with a potentiometer circuit. The system consists of two main elements; a converter-amplifier unit and a two-phase balancing motor. When the potentiometer becomes unbalanced due to a change of temperature, which alters the thermo-couple e.m.f., the converter contacts alternately connect the unbalance potential across the primary winding of the amplifier input transformer. The current induced



(*Foster Instrument Co.*)

FIG. 313. "RESILIA" INDICATOR MOVEMENT

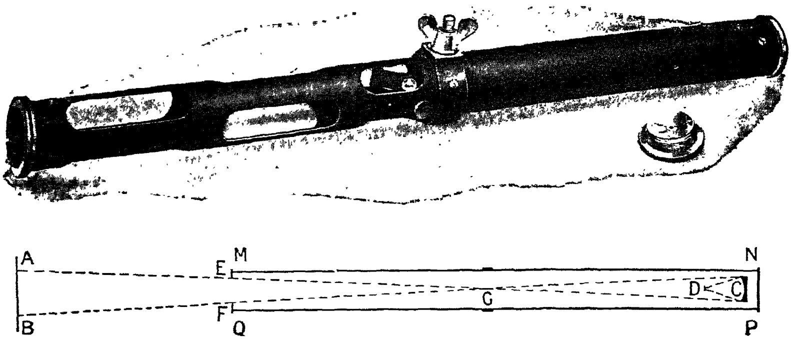


(*Leeds & Northrup Co.*)

FIG. 314. "SPEEDOMAX" POTENTIOMETER CIRCUIT FOR MEASURING TEMPERATURE WITH THERMO-COUPLES in the transformer secondary winding is amplified to energize the control winding of the balancing motor. This starts the motor which moves the contact along the measuring slide-wire, and the indicating pointer and recording pen across the scale and chart, until balance is obtained when the motor torque becomes zero.

Another self-balancing potentiometer recorder for use with a thermo-couple circuit is made by Elliott Brothers (London), Ltd. and is fully described in their publication G.126A.

Radiation Pyrometers. For temperatures above about 1,200° C. it is convenient, if not absolutely necessary, to employ some method of measurement in which the measuring apparatus is not subjected to the full heating effect of the source whose temperature is to be measured. Pyrometers for such measurements depend, for their action, upon the heat radiated from the source, and are therefore called "radiation" pyrometers. Their upper limit of measurement depends upon the possibility of calibrating the instruments at very high temperatures, using tables of standardization temperatures such as already mentioned on page 498.



(*Foster Instrument Co.*)

FIG. 315. FOSTER FIXED-FOCUS TYPE OF RADIATION PYROMETER

Radiation pyrometers are either of the Foster fixed-focus type or of the Féry variable-focus type.

FIXED-FOCUS TYPE. Fig. 315 shows a pyrometer of this type. It consists essentially of a long tube *MNPQ* containing a concave mirror *C* which is adapted to focus the heat rays which pass into the tube (from the source *AB*, through the narrow aperture *EF*) on a sensitive thermo-couple *D*. The image formed at *D* is a "heat image," and is always in focus, without adjustment, over a wide range of distances between the pyrometer and the source of heat. This is brought about by making the tube long and the aperture small.

Suppose the solid angle subtended at *G* is ϕ; then, provided the radiating surface of the hot body is large enough to subtend this angle ϕ at G at any given distance of the pyrometer tube from the hot body, the temperature measured will be the true one. If, however, the distance between the hot body and pyrometer is so great that the total radiating surface subtends an angle less than ϕ at G, the measured temperature will be low.

The e.m.f. produced in the thermo-couple is measured as previously described. If a galvanometer or millivoltmeter is used for this purpose, it may be calibrated to read temperatures directly.

The Stefan-Boltzmann Law of Radiation states that the energy radiated by a heated black body (i.e. a body for which the radiated energy for a given temperature is a maximum) is proportional to the fourth power of its absolute temperature.

If T_2 is the absolute temperature of the hot body and T_1 is the absolute temperature of a colder body near to it, then the law of radiation is

$$W = k\,(T_2^4 - T_1^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (261)$$

where W is the energy, per sq. cm. per sec., received by the cooler body and k is a constant. The experimental value of k is $1{\cdot}279 \times 10^{-12}$ gm.-cal. per sec. per sq. cm.

Again, the energy W' received by the body at absolute temperature T_1, from a body at temperature T_3, is given by

$$W' = k\,(T_3^4 - T_1^4)$$

Thus,
$$\frac{W'}{W} = \frac{T_3^4 - T_1^4}{T_2^4 - T_1^4}$$

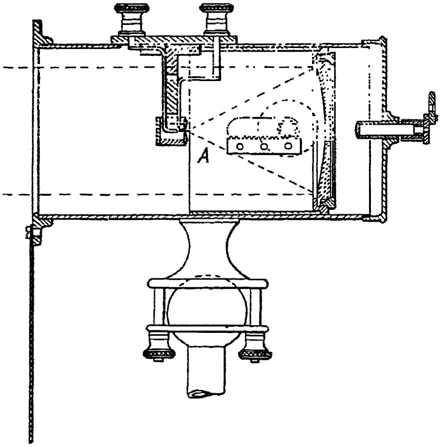
If T_1 is small compared with T_2 and T_3, we have, very approximately,

$$\frac{W'}{W} = \frac{T_3^4}{T_2^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (262)$$

The receiver at D is blackened so as to approach as nearly as possible to the theoretical "black body," and absorbs all, or almost all, of the radiations falling upon it. Its temperature rise is thus proportional to the fourth power of the absolute temperature of the hot body, whose temperature is to be measured. For this reason the scale of the indicating instrument used with radiation pyrometers is cramped at the lower end, and open at the upper end.

Féry Variable-focus Pyrometer. The construction of the Féry variable focus pyrometer, manufactured by the Cambridge Instrument Co., is shown in Fig. 316. A concave mirror, with an opening at its centre, through which the hot body may be observed through an eyepiece, is used to focus the heat radiations upon a sensitive thermo-couple A. The focusing is carried out by means of a rack and pinion, which moves the mirror as required. A small mirror is placed directly behind the thermo-couple. The position of the concave mirror is adjusted, by the focusing arrangement, until the two images of the hot body, which are reflected by the concave mirror upon the small mirror behind the thermo-couple, overlap the couple itself. The distance of the pyrometer from the hot body is not important, provided it is small enough for this overlap to be obtained. As in the fixed-focus instrument, the e.m.f. set up in the thermo-couple is measured by a sensitive moving-coil instrument whose scale may be graduated in temperatures directly.

Use of Radiation Pyrometers. In both of the above pyrometers the terminals which form the cold junction of the couple must be protected from the heat from the hot body whose temperature is to be measured. Base metal couples, such as copper-constantan, are usually employed in both cases.



(*Cambridge Instrument Co., Ltd.*)
FIG. 316. FÉRY VARIABLE-FOCUS PYROMETER

Although the laws of radiation stated above only hold, strictly, for perfectly black bodies, only small errors are usually introduced, since the departure of the actual hot bodies from perfect black body conditions is in most cases for which such pyrometers are used, slight.

Radiation pyrometers are used for temperatures above 600° C., since at lower temperatures errors may be introduced by the fact that the temperature of the pyrometer itself may not be negligible compared with that of the hot body. They are often used for the measurement of furnace temperatures. If viewed through a small opening, a furnace which is surrounded by walls which are at approximately the same temperature as itself behaves as a perfectly black body.

Such pyrometers may be used for recording purposes by using the thermo-couple in conjunction with a recorder of the type described earlier in the chapter.

The advantages of these pyrometers are their high upper limit of temperature measurement, and their comparative independence upon the distance of the instrument from the hot body. They have, however, the disadvantage of having to be calibrated individually, using hot bodies whose temperatures are known.

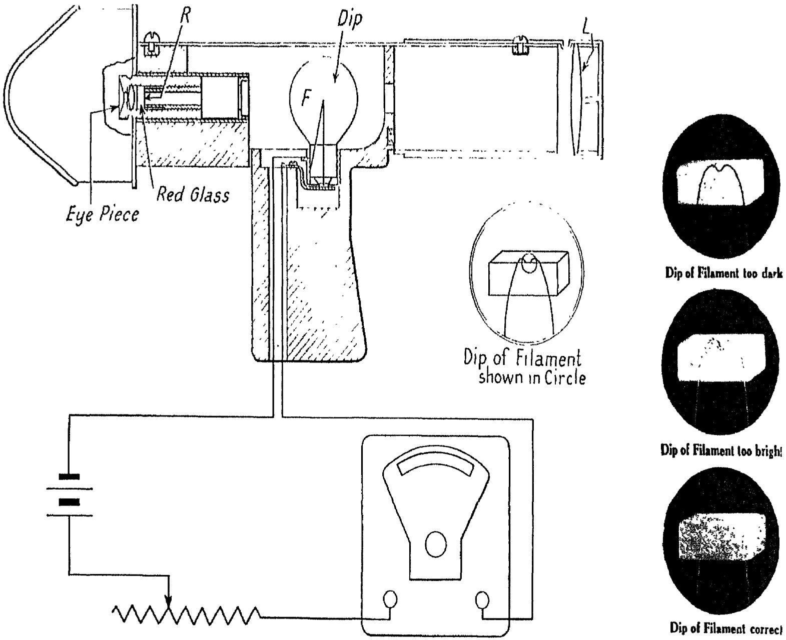
Optical Pyrometers. The optical system of the "disappearing filament" type of optical pyrometer as manufactured by the Leeds and Northrup Co. is shown in Fig. 317. *F* is a standard lamp, and *L* is a lens. This lens focuses light, radiated from the hot body, upon the plane containing the lamp filament. The light from the hot body, and that from the lamp, are both viewed through a piece of red glass *R* which renders the comparison of these monochromatic, i.e. using light of one wave-length only, and not the whole of the light. The intensity of the light of any one wave-length depends upon the temperature of the hot body. In this case red light is used for the comparison.

The current through the lamp filament is adjusted until the brightness of the filament is equal to that of the source, as viewed through *R*. When these two brightnesses are equal the outline of the filament disappears into the surrounding field of light from the hot body. Otherwise the filament shows either brighter or duller than the surrounding light, according as its temperature is greater or less than that required for equality of brightness.

The current through the lamp for equality of brightness is measured either by an ammeter, or galvanometer, whose scale gives the temperature of the hot body directly as a result of previous calibration against hot bodies of known temperatures, or by including the lamp in one arm of a Wheatstone bridge network. In the Leeds and Northrup instrument shown the former method is used. The bridge network method is adopted in an optical pyrometer manufactured by Messrs. H. Tinsley & Co. The lamp current is adjusted by variation of the current supplied to the bridge network, and the out-of-balance currents at the increased lamp temperatures (above the temperature for which the network is balanced) are indicated on the bridge galvanometer and give a measure of the temperature.

In the calibration of the standard lamp the hot body is assumed to be perfectly "black." Even if this condition is not fulfilled, the errors caused thereby are less in the case of an optical pyrometer, when monochromatic light is used for comparison purposes, than when radiation pyrometers are used. Another advantage is that the hot body sighted need not be large.

These pyrometers, which are used for work of the highest precision, may be used for temperatures up to about 3,500° C., but above 1,400° C. an absorption screen is interposed between the hot



(*Leeds & Northrup Co.*)

FIG. 317. OPTICAL PYROMETER—DISAPPEARING-FILAMENT TYPE

body and the lamp as the latter should not be run above this temperature (1,400° C.).

The measurement of the increase in temperature of coils in electrical machinery and apparatus by measurement of the increase of resistance and otherwise, have not been discussed in the foregoing pages, but such methods are discussed in several of the publications mentioned below.

BIBLIOGRAPHY AND REFERENCES

(1) *Resistance and Its Measurement*, E. F. Northrup.

(2) *Electrical Engineers Data Book*, Vol. II, Wedmore and Onslow.

(3) *Electrical Measuring Instruments*, D. J. Bolton.

(4) H. L. Callendar, *Phil. Mag.* (July, 1891; February, 1899; December, 1899), *Proc. Roy. Inst.*, Vol. XVI (1901).

(5) "On the Boiling-point of Sulphur," H. L. Callendar and E. H. Griffiths, *Proc. Roy. Soc.*, Vol. XLIX.

(6) "The Measurement of High Temperatures and Stefan's Law," C. Féry, *Journ. de Phys.* (September, 1904).

(7) "On the Determination of some Boiling- and Freezing-points by means of the Platinum Thermometer," E. H. Griffiths, *Proc. Roy. Soc.*, Vol. XLVIII.

(8) *High Temperature Measurements*, H. Le Chatelier and O. Bondonard (translated by G. K. Burgess).

(9) "Optical Pyrometry," by C. W. Waidner and G. K. Burgess, *Bull. No. 2, Bur. Stands.* (1905).

(10) "Thermo-electric Couple Potentiometer," *The Electrician* (31st January, 1908).

(11) "Modern Methods of Measuring Temperature," by R. S. Whipple, *The Mechanical Engineer* (August, 1913).

(12) "Temperature Measurement with the Einthoven Galvanometer," by F. Adcock and E. H. Wells, *Jour. I.E.E.*, Vol. LXI, p. 1115.

(13) "The Loading of Transformers on a Thermal Basis with Special Reference to Temperature Indicators," by T. A. Long, *Jour. I.E.E.*, Vol. LXVIII, p. 1516.

(14) "An Analysis of Heating Tests on Electrical Machines," by E. Hughes, *Jour. I.E.E.*, Vol. LXVIII, p. 912.

(15) "The Measurement of Temperature by Thermo-couples," *ASEA Journal*, Vol. VI, No. 6 (1929).

(16) "The Rise and Distribution of Temperature in Small Electrical Machines," by E. Hughes, *Jour. I.E.E.*, Vol. LXII, p. 628.

(17) "Heating Time Constants: Their Variation in Forced Cooled Machinery," D. H. Wills, *World Power*, Vol. VIII, No. XLIII.

(18) "The Peltier Effect," T. F. Wall, *Electrician* (16th March, 1928).

(19) *Methods of Measuring Temperature*, Ezer Griffiths.

(20) *Pyrometers*, Ezer Griffiths.

(21) *Pyrometers*, W. P. Wood and J. M. Cook.

CHAPTER XIV

EDDY CURRENTS

The Nature of Eddy Currents. The term "eddy currents" is applied to those electric currents which circulate within a mass of conducting material when the latter is situated in a varying magnetic field. The conducting material may be considered as consisting of a large number of closed conducting paths, each of which behaves like the short-circuited winding of a transformer of which the varying magnetic field is the working flux.

"Eddy" e.m.f's are induced in these elemental paths, by the varying magnetic field, giving rise to the eddy currents. Fig. 318 illustrates this induction.

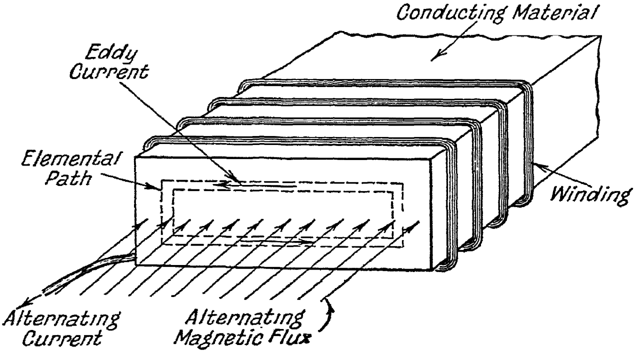


FIG. 318. INDUCTION OF EDDY CURRENTS

Effects of Eddy Currents. Eddy currents result in a loss of power, with consequent heating of the material, and the magnitude of this power loss is often a matter of considerable importance in electrical engineering.

The eddy currents, since they flow in closed paths in the material —usually iron—have an axial magnetic field of their own which is in opposition to the inducing magnetic field, and so reduces its strength. The reduction is greatest at the centre of the core since the eddy currents in *all* the elemental paths, from the centre of the material to the outside surface, are effective in producing the opposing magnetic field there. This results in a flux distribution which is not uniform, the flux density in the outer portions of the conductor being greater than that at its centre, which is screened by the eddy currents. There is thus a reduction of the effective cross-section of the core.

The effect of eddy currents upon the flux distribution is chiefly of importance in transformers and other apparatus where the iron used would otherwise be worked at uniform flux density. Not only is the distribution of flux in such cores affected by eddy currents, but its magnitude, for a given value of magnetizing current, is obviously reduced thereby, and the phase of the resultant flux is not the same as that of the magnetizing current.

Eddy Current Loss in Cylindrical Conductors. Fig. 319 represents ortion of a conductor which carries an alternating magnetic flux in a direction parallel to its axis. Eddy currents will flow in elemental circular paths, as shown.

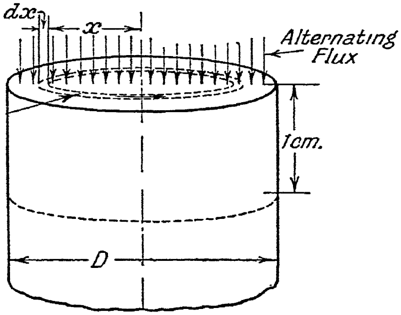


FIG. 319. EDDY CURRENTS IN A CYLINDRICAL CONDUCTOR

Assume—as an approximation—that the flux distribution is uniform and that it alternates according to the law

$$B = B_{max} \sin \omega t$$

where B is the flux density at any time t and

$$\omega = 2\pi \times \text{frequency} = 2\pi f$$

Consider, first, 1 cm. axial length of the conductor. The flux enclosed by an elemental circular path, of radius x and radial width dx, at any instant is

$$\pi x^2 \, . \, B$$

Thus, the e.m.f. induced in this path is given by

$$e = \frac{d(\pi x^2 \, . \, B)}{dt} \times 10^{-8} \text{ volts}$$
$$= \pi x^2 \, . \, B_{max} \, \omega \cos \omega t \times 10^{-8}$$
$$= 2\pi^2 x^2 f B_{max} \cos \omega t \times 10^{-8}$$

The r.m.s. value of this voltage is

$$E_x = \sqrt{2}\pi^2 x^2 f B_{max} \times 10^{-8}$$

If ρ is the resistivity of the material of the conductor (in ohms per centimetre cube), the resistance of the elemental path is

$$r_x = \frac{2\pi x \times \rho}{dx \times 1} \text{ (for an axial length of 1 cm.)}$$

Neglecting its inductance, the r.m.s. value of the eddy current in this path is

$$i_x = \frac{\sqrt{2}\pi^2 x^2 f B_{max} \times 10^{-8}}{\frac{2\pi\, x\, \rho}{dx}}$$

$$\therefore\ i_x = \frac{\pi x f B_{max} dx}{\sqrt{2}\rho} \times 10^{-8} \text{ amp.}$$

The eddy current loss in watts in this elemental path is thus

$$i_x^2 r_x = \frac{\pi^2 x^2 f^2 B_{max}^2 dx^2 \times 10^{-16}}{2\rho^2} \times \frac{2\pi x \rho}{dx}$$

$$= \frac{\pi^3 \,.\, x^3 f^2 B_{max}^2}{\rho}\, dx \times 10^{-16} \text{ watts}$$

The total loss per centimetre axial length, obtained by integration between the limits $\frac{D}{2}$ and zero, is

$$P = \int_{x=o}^{x=\frac{D}{2}} \frac{\pi^3 x^3 f^2 B_{max}^2}{\rho} \times 10^{-16} \,.\, dx$$

or

$$P = \frac{\pi^3 f^2 B_{max}^2 D^4}{64\rho} \times 10^{-16} \text{ watts per cm. axial length} \quad . \quad (263)$$

If the flux wave-form is not sinusoidal, but has a form-factor k_f instead of $\frac{\pi}{2\sqrt{2}}$ (as in the case of the sine wave), the above expression becomes

$$P = \frac{\pi k_f^2 \,.\, f^2 B_{max}^2 D^4}{8\rho} \times 10^{-16} \quad . \quad . \quad (264)$$

Now, the eddy e.m.f. is independent of the length of the cylinder (provided the flux is parallel to the axis throughout), and since the eddy current loss in watts $= i_x^2 r_x = \frac{E_x^2}{r_x}$ this loss is proportional to $\frac{1}{r_x}$.

Thus, eddy current loss $\propto \frac{1}{r_x}$, $\propto \frac{1}{\frac{2\pi x \rho}{dx \,.\, l}} \propto l$

where l is the total axial length of the cylinder and r_x' the resistance of an elemental path of radius x, radial thickness dx, and axial length l.

Summarizing, the total eddy current loss in a cylindrical conductor of any given diameter and material, is directly proportional to its axial length.

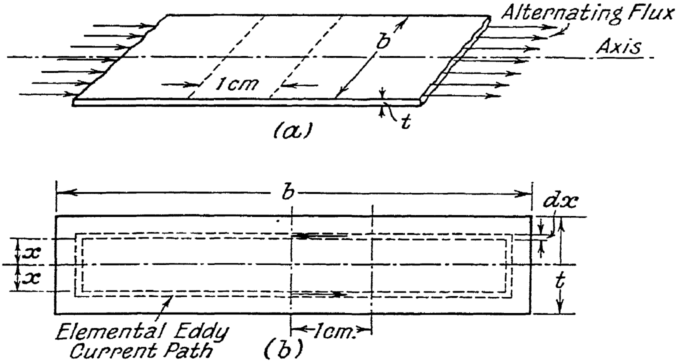


FIG. 320. EDDY CURRENTS IN THIN SHEETS

The eddy current loss per cubic centimetre of the conductor is, therefore, in the general case,

$$P = \frac{\dfrac{\pi k_f^2 \,.\, f^2 \,.\, B_{max}^2 D^4}{8\rho}}{\dfrac{\pi D^2}{4}} \times 10^{-16}$$

or

$$P = \frac{k_f^2 \,.\, f^2 \,.\, B_{max}^2}{2 \,.\, \rho} D^2 \times 10^{-16} \text{ watts per c.c.} \quad . \quad (265)$$

In general, the eddy current loss per cubic centimetre, for a given material and form-factor of the flux-wave is proportional to

$$f^2 B_{max}^2 D^2$$

Eddy Current Loss in Thin Sheets. Fig. 320(*a*) shows a thin plate, of thickness t and width b, the thickness being small in comparison with the width. Suppose that this plate carries an alternating magnetic flux $B = B_{max} \sin \omega t$, and that the flux density is uniform over the cross-section, the flux running parallel to the axis of the plate as shown. Eddy currents will flow in the plate in elemental paths, as shown in Fig. 320(*b*).

Consider 1 cm. axial length, and also 1 cm. of the width of the plate. In the case of an elemental path whose long sides are each distant x cm. from the centre of the plate, and whose depth is dx (Fig. 320(b)), the flux enclosed within 1 cm. width of this path is, at any instant,

$$2x \times 1 \times B = 2xB_{max} \sin \omega t$$

Thus, the e.m.f. induced in this portion of the elemental path is

$$e_x = \frac{d}{dt}(2xB_{max} \sin \omega t) \times 10^{-8}$$

or

$$e_x = 2xB_{max}\omega \cos \omega t \times 10^{-8} \text{ volts}$$

Its r.m.s. value is

$$E_x = \frac{4\pi fxB_{max}}{\sqrt{2}} \times 10^{-8} \text{ volts}$$

If ρ is the resistivity of the material of the plate (in ohms per centimetre cube), the resistance of this portion of the path is $\frac{2 \times \rho}{dx \times 1}$ or $\frac{2\rho}{dx}$ ohms. Then, neglecting the end portions of the path (parallel to the edges of the plate), the eddy current in the path is

$$i_x = \frac{E_x}{\frac{2\rho}{dx}} = \frac{4\pi fxB_{max}}{\frac{2\sqrt{2}\rho}{dx}} \times 10^{-8} \text{ amp.}$$

or

$$i_x = \frac{\sqrt{2}\pi fxB_{max}dx}{\rho} \times 10^{-8} \text{ amp.}$$

Thus, the eddy current loss in this path per centimetre length and breadth of the plate is

$$P_x = \frac{2\pi^2 f^2 x^2 B_{max}{}^2 dx^2 \times 10^{-16}}{\rho^2} \times \frac{2\rho}{dx} \text{ watts}$$

$$= \frac{4\pi^2 f^2 x^2 B_{max}{}^2\, dx}{\rho} \times 10^{-16}$$

Integrating between the limits $x = \frac{t}{2}$ and $x = 0$, we have, for the total loss per centimetre length and breadth of the plate,

$$P = \int_{x=0}^{x=\frac{t}{2}} \frac{4\pi^2 f^2 B_{max}{}^2}{10^{16} \cdot \rho} x^2 \,.\, dx$$

or

$$P = \frac{\pi^2 f^2 B_{max}{}^2}{10^{16} \rho} \times \frac{t^3}{6} \quad . \quad . \quad . \quad . \quad . \quad (266)$$

watts per cm. length and breadth (i.e. for t cu. cm.).

Thus the loss per cubic centimetre is

$$P = \frac{\pi^2 f^2 B_{max}{}^2 t^2}{6\rho \times 10^{16}} \text{ watts} \quad . \quad . \quad . \quad . \quad (267)$$

If the wave-form of the flux is not sinusoidal, but has a form-factor k_f, the loss per cubic centimetre is

$$\frac{4k_f^2 f^2 B_{max}^2 t^2}{3\rho \times 10^{16}} \text{ watts} \quad . \quad . \quad . \quad . \quad (268)$$

Influence of Eddy Currents upon the Phase and Magnitude of the Flux in an Iron Core. Consider an iron plate in which a magnetic flux, parallel to the axis of the plate, exists and alternates according to a sinusoidal law.

Assume, also, for simplicity in the following discussion, that all the eddy currents are in the same phase relative to the working magnetic flux in the iron core—and therefore relative also to the eddy e.m.f. Although this assumption is not correct, owing to the different phase angles of the various eddy current paths in the core, it will suffice for the discussion of the effect of the eddy currents upon the core flux. Referring to Fig. 321 (*a*), let i represent, in magnitude and phase, the resultant of all the eddy currents in a portion of the iron core 1 cm. wide and of 1 cm. axial length (see Fig. 320). This resultant is actually the vector resultant of the eddy currents in the core and could be obtained by constructing a polygon, as in Fig. 321 (*b*), if the magnitudes and phases of the individual eddy currents were known. The closing line of the polygon gives the resultant eddy current.

It will be observed that the vector diagram of Fig. 321 (*a*) is similar to that of a transformer. The similarity is explained by the fact that the iron core, with its magnetizing winding, does actually constitute a transformer, the secondary "winding" of which is permanently short-circuited and exists *inside* the core itself. This secondary "winding" is, of course, the eddy current path in the core.

The vector ϕ represents the working flux in the core. It is the flux which the magnetizing current I_m—supposed to be flowing in a coil of N turns wound on the iron core—would produce if no eddy currents were present. The eddy e.m.f.'s, represented by the single vector E_e, will lag 90° behind ϕ as shown. The resultant eddy current i lags behind this voltage, owing to the inductance of the eddy current paths, and has, in phase with it, a flux ϕ_e—produced by the eddy currents. Thus the resultant flux in the iron is the vector sum of ϕ and ϕ_e, and is represented by ϕ_r.

The effect of the eddy currents upon the flux in the iron is, therefore, twofold; they both reduce the working flux—ϕ_r being less than ϕ—and also cause the resultant flux to lag behind the magnetizing current.

The vector E represents the e.m.f. required to overcome the inductive volt drop in the magnetizing winding—i.e. E is the vector difference between the e.m.f. applied to this winding and the voltage drop in the resistance of the winding. On account of the eddy current i, the magnetizing winding must carry a current i'—in phase opposition to i—in addition to the magnetizing current I_m. (The current required to supply the power lost in hysteresis in the iron is omitted in the diagram to avoid complication.) The current i' has two components, i_m in phase with ϕ and i_e in phase with E. The former compensates for the demagnetizing effect of the eddy currents, and maintains the flux ϕ at the same value as it would have if there were no such currents, while the latter supplies the power which is lost in eddy current heating of the iron.

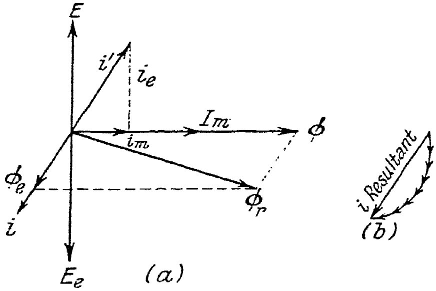


FIG. 321

If there are n turns per centimetre axial length on the magnetizing winding, the e.m.f. induced in this winding per centimetre breadth of the core is

$$\frac{\sqrt{2}\pi ftnB}{10^8} \cdot \text{volts (r.m.s.)}$$

As already seen, the power loss per centimetre axial length and breadth of the core plate (of thickness t cm.) is

$$\frac{\pi^2 f^2 B_{max}{}^2 t^3}{6\rho \times 10^{16}} \text{ watts (assuming uniform flux distribution)}$$

Thus, the component i_e of the current in the magnetizing winding is given by

$$i_e = \frac{\dfrac{\pi^2 f^2 B_{max}{}^2 t^3}{6\rho \times 10^{16}}}{\dfrac{\sqrt{2}\pi ftnB_{max}}{10^8}} = \frac{\pi f B_{max} t^2}{\sqrt{2} \times 6\rho n \times 10^8} \text{ amp.}$$

or

$$i_e = \frac{0{\cdot}118\pi f B_{max} t^2}{\rho n \times 10^8} \text{ amp.} \quad . \quad . \quad . \quad . \quad . \quad (269)$$

If all the eddy currents are assumed to be in phase, the resultant eddy current i is given by

$$\int_0^{\frac{t}{2}} i_x dx = \int_0^{\frac{t}{2}} \frac{\sqrt{2}\pi f x B_{max}}{10^8 \rho} dx$$

$$= \frac{\sqrt{2}\pi f B_{max}t^2}{8\rho \times 10^8} \text{ amp.}$$

Thus $$i' = \frac{\sqrt{2}\pi f B_{max}t^2}{8\rho n \times 10^8} \text{ amp.}$$

$$= \frac{0{\cdot}177\pi f B_{max}t^2}{\rho n \times 10^8} \text{amp.} \quad . \quad . \qquad (270)$$

Since $$(i')^2 = i_e^2 + i_m^2$$

$$i_m = \frac{0{\cdot}132\pi f B_{max}t^2}{\rho n \times 10^8} \text{amp.} \quad . \quad . \quad . \qquad (271)$$

The above discussion is based upon the following assumptions—

(*a*) that uniform flux distribution exists,

(*b*) the eddy currents are all in phase,

(*c*) the thickness of the iron plate is small compared with its length and breadth.

The flux distribution, when such assumptions are not made, will now be discussed.

Flux Distribution in Thick Iron Plates. Consider 1 cm. axial length and breadth of an iron plate with an alternating magnetic flux (of sinusoidal wave-form) running parallel to the axis of the plate. Let ϕ_x be the maximum value of the flux enclosed per centimetre breadth within the two inner sides of an elemental path of width dx, these sides each being parallel to the centre line of the plate and at distance x from it, as in Fig. 320. Then the total e.m.f. induced, per centimetre breadth, for the two inner sides is

$$E_{x\ max} = \frac{2\pi f \phi_x}{10^8} \text{ volts}$$

Let $B_{x\ max}$ be the maximum value of the flux density—considered uniform—over the width dx. This is justifiable if dx is very small. The maximum e.m.f. induced in the two outer sides of the path is then

$$E_{(x+dx)\ max} = \frac{2\pi f}{10^8}(\phi_x + 2B_{x\ max}\,dx)$$

The difference between the e.m.f's induced at distances $(x + dx)$ and x from the centre of the plate is thus

$$E_{(x+dx)\ max} - E_{x\ max} = \frac{4\pi f B_{x\ max}}{10^8}\,dx$$

Now the maximum value of the eddy current at distance x from the centre of the plate is

$$I_{x\ max} = \frac{E_{x\ max}}{\dfrac{2\rho}{dx}} = \frac{E_{x\ max}}{2\rho}\,dx$$

The maximum m.m.f., per centimetre axial length of the plate, due to this eddy current, is

$$\frac{4\pi}{10} \cdot I_{x\,max} = \frac{4\pi}{10} \cdot \frac{E_{x\,max}}{2\rho}\,dx$$

The maximum value of the flux density, $dB_{x\,max}$, produced by this m.m.f., is given by

$$dB_{x\,max} = \frac{4\pi}{10} \cdot \frac{E_{x\,max}}{2\rho}\,dx \cdot \mu$$

where μ is the permeability of the iron.

Since the flux density and eddy e.m.f. are not in phase with one another, symbolic notation is necessary for simplification of the calculation. Thus, denoting the symbolic value by a bar thus, $\bar{B}$, $\bar{E}$, we have

$$\frac{d\bar{B}_{x\,max}}{dx} = \frac{2\pi}{10} \cdot \frac{\bar{E}_{x\,max}}{\rho} \cdot \mu$$

and
$$\frac{d\bar{E}_{x\,max}}{dx} = -j\,\frac{4\pi f}{10^8}\,\bar{B}_{x\,max}$$

whence by differentiation of $\frac{d\bar{B}_{x\,max}}{dx}$ and by substitution for $\frac{d\bar{E}_{x\,max}}{dx}$ we have

$$\frac{d^2\bar{B}_{x\,max}}{dx^2} = -j\,\frac{8\pi^2 f\bar{B}_{x\,max}\,\mu}{10^9\rho}$$

$$= -j \cdot 2\gamma^2\bar{B}_{x\,max}$$

where
$$\gamma = \sqrt{\frac{4\pi^2 f\mu}{10^9\rho}}$$

The solution of this differential equation for $\bar{B}_{x\,max}$ is

$$\bar{B}_{x\,max} = \bar{M}\varepsilon^{x\sqrt{-j2\gamma^2}} + \bar{N}\varepsilon^{-x\sqrt{-j2\gamma^2}} \qquad . \quad . \quad (272)$$

$\bar{M}$ and $\bar{N}$ are complex constants which are equal, since $\bar{B}_{x\,max}$ has the same value, but is opposite in sign on the two sides of the centre line of the plate and at distances of x from this centre line.

Let $\bar{B}_{max}$ = the flux density at the surface of the plate $\left(\text{i.e. when } x = \frac{t}{2}\right)$

Then
$$\bar{B}_{max} = \bar{M}\left(\varepsilon^{\frac{t}{2}\sqrt{-j2\gamma^2}} + \varepsilon^{-\frac{t}{2}\sqrt{-j2\gamma^2}}\right)$$

and
$$\frac{\bar{B}_{x\,max}}{\bar{B}_{max}} = \frac{\varepsilon^{\gamma x\sqrt{-2j}} + \varepsilon^{-\gamma x\sqrt{-2j}}}{\varepsilon^{\frac{t}{2}\gamma\sqrt{-2j}} + \varepsilon^{-\frac{t}{2}\gamma\sqrt{-2j}}}$$

Now,
$$\sqrt{-2j} = \sqrt{2}\,\sqrt{-j} = \sqrt{2}\,\sqrt{\varepsilon^{-j\frac{\pi}{2}}} = \sqrt{2}\,\varepsilon^{-j\frac{\pi}{4}}$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right)$$
$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - j\,\frac{1}{\sqrt{2}}\right) = 1 - j$$

Hence,
$$\frac{\bar{B}_{x\ max}}{\bar{B}_{max}} = \frac{\varepsilon^{\gamma x(1-j)} + \varepsilon^{-\gamma x(1-j)}}{\varepsilon^{\frac{t}{2}\gamma(1-j)} + \varepsilon^{-\frac{t}{2}\gamma(1-j)}} \quad . \quad . \quad . \quad . \quad (273)$$

Using the expansions,
$$\varepsilon^{+j\theta} = \cos\theta + j\sin\theta$$
and
$$\varepsilon^{-j\theta} = \cos\theta - j\sin\theta$$
we have,
$$\frac{\bar{B}_{x\ max}}{\bar{B}_{max}} = \frac{(\varepsilon^{\gamma x} + \varepsilon^{-\gamma x})\cos\gamma x - j\,(\varepsilon^{\gamma x} - \varepsilon^{-\gamma x})\sin\gamma x}{\left(\varepsilon^{\frac{t}{2}\gamma} + \varepsilon^{-\frac{t}{2}\gamma}\right)\cos\frac{t}{2}\gamma - j\left(\varepsilon^{\frac{t}{2}\gamma} - \varepsilon^{-\frac{t}{2}\gamma}\right)\sin\frac{t}{2}\gamma}$$
or,
$$\bar{B}_{x\ max} = \frac{\bar{B}_{max}[(\varepsilon^{\gamma x} + \varepsilon^{-\gamma x})\cos\gamma\, x - j\,(\varepsilon^{\gamma x} - \varepsilon^{-\gamma x})\sin\gamma\, x]}{\left(\varepsilon^{\frac{t}{2}\gamma} + \varepsilon^{-\frac{t}{2}\gamma}\right)\cos\frac{t}{2}\gamma - j\left(\varepsilon^{\frac{t}{2}\gamma} - \varepsilon^{-\frac{t}{2}\gamma}\right)\sin\frac{t}{2}\gamma}$$
$$. \quad . \quad . \quad . \quad (274)$$

This gives the law of variation of flux density across the section of the plate in terms of the flux density at the surface.

Oberbeck and J. J. Thomson first investigated this effect of eddy currents upon the flux density in plates of various thicknesses. The curves of Fig. 322 show the order of the variation in maximum flux density across the section of the plate for plates of three different thicknesses, the material of the plates being transformer iron of ordinary grade and the frequency 100 cycles per second. The permeability μ (assumed constant) is taken as 2,500, and the resistivity $\frac{1}{10^5}$ ohm per centimetre cube.

It can be seen from these curves that, if approximately uniform flux distribution is to be obtained, the thickness of plate used, at 100 cycles per second frequency, should be less than ½ mm. Actually, plates about 0·35 mm. (0·014 in.) are used for this frequency, and for 50 cycles per second, although the variation in flux density is less than that shown when the frequency is less than 100 cycles per second.

By the use of thin plates, or laminations, the adjacent sheets being insulated from one another, the iron is used most economically, since almost uniform flux density is obtained—which means that no portion of the iron core is under-worked—and also the eddy current loss, which is proportional to the square of the thickness of the plate, is thereby reduced.

When plates of this thickness are used for work at commercial frequencies, the assumption that the flux density is uniform throughout the cross-section is justifiable, and the eddy current loss may be calculated with reasonable accuracy by the methods previously given.

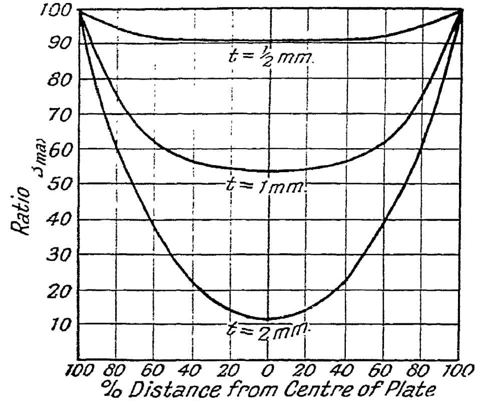


FIG. 322. VARIATION IN FLUX DENSITY IN THE CROSS-SECTION OF A THICK PLATE

For further consideration of the question the reader should refer to the works quoted in Refs. (1), (2), (5). The first of these three works discusses also the eddy current loss in iron sheets when the magnetization is rotating, as is the case in rotating electrical machinery.

Eddy Current Losses in Armature Conductors. Fig. 323 shows the distribution of flux in and around a stator slot on an alternator. The slot shown contains three conductors, and the flux is shown as though it consisted of two quite independent components. Actually, these components combine to form one resultant flux.

Both of these fluxes will induce eddy currents in the conductors. The flux ϕ_M, flowing parallel to the sides of the slot, will be small unless the teeth on each side of the slot are highly saturated, in which case the reluctance of the path in the slot becomes comparable with that of the paths in the teeth. In this case the conductors should be made narrow in the direction across the slot—perpendicular to the slot sides.

The flux ϕ_C is produced by the current in the conductors in the slot. Some of the flux surrounding these conductors, and proportional to the current in them at any instant, will be carried in the iron of the teeth and will pass, also, across the mouth of the slot and across the iron under the bottom of the slot. The remainder passes across the slot itself, and is responsible for the induction of eddy currents in the conductors.

The eddy currents in the individual conductors depend, therefore, upon the position, in the slot, of the conductor considered, and also upon the ratio of the width of the conductor to the width of the slot. The conductor nearest the mouth of the slot has the greatest eddy-current loss.

Messrs. A. B. and M. B. Field (Refs. (9) and (10)) have both investigated the question of eddy-current losses in such conductors. The former, in his paper (*loc. cit.*) gives sets of curves from which the eddy-current loss in any particular conductor in a slot may be determined. Each curve in the set refers to the conductor in some particular position in the slot, one curve referring to the bottom conductor, one to the bottom but one, and so on. These curves give the ratio $\frac{R_{eff}}{R}$ for the conductor, corresponding to various values of ad, where d is the depth of the conductor itself in centimetres, and a is a number which depends upon the ratio r of the width of the conductor to the width of the slot, and upon the frequency.

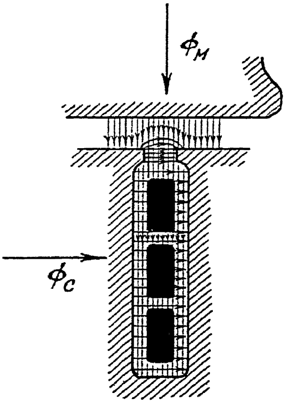


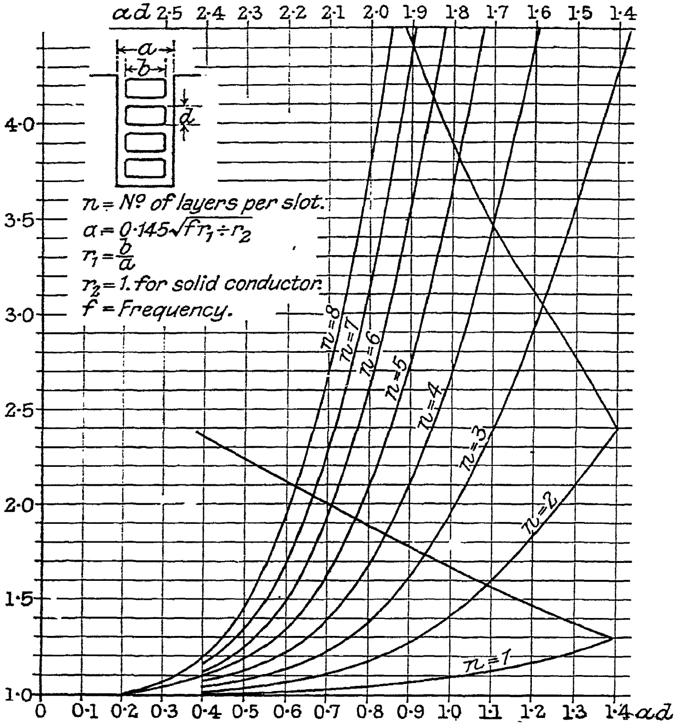
FIG. 323. FLUX IN AN ARMATURE SLOT

S. P. Smith (Ref. (12)) has redrawn these curves so that they give the mean value of the ratio $\frac{R_{eff}}{R}$ for all the conductors in the slot, instead of the ratio for each individual conductor. This set of curves is reproduced, by permission, in Fig. 324.

R_{eff} is the effective resistance of the conductor as influenced by eddy currents—which have the effect of increasing the effective resistance, since they increase the copper loss. R is the resistance of the conductor with direct current flowing in it (i.e. when there are no eddy currents). The power loss in the conductor with alternating current of r.m.s. value I flowing in it is thus I^2R_{eff}, whereas the power loss with an equal direct current I flowing is I^2R, the eddy current loss being responsible for the increase of the a.c. power loss over that with direct current.

The number a is given by

$$\frac{2\pi}{10}\sqrt{\frac{\mu f r}{10\rho}}$$



(*By the courtesy of Prof. S. P. Smith*

FIG. 324

where μ is the permeability of the material of the conductor, ρ its resistivity, and f is the frequency of the current in the conductors.

Thus, for copper, $a = 0{\cdot}145\sqrt{fr}$.

As an example of the use of such curves, suppose we have three conductors, each 0·5 cm. wide, and 1·25 cm. deep, in a slot of width 0·9 cm., and that the frequency is 50 cycles per second.

Then $$r = \frac{0{\cdot}5}{0{\cdot}9} = 0{\cdot}555$$

$$\therefore a = 0{\cdot}145\sqrt{0{\cdot}555 \times 50} = 0{\cdot}764$$

Then the ratio K_{mean} $\left(\text{i.e. the ratio } \frac{R_{eff}}{R}\right)$ is, from the curves—1·8.

Thus the mean value of R_{eff} for the three conductors is 1·8 R where R is the d.c. resistance of each conductor. This means that the copper loss with alternating current of 50 cycles per second is 1·8 times the copper loss with direct current. Hence, the eddy current loss is 0·8 of the d.c. copper loss.

Eddy Current Damping. Advantage is taken of the fact that eddy currents are induced in metal parts when they cut through magnetic lines of force by using these currents as a means of damping electrical indicating instruments. In the induction type instruments, for alternating current work, the eddy currents induced in a thin metal disc, together with the magnetic field of an electromagnet, produce also the operating, or deflecting, torque.

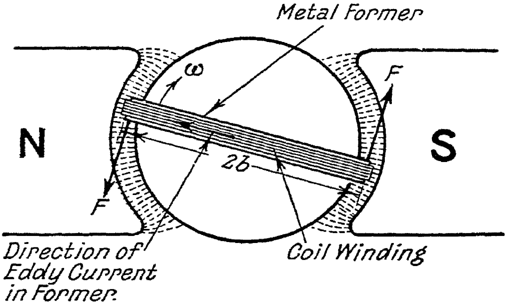


FIG. 325. EDDY CURRENT DAMPING IN A MOVING-COIL INSTRUMENT

The two commonest forms of eddy-current damping device are a metal former, upon which the working coil of the instrument is wound, or a thin aluminium disc which is attached to the moving system and moves in the field of a permanent magnet.

EDDY CURRENT DAMPING TORQUE WITH A METAL FORMER. Fig. 325 gives an outline diagram of the working portion of a moving-coil, permanent-magnet, instrument whose coil is wound upon a metal former. Suppose that the axial length of the former is l cm. and its breadth $2b$ cm., and let it rotate in a permanent magnet field which is radial and of uniform flux density B lines per square centimetre. If the former moves with angular velocity ω radians per second, its linear velocity is $b\omega$ cm. per second, and the e.m.f.

induced in each side of it is $\frac{Blb\omega}{10^8}$ volts. Thus the total e.m.f. induced in the former is

$$\frac{2Blb\omega}{10^8} \text{ volts}$$

The resistance of the path of the resulting current in the former is $\frac{(4b + 2l)\rho}{td}$ where t is the thickness and d the width of the section of the metal of the former, td being thus the cross-sectional area of the current-path. ρ is the resistivity in ohms per centimetre cube of the material of the former.

Then, the eddy current is given by

$$I_e = \frac{2Blb\omega td}{10^8(4b + 2l)\rho} \quad . \quad . \quad . \quad . \quad (275)$$

The force on each side in the radial field is $\mathrm{F} = \frac{BI_e l}{10}$ dynes. These forces produce a torque which opposes the motion of the former. This torque is $2b\mathrm{F}$ dyne-cm., or

$$\text{Damping torque } T_{\mathrm{D}} = 2b\mathrm{F}$$

$$= 2b \cdot \frac{BI_e l}{10}$$

$$= \frac{2b \cdot B \cdot l}{10}\left(\frac{2Blb\omega td}{10^8(4b + 2l)\rho}\right)$$

or

$$T_{\mathrm{D}} = \frac{4B^2b^2l^2\omega td}{10^9(4b + 2l)\rho} \text{ dyne-cm.} \quad . \quad . \quad (276)$$

The *damping constant*, expressed in dyne-centimetres per radian per second is

$$K_{\mathrm{D}} = \frac{T_{\mathrm{D}}}{\omega} = \frac{2B^2b^2l^2td}{10^9(2b + l)\rho} \quad . \quad . \quad . \quad (277)$$

The damping can obviously be varied by varying the thickness t of the metal of the former.

EDDY CURRENT DAMPING TORQUE WITH A METAL DISC. Although, in this case, the exact calculation of the eddy current is a matter of considerable difficulty, simple methods of calculating the approximate values of such currents, and of their damping effect, have been given by Evershed (Ref. (11)) and by Drysdale and Jolley (Ref. (7)). The method given by the latter is followed below.

Fig. 326 shows a metal disc of thickness t cm., which rotates with angular velocity ω radians per second, and cuts through the magnetic field of a permanent magnet whose poles are NS.

Then, if the flux density under the pole is B lines per square centimetre, an e.m.f. $e = \frac{Bb\omega r}{10^8}$ volts will be induced in the portion of the disc which is, at any instant, in the inter-polar gap. The radius r is measured from the centre of the disc to the centre of the pole face. If we consider only the portion of the disc which is immediately under the pole, the resistance of the eddy current path is $\frac{b\rho}{dt}$, where b is the radial length of the pole (which is the length of the portion of the disc in which the e.m.f. is induced), and d is the width of the pole. ρ is the resistivity of the material of the disc in ohms per centimetre cube.

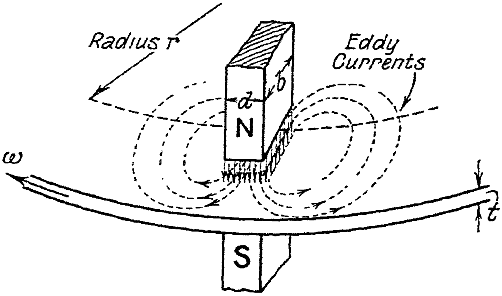


FIG. 326. EDDY CURRENT DAMPING WITH A METAL DISC

The actual resistance of the total eddy current path depends upon the radial position of the pole, and is somewhat greater than $\frac{b\rho}{dt}$. Let this resistance be $k \, . \, \frac{b\rho}{dt}$ where k is a constant (greater than unity) for any given radial position of the poles.

The eddy current is thus given by

$$i_e = \frac{Bb\omega r dt}{10^8 k \, . \, b \, . \, \rho} = \frac{B\omega r dt}{10^8 k \, . \, \rho} \text{ amp.} \quad . \quad . \quad (278)$$

and the retarding force by

$$\text{F} = \frac{Bi_e \, . \, b}{10} \text{ dynes}$$

The damping torque is thus

$$T_{\text{D}} = \text{F}r = \frac{B^2 \, . \, b \, . \, \omega r^2 dt}{10^9 k \, . \, \rho} \text{ dyne-cm.} \quad . \quad . \quad (279)$$

Substituting ϕ for the total flux passing between the poles, we have, since $\phi = Bbd$,

$$T_{\text{D}} = \frac{\phi^2 r^2 t\omega}{10^9 k \, . \, \rho \, . \, A} \quad . \quad . \quad . \quad . \quad . \quad (280)$$

where A is the area of the pole face and equals bd.

The damping constant, in dyne-cm. per radian per second, is therefore

$$K_D = \frac{T_D}{\omega} = \frac{}{10^9 k} \qquad (281)$$

dyne-cm. per radian per sec.

The damping may be altered by adjustment of the radial position of the poles, and is zero when the centres of the poles are at the edge of the disc. Drysdale and Jolley (*loc. cit.*) give the results of measurements of damping constants for discs of different materials and thicknesses, and their book should be consulted for further information on the subject.

BIBLIOGRAPHY AND REFERENCES

(1) *Theory and Calculation of Electric Currents*, J. L. La Cour and O. S. Bragstad.

(2) *Alternating Currents*, A. Russell, Vol. I, Chap. XVI.

(3) *Alternating Current Phenomena*, C. P. Steinmetz.

(4) *Theory and Calculation of Transient Electric Phenomena and Oscillations*, C. P. Steinmetz.

(5) *Theory and Practice of Alternating Currents*, A. T. Dover.

(6) *Specification and Design of Dynamo-electric Machinery*, Miles Walker.

(7) *Electrical Measuring Instruments*, C. V. Drysdale and A. C. Jolley, Vol. I.

(8) "Das Nutenfeld in Zahnarmaturen und die Wirbelstromverluste in massive Armatur-Kupferleitern," *Sammlung electrotechnischer Vorträge* (Stuttgart, 1903).

(9) A. B. Field, *Proc. Amer. Inst. Elect. Engr.*, Vol. XXIV, p. 761 (1905), and *Electrical World*, Vol. XLVIII (29th September, 1906).

(10) M. B. Field, *Jour. I.E.E.*, Vol. XXXIII, p. 1125 (1906).

(11) "A Frictionless Motor Meter," S. Evershed, *Jour. I.E.E.*, Vol. XXIX, p. 772.

(12) "Papers on The Design of Alternating Current Machinery," by C. C. Hawkins, S. P. Smith, S. Neville. First Paper.

(13) "An Experimental Investigation of the Theory of Eddy Currents in Laminated Cores of Rectangular Section," M. Reed, *Jour. I.E.E.*, Vol. LXXX, p. 567.

(14) "Intersheet Eddy-current Loss in Laminated Cores," L. V. Bewley and H. Poritsky, *Trans. A.I.E.E.*, Vol. LVI, p. 344.

CHAPTER XV

WAVE-FORMS AND THEIR DETERMINATION

Wave-form. Alternating current calculations generally are carried out upon the assumption that the wave-form of the voltage supplied to the circuit is sinusoidal; i.e. the law of variation of the voltage with time may be written

$$e = E_{max} \sin \omega t$$

e being the instantaneous value of the voltage at any time t and E_{max} being its maximum value. ω is, of course, $2\pi \times$ frequency.

Special precautions must, however, be taken, in constructing the supply alternator, if the voltage wave-form is to approximate reasonably closely to a purely sinusoidal form.

Consider an alternator with a revolving field and with an armature —or stator—the surface of which is smooth instead of containing slots and teeth, which render its iron surface discontinuous. In such a machine a sinusoidal voltage wave-form could be obtained by shaping the poles so that the length of the air-gap at any point is proportional to $\frac{1}{\cos \theta}$ where θ is the angle—measured in electrical degrees—between the point in question and the centre of the pole.

If, however, the stator has open slots, these will affect the flux distribution from the poles, the air-gap reluctance at any point being dependent upon the position of the point, at any instant, relative to the adjacent stator teeth. Thus, the movement of the poles past the teeth is accompanied by small variations of flux distribution which are superposed upon the main flux, and these produce corresponding small e.m.f's in the conductors, the frequency of which is higher than, and some multiple of, that of the main e.m.f. These small e.m.f's produce "harmonics" in the e.m.f. wave, and cause it to depart from purely sinusoidal wave-form. Again, the magnetizing current of a transformer, the core of which is worked at a flux density sufficiently high to produce saturation, contains a pronounced "third harmonic." This means that it consists of a pure sine wave of current of frequency f (say), together with another sine wave of frequency $3f$. This distortion of the current wave is a result of hysteresis in the iron core of the transformer. One example of a distorted wave-form is shown in Fig. 327.

This wave can be split up into the two pure sine curves shown dotted.

Thus, $e_1 + e_2 = e$

The sine curve whose frequency is the same as the "complex" wave (i.e. the original wave) is called the "fundamental," and the other sine curve is called the "third harmonic," since its frequency is three times that of the fundamental.

In the figure, the maximum value of the fundamental is four times that of the third harmonic. Hence, calling the former E_1, the equation for the complex wave may be written

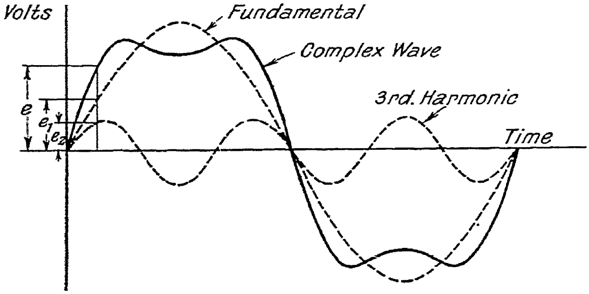


FIG. 327. WAVE-FORM WITH A THIRD HARMONIC

$$= E_1 \sin 2\pi ft \qquad \frac{E_1}{4}$$

(where f is the frequency of the fundamental),

or, $$e = E_1 \sin \omega t + \frac{E_1}{4} \sin 3\omega t \qquad . \quad (282)$$

If the third harmonic had been displaced in phase by 180° relative to the fundamental, the effect would have been to make the wave-form peaked as shown by Fig. 328, instead of flat.

The Composition of Complex Wave-forms. *Any single-valued periodic function* (i.e. a function which can have only one value for any given value of the independent variable) *can be resolved into a fundamental periodic function and a number of harmonics.* This is known as Fourier's theorem, and is true no matter how complicated the complex function may be.

Expressed mathematically, this means that the equation of any complex wave e may be written

$$\begin{aligned} e = E_0 + E_1 \sin (\omega t + \phi \;\,) + E_2 \sin (2\omega t + \phi_2) \\ + E_3 \sin (3\omega t + \quad {}_3) + \ldots E_n \sin (n\omega t + \phi_n) \qquad (283) \end{aligned}$$

where E_1, E_2, etc., are the maximum values of the various component sine waves, and the angles ϕ_1, ϕ_2, etc., are the phases of these waves when time is zero.

$\omega = 2\pi \times$ frequency, and therefore the frequencies of the terms from the second onwards are in the ratio $1 : 2 : 3 : \ldots : n$. $E_1 \sin (\omega t \dotplus \phi_1)$ represents the fundamental wave, $E_2 \sin (2\omega t + \phi_2)$ the second harmonic, and so on.

E_0 is independent of time and represents a constant component of the complex wave. If the time axis is drawn so that the mean ordinate of the complex wave is zero, E_0 is zero. This is the case in alternating current and e.m.f. waves.

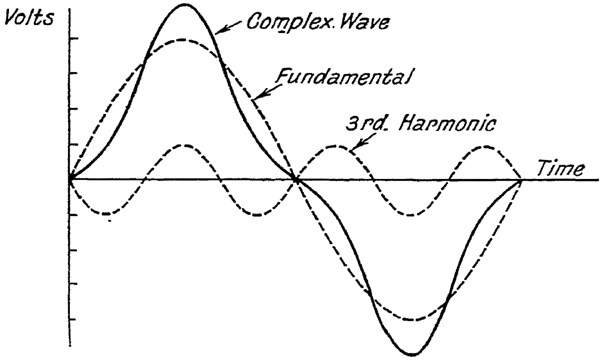


FIG. 328

The above general expression may be rewritten in another form if the trigonometrical expansion

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

is applied to it.

Then, omitting the E_0 term, as being zero,

$$\begin{aligned} e = {} & E_1(\sin \omega t \cos \phi_1 + \cos \omega t \sin \phi_1) \\ & + E_2(\sin 2\omega t \cos \phi_2 + \cos 2\omega t \sin \phi_2) \\ & + E_3(\sin 3\omega t \cos \phi_3 + \cos 3\omega t \sin \phi_3) + \ldots \\ & \ldots E_n(\sin n\omega t \cos \phi_n + \cos n\omega t \sin \phi_n) \end{aligned}$$

$$\begin{aligned} \text{or, } e = {} & A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t \\ & + B_1 \cos \omega t + B_2 \cos 2\omega t \qquad {}_3 \cos 3\omega t \\ & + \ldots B_n \cos n\omega t \quad . \end{aligned} \tag{284}$$

where the coefficients A_1, A_2, A_3, etc., are respectively $E_1 \cos \phi_1$, $E_2 \cos \phi_2$, $E_3 \cos \phi_3$, etc., and B_1, B_2, B_3, are $E_1 \sin \phi_1$, $E_2 \sin \phi_2$, $E_3 \sin \phi_3$, respectively.

If the fundamental and all the harmonics start from zero together, their phase differences with respect to the complex wave are all zero and

$$e = E_1 \sin \omega t + E_2 \sin 2\omega t + E_3 \sin 3\omega t + \ldots$$
$$+ \ldots E_n \sin n\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (285)$$

Even Harmonics. Since all the poles forming the field of an alternator are similarly constructed and shaped, it follows that the half-wave of e.m.f. generated in a conductor during the passage of (say) a south pole past it, is exactly similar in shape to the half-wave generated during the passage of the preceding north pole. In general, therefore, it follows that the negative halves of the waveform of alternating currents and voltages are identical in shape with the positive halves.

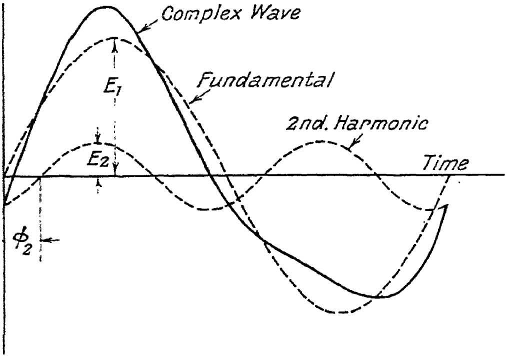


FIG. 329. WAVE-FORM CONTAINING A SECOND HARMONIC

It will be shown below that the effect of even harmonics in a complex wave is to make the shape of the negative half-wave different from that of the positive half. From the above reasoning, it is thus obvious that *alternating current and voltage wave-forms do not, in general, contain even harmonics.*

Consider two sine waves, of frequency f and $2f$, such as those shown dotted in Fig. 329. When added together, these give a complex wave whose law of variation with time is given by

$$e = E_1 \sin 2\pi ft + E_2 \sin (4\pi ft - \phi_2)$$

where E_1 and E_2 are the amplitudes of the fundamental and second harmonic respectively. (The time is, in this case, reckoned from the instant when the fundamental is zero and the phase angle ϕ_2 is measured on the scale of degrees applying to the second harmonic.)

It is seen that the resultant complex wave has a negative half which is of different shape from the positive half. A similar effect is produced whatever the relative phases of the fundamental and harmonic, and with all even harmonics, whatever their frequency. Odd harmonics do not produce the effect.

Since there can be no even harmonics present in alternating current and voltage waves whose two halves are identical in shape, the equations of such waves may be written in the form

$$e = E_1 \sin(\omega t + \phi_1) + E_3 \sin(3\omega t + \phi_3) + E_5 \sin(5\omega t + \phi_5) + \ldots E_n \sin(n\omega t + \phi_n) \quad . \quad . \quad . \quad (286)$$

where n is odd.

Harmonic Analysis. The process of splitting-up a complex wave into its fundamental and harmonics is called "harmonic analysis." The number of terms to be obtained in such an analysis depends upon the accuracy with which it is desired to express, mathematically, the complex wave. The third and fifth harmonics are the most important in alternating current and voltage wave-forms, and the analysis is, therefore, often carried no further than the third term of the expression for the complex wave, although in special cases it may be necessary to determine the amplitudes of harmonics up to (say) the seventeenth.

METHOD 1. This method was given originally by Perry (Ref. (9)). It depends upon the following theorems in the integral calculus.

$$(a) \quad \int_0^{\pi} \sin\theta d\theta = 2$$

$$(b) \quad \int_0^{\pi} \sin^2\theta d\theta = \frac{\pi}{2}$$

$$(c) \quad \int_0 \cos^2\theta d\theta = \frac{\pi}{2}$$

$$(d) \quad \int_0^{\pi} \sin m\theta \cos n\theta d\theta = 0$$

$$(e) \quad \int_0^{\pi} \sin m\theta \sin n\theta d\theta = 0$$

$$(f) \quad \int_0^{\pi} \cos m\theta \cos n\theta d\theta = 0$$

where m and n are positive integers which are unequal.

Consider a complex wave whose ordinate y varies with the abscissa θ according to the law

$$y = A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots$$
$$+ B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots$$

The ordinate is given the symbol y in order to make the equation general in its application to such waves.

If each side of this equation is multiplied by sin θ and integration of each side is carried out between the limits 0 and π, we have,

$$\int_0^\pi y \sin \theta d\theta = \int_0^\pi A_1 \sin^2 \theta d\theta + \int_0^\pi A_3 \sin \theta \sin$$

$$+ \int_0^\pi A_5 \sin \theta \sin 5\theta d\theta + \dots$$

$$+ \int_0^\pi B_1 \sin \theta \cos \theta d\theta + \int_0^\pi B_3 \sin \theta \cos 3\theta d\theta$$

$$+ \int_0^\pi B_5 \sin \theta \cos 5\theta d\theta + \dots \quad . \quad (287)$$

From the theorems already quoted, all terms on the right-hand side of the equation, except the first, are equal to zero. Therefore,

$$\int_0^\pi y \sin \theta \,.\, d\theta = \int_0^\pi A_1 \sin^2 \theta \,.\, d\theta = A_1 \frac{\pi}{2}$$

Hence, $$A_1 = \frac{2}{\pi} \int_0^\pi y \sin \theta \,.\, d\theta$$

Now, $\frac{1}{}$ $\theta \,.\, d\theta$ is obviously the mean value of the function y sin θ for one half-period of the complex wave, and therefore the amplitude A_1 of the fundamental of the complex wave is given by twice the mean value of y sin θ for one half-wave.

In the same way the value of A_3, A_5, etc., are obtained by multiplying each side of the equation of the complex wave by sin 3θ, sin 5θ, and so on, and integrating each side between the limits 0 and π. To obtain B_1, B_3, etc., multiply each side of the equation by cos θ, cos 3θ, and so on, and integrate.

The values of the coefficients so determined are—

$A_1 = 2 \times$ mean value of $y \sin \theta$ for $\frac{1}{2}$-period
$A_3 = 2 \times$,, ,, $y \sin 3\theta$,, ,,
$A_5 = 2 \times$,, ,, $y \sin 5\theta$,, ,,
$\dots = \dots$
$B_1 = 2 \times$ mean value of $y \cos \theta$ for $\frac{1}{2}$-period
$B_3 = 2 \times$,, ,, $y \cos 3\theta$,, ,,

and so on.

If the equation is required in the form

$$y = Y_1 \sin (\theta + \phi_1) + Y_3 \sin (3\theta + \phi_3) + \dots Y_n \sin (n\theta + \phi_n)$$

the coefficients Y_1, Y_3, etc., and the phase angles ϕ_1, ϕ_3, etc., may be obtained from the values of the predetermined coefficients $A_1, A_3, \ldots$ and $B_1, B_3, \ldots$, as follows—

From equation (284),

$$A_1 = Y_1 \cos \phi_1$$
$$A_3 = Y_3 \cos \phi_3$$
$$\ldots = \ldots\ldots$$

also

$$B_1 = Y_1 \sin \phi_1$$
$$B_3 = Y_3 \sin \phi_3$$
$$\ldots = \ldots\ldots$$

Hence, $A_1^2 + B_1^2 = (Y_1 \cos \phi_1)^2 + (Y_1 \sin \phi_1)^2 = Y_1^2$

or,

$$Y_1 = \sqrt{A_1^2 + B_1^2}$$

Similarly,

$$Y_3 = \sqrt{A_3^2 + B_3^2}$$
$$Y_5 = \sqrt{A_5^2 + B_5^2} \text{ and so on.}$$

Also,

$$\frac{B_1}{A_1} = \frac{Y_1 \sin \phi_1}{Y_1 \cos \phi_1} = \tan \phi_1$$

so that

$$\phi_1 = \tan^{-1} \frac{B_1}{A_1}$$

Similarly,

$$\phi_3 = \tan^{-1} \frac{B_3}{A_3}$$
$$\phi_5 = \tan^{-1} \frac{B_5}{A_5} \text{ and so on.}$$

In this way, the equation may be written in the desired form by substituting these values of the amplitudes and phase angles in the equation.

Procedure for the Determination of the Coefficients by a Summation Process. From the fact that

$$A_1 = 2 \times \text{mean value of } y \sin \theta \text{ for one half-period}$$

it is obvious that A_1 could be obtained by a graphical method as follows.

Take any ordinate y_1, corresponding to an angle θ_1 and multiply it by $\sin \theta_1$, setting up an ordinate representing $y_1 \sin \theta_1$, at abscissa θ_1. By repeating the process for a large number of values of y, a new curve $y \sin \theta$, plotted against θ, is obtained. The area under one half-wave of this curve could be measured, to scale, using a planimeter, and its mean value obtained by dividing this area by the length of the base π. By this means A_1 could be found. The other coefficients $A_3, A_5, \ldots$, and also $B_1, B_3, \ldots$, could also be found by similar methods. Such a procedure would, however, be very laborious and a tabulation method is therefore used, as below, in order to simplify it somewhat. In the tabular method the base of the half-wave is divided into a number of equal parts—the greater the number of parts the greater the accuracy of the analysis—and ordinates are set up at the dividing points.

Let the number of these parts be n. Then each part corresponds to an angle of $\frac{\pi}{n}$. If $y_{\frac{\pi}{n}}$, $y_{\frac{2\pi}{n}}$, . . ., are the ordinates at the dividing points, then the corresponding products—when A_1 is being determined—are $y_{\frac{\pi}{n}} \sin \frac{\pi}{n}$, $y_{\frac{2\pi}{n}} \sin \frac{2\pi}{n}$, ...

The mean value of $y \sin \theta$ for the half-wave is then obtained by dividing the sum of these products by the number of parts into which the base is divided.

The other A coefficients are obtained in a similar way, but the y ordinates are then multiplied by $\sin 3\theta$, $\sin 5\theta$. . . . The B coefficients are determined in the same way, when the y ordinates are multiplied by $\cos \theta$, $\cos 3\theta$. . . .

The accuracy of the results of such an analysis may be checked from the facts that—

(i) The ordinate of the complex wave corresponding to $\theta = 0$ should equal $B_1 + B_3 + B_5 + \ldots .$

(ii) The ordinate corresponding to $\theta = \frac{\pi}{2}$ should equal $A_1 - A_3 + A_5 - A_7 + \ldots$

These relations are obtained as below—

When $\theta = 0$

$$y = A_1 \sin 0 + A_3 \sin 0 + A_5 \sin 0 + \ldots + B_1 \cos 0 + B_3 \cos 0 + B_5 \cos 0 + \ldots$$

or

$$y = B_1 + B_3 + B_5 + \ldots$$

When $\theta = \frac{\pi}{2}$

$$\begin{aligned} y &= A_1 \sin \frac{\pi}{2} + A_3 \sin \frac{3\pi}{2} + A_5 \sin \frac{5\pi}{2} + \ldots \\ &\quad + B_1 \cos \frac{\pi}{2} + B_3 \cos \frac{3\pi}{2} + B_5 \cos \frac{5\pi}{2} + \ldots \\ &= A_1 - A_3 + A_5 - \ldots \end{aligned}$$

To illustrate the method of tabulation for the determination of the coefficients, consider, as an example, the complex wave shown in Fig. 330, the analysis of which is to be carried out for the determination of harmonics up to, and including, the fifth.

In Table XIV, the base of the half-wave is divided into twelve equal intervals, each of 15°, as shown in column (a). The corresponding values of y, taken from the curve, are shown in column (b). The coefficients A_1, A_3, and A_5 are determined from the mean values of the products given in columns (*d*), (*f*), and (*h*), and the coefficients B_1, B_3, and B_5 from the products given in columns (*k*), (*m*), and (*p*). As previously explained, checks are applied in order to test the accuracy of the determination. The curve to be analysed contains, in this case, only third and fifth harmonics, in addition to the fundamental. This is shown by the checks, since $A_1 - A_3 + A_5$ is equal to the ordinate of the complex wave at $\theta = 90°$, and $B_1 + B_3 + B_5$ equal to the ordinate at 0° (to within the limits of error obtainable with a graph of this size).

If other harmonics had been present, $A_1 - A_3 + A_5$ could not have equalled the ordinate at 90°, nor would $B_1 + B_3 + B_5$ have equalled the ordinate at 0°. These checks give a guide as to whether further analysis, beyond the fifth harmonic, is necessary, in order to represent the complex wave mathematically with sufficient accuracy.

The equation of the complex wave is, therefore, from the results obtained in the table,

$$y = 199{\cdot}4 \sin\theta + 35{\cdot}8 \sin 3\theta + 13{\cdot}2 \sin 5\theta - 2{\cdot}92 \cos\theta + 18{\cdot}8 \cos 3\theta - 15 \cos 5\theta \quad . \quad . \quad . \quad . \quad . \quad (288)$$

In order to write the expression for y in terms of sine functions only, the coefficients Y_1, Y_3, and Y_5 must be determined, and also the phase angles ϕ_1, ϕ_3, and ϕ_5, relative to the phase of the complex wave.

Then,

$$Y_1 = \sqrt{A_1^2 + B_1^2} = 200$$

$$Y_3 = \sqrt{A_3^2 + B_3^2} = 40{\cdot}3$$

$$Y_5 = \sqrt{A_5^2 + B_5^2} = 20$$

Also,

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = -0°\ 50'$$

$$\phi_3 = \tan^{-1}\frac{B_3}{A_3} = 27°\ 42'$$

$$\phi_5 = \tan^{-1}\frac{B_5}{A_5} = -48°\ 39'$$

The equation of the complex wave is thus,

$$y = 200 \sin(\theta - 0°\ 50') + 40{\cdot}3 \sin(3\theta + 27°\ 42') + 20 \sin(5\theta - 48°\ 39') \quad . \quad . \quad . \quad . \quad . \quad (289)$$

The three component waves are shown dotted in Fig. 330.

METHOD 2. *Summation Method with Grouped Terms.* This method is similar to the one given above, but the labour involved in the summation for the determination of the coefficients is reduced by

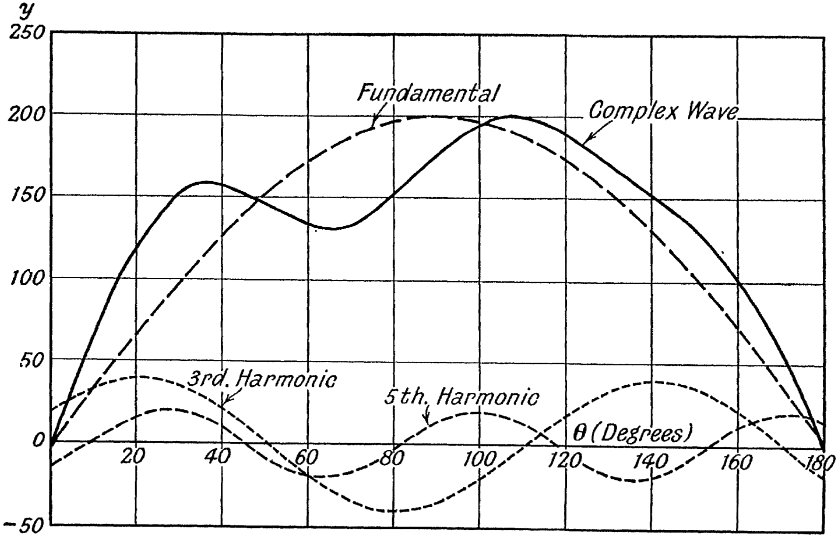


FIG. 330

TABLE XIV
ANALYSIS OF WAVE SHOWN IN FIG. 330

Complex Wave		Analysis of Sine Terms						Analysis of Cosine Terms					
(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	(*h*)	(*j*)	(*k*)	(*l*)	(*m*)	(*n*)	(*p*)
θ	y	$\sin\theta$	$y\sin\theta$	$\sin 3\theta$	$y\sin 3\theta$	$\sin 5\theta$	$y\sin 5\theta$	$\cos\theta$	$y\cos\theta$	$\cos 3\theta$	$y\cos 3\theta$	$\cos 5\theta$	$y\cos 5\theta$
15°	97	0·259	25	0·707	68·5	0·966	94	0·966	94	0·707	68·5	0·259	25
30°	152	0·5	76	1·0	152	0·5	76	0·866	132	0	0	– 0·866	– 132
45°	152	0·707	107·5	0·707	107·5	– 0·707	– 107·5	0·707	107·5	– 0·707	– 107·5	– 0·707	– 107·5
60°	134	0·866	116	0	0	– 0·866	– 116	0·5	67	– 1·0	– 134	0·5	67
75°	142	0·966	137	– 0·707	– 100·5	0·259	36·8	0·259	36·8	– 0·707	– 100·5	0·966	137
90°	176	1·0	176	– 1·0	– 176	1·0	176	0	0	0	0	0	0
105°	200	0·966	193	– 0·707	– 141·4	0·259	51·8	– 0·259	– 51·8	0·707	141·4	– 0·966	– 193
120°	189	0·866	164	0	0	– 0·866	– 164	– 0·5	– 94·5	1·0	189	– 0·5	– 94·5
135°	161	0·707	114	0·707	114	– 0·707	– 114	– 0·707	– 114	0·707	114	0·707	114
150°	133	0·5	66·5	1·0	133	0·5	66·5	– 0·866	– 115	0	0	0·866	115
165°	82	0·259	21·2	0·707	58	0·966	79·5	– 0·966	– 79·5	– 0·707	– 58	– 0·259	– 21·2
180°	0	0	0	0	0	0	0	– 1·0	0	– 1·0	0	– 1·0	0
		Sum	1196·2	Sum	215·1	Sum	79·1	Sum	– 17·5	Sum	112·9	Sum	– 90·2
		Mean Value	99·7	Mean Value	17·9	Mean Value	6·6	Mean Value	– 1·46	Mean Value	9·4	Mean Value	– 7·5
		$A_1 =$	199·4	$A_3 =$	35·8	$A_5 =$	13·2	$B_1 =$	– 2·92	$B_3 =$	18·8	$B_5 =$	– 15
Check		$A_1 - A_3 + A_5 = 176{\cdot}8 \doteqdot$ the value of y at $\theta = 90°$ (i.e. 176)						$B_1 + B_3 + B_5 = 0{\cdot}88 \doteqdot$ the value of y at $\theta = 0°$ (i.e. 0)					

Equation of wave: $y = 200 \sin(\theta - 0° 50') + 40{\cdot}3 \sin(3\theta + 27° 42') + 20 \sin(5\theta - 48° 39')$

the grouping of terms. The method is due to S. P. Thompson (Ref. (10)) and to Runge (Ref. (11)).

If the nth harmonic is to be determined, the base of the half-wave is divided into $n + 1$ equal parts, giving n equidistant ordinates. These ordinates are treated in supplemental pairs, the ordinate y_1 at θ_1 being used in conjunction with the ordinate y_n at $(180 - \theta_1)$ as below.

First consider the sum of two such ordinates,

$$\begin{aligned} y_1 + y_n = A_1 \sin \theta_1 &+ A_3 \sin 3\theta_1 + A_5 \sin 5\theta_1 + \dots \\ &+ B_1 \cos \theta_1 + B_3 \cos 3\theta_1 + B_5 \cos 5\theta_1 + \dots \\ &+ A_1 \sin (180 - \theta_1) + A_3 \sin 3(180 - \theta_1) \\ &+ A_5 \sin 5(180 - \theta_1) + \dots \\ &+ B_1 \cos (180 - \theta_1) + B_3 \cos 3(180 - \theta_1) \\ &+ B_5 \cos 5(180 - \theta_1) + \dots \end{aligned}$$

All the terms containing coefficients B disappear, since

$$\begin{aligned} \cos \theta_1 &= - \cos (180 - \theta_1) \\ \cos 3\theta_1 &= - \cos 3(180 - \theta_1) \\ \dots &= \dots . \end{aligned}$$

Also,

$$\begin{aligned} \sin \theta_1 &= \sin (180 - \theta_1) \\ \sin 3\theta_1 &= \sin 3(180 - \theta_1) \\ \dots &= \dots . \end{aligned}$$

Hence,

$$y_1 + y_n = 2A_1 \sin \theta_1 + 2A_3 \sin 3\theta_1 + 2A_5 \sin 5\theta_1 + \dots \qquad (290)$$

Similarly, if the difference of these ordinates is considered, we have

$$y_1 - y_n = 2B_1 \cos \theta_1 + 2B_3 \cos 3\theta_1 + 2B_5 \cos 5\theta_1 + \dots \qquad (291)$$

Ordinates are grouped in this way before multiplying by the sine functions for the determination of the individual coefficients. This obviously saves labour.

Taking as an example the curve (Fig. 330) previously analysed, the procedure, in an analysis up to the fifth harmonic, is to divide the base into six equal parts, the five ordinates set up at the dividing points being measured and arranged in rows as under—

	y_{30}	y_{60}	y_{90}
	y_{150}	y_{120}	
Sum . . .	a_1	a_2	a_3
Difference . .	d_1	d_2	

The notation "y_{30}" means the ordinate at $\theta = 30°$.
From the previous method, we have

$$A_1 = 2 \times \text{the mean value of } y \sin\theta \text{ for half-period}$$

In this case, therefore,

$$A_1 = \frac{2\,[y_{30}\sin 30° + y_{60}\sin 60° + y_{90}\sin 90° + y_{120}\sin 120°}{6}$$
$$\frac{+\,y_{150}\sin 150° + y_{180}\sin 180°]}{6}$$

or, since sin 180° and y_{180} are both zero

$$A_1 = \frac{1}{3}\,[y_{30}\sin 30° + y_{60}\sin 60° + y_{90}\sin 90° + y_{120}\sin 120° + y_{150}\sin 150°]$$
$$= \frac{1}{3}\,[a_1\sin 30° + a_2\sin 60° + a_3\sin 90°]$$

since sin 30° = sin 150°
and sin 60° = sin 120°

In the same way it can be shown that

$$A_3 = \frac{1}{3}\,(a_1 - a_3)\sin 90°$$

and $A_5 = \frac{1}{3}\,(a_1\sin 30° - a_2\sin 60° + a_3\sin 90°)$

Again,

$$B_1 = \frac{2\,[y_{30}\cos 30° + y_{60}\cos 60° + y_{90}\cos 90° + y_{120}\cos 120°}{6}$$
$$\frac{+\,y_{150}\cos 150° + y_{180}\cos 180°]}{6}$$
$$= \frac{1}{3}\,[y_{30}\cos 30° + y_{60}\cos 60° + y_{120}\cos 120° + y_{150}\cos 150°]$$

since cos 90° and y_{180} are both zero.

Hence, $B_1 = \frac{1}{3}\,(d_1\cos 30° + d_2\cos 60°)$

It can be shown also that

$$B_3 = \frac{1}{3}\,d_2\cos 180° = -\frac{1}{3}\,d_2$$

and $B_5 = \frac{1}{3}\,(-d_1\cos 30° + d_2\cos 60°)$

TABLE XV

Analysis of Wave shown in Fig. 330

θ	$\sin\theta$	$\cos\theta$	Sine Terms			Cosine Terms		
			Fundamental	5th Harmonic	3rd Harmonic	Fundamental	5th Harmonic	3rd Harmonic
0°	0	1·0						55 [$= -d_2 \cos 0°$]
30°	0·50	0·866	142·5 [$= a_1 \sin 30°$]				16·45 [$= d_1 \cos 30°$]	
60°	0·866	0·50		280 [$= a_2 \sin 60°$]		−27·5 [$= d_2 \cos 60°$]		
90°	1·0	0	176 [$= a_3 \sin 90°$]		109 [$= (a_1 - a_3) \sin 90°$]			
Total, 1st Column . .			318·5		109	−27·5		55
Total, 2nd Column . .			280			16·45		
Sum of 1st and 2nd Column totals			598·5 $\therefore A_1 = \frac{598·5}{3} = 199·5$		$A_3 = \frac{109}{3} = 36·3$	$-11·05 \therefore B_1 = \frac{-11·05}{3} = -3·68$		55
Difference of 1st and 2nd Column totals . .			38·5 $\therefore A_5 = \frac{38·5}{3} = 12·8$			$-43·95 \therefore B_5 = \frac{-43·95}{3} = -14·65$		$\therefore B_3 = \frac{55}{3} = 18·33$

Check $A_1 - A_3 + A_5 = 176$ = ordinate of complex wave at $\theta = 90°$. $B_1 + B_3 + B_5 = 0$ = ordinate of complex wave at $\theta = 0°$

Equation of complex wave: $y = 200 \sin (\theta - 1° 3' + 40·6 \sin (3\theta + 26° 42') + 19·5 \sin (5\theta - 48° 45')$

Proceeding with the analysis we have, therefore

	$y_{30} = 152$	$y_{60} = 134$	$y_{90} = 176$
	$y_{150} = 133$	$y_{120} = 189$	
Sum	$a_1 = 285$	$a_2 = 323$	$a_3 = 176$
Difference	$d_1 = 19$	$d_2 = -55$	

The values a_1, a_2, a_3, d_1 and d_2 are then used in the table as shown.

Using the coefficients A_1, A_3, etc., as obtained in the table, we have, for the equation of the complex wave,

$$y = 199{\cdot}5 \sin \theta + 36{\cdot}3 \sin 3\theta + 12{\cdot}8 \sin 5\theta - 3{\cdot}68 \cos \theta + 18{\cdot}33 \cos 3\theta - 14{\cdot}65 \cos 5\theta$$

From these coefficients

$$Y_1 = \sqrt{A_1^2 + B_1^2} = 200$$

$$Y_3 = \sqrt{A_3^2 + B_3^2} = 40{\cdot}6$$

$$Y_5 = \sqrt{A_5^2 + B_5^2} = 19{\cdot}5$$

Also,

$$\phi_1 = \tan^{-1} \frac{B_1}{A_1} = -1^\circ\, 3'$$

$$\phi_3 = \tan^{-1} \frac{B_3}{A_3} = 26^\circ\, 42'$$

$$\phi_5 = \tan^{-1} \frac{B_5}{A_5} = -48^\circ\, 45'$$

Hence, the equation of the wave, written in sine terms only, becomes

$$y = 200 \sin (\theta - 1^\circ\, 3') + 40{\cdot}6 \sin (3\theta + 26^\circ\, 42') + 19{\cdot}5 \sin (5\theta - 48^\circ\, 45') \quad . \quad . \quad . \quad . \quad (292)$$

The checks in this analysis (see Table XV) show that there are no other components of the wave beside the fundamental, and 3rd and 5th harmonics.

It will be noted that the above equation is not quite the same as that obtained in the previous method of analysis, but the differences are within the limits of error to be expected when such methods are applied to a wave of the size shown.*

In the above, it has been assumed that the wave-form of a current or voltage is known and that the wave is to be split up, by mathematical or graphical methods, into its component fundamental and harmonic waves.

Harmonic Analyser. A complex wave can be split up into its components by experimental methods, the magnitude of the various harmonics being measured directly. An electric harmonic analyser has been described by J. D. Cockcroft, R. T. Coe, J. A. Tyacke, and Miles Walker (Ref. (12)). A portable form of this instrument is manufactured by H. Tinsley & Co. and has been described by Coe (Ref. 18). The essential features of the analyser are shown in Fig. 331 in which D is an astatic dynamometer instrument whose moving coil carries a current proportional to the voltage under test (i.e. whose wave-form is to be analysed). The fixed coil of this instrument carries a current I_a, called the *analysing current*, the frequency of which can be varied so as to be exactly equal to that of the fundamental or of any harmonic of the wave under test. This frequency variation is carried out by means of a synchronously-driven contact disc having a series of concentric rings of contacts, one ring for each harmonic, the number of contacts in the ring being sufficient to give the frequency of the harmonic concerned. This disc is used in conjunction with a triode valve circuit as shown, the analysing current in the dynamometer fixed coil being induced through coupling with the anode circuit. Measurement of all the odd harmonics up to the thirty-fifth is possible with the normal form of contact disc.

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* For graphical methods of analysis and for other methods, including analyses up to the eleventh or seventeenth harmonic, the reader is referred to the works quoted in Refs. (2), (3), (5), (6).

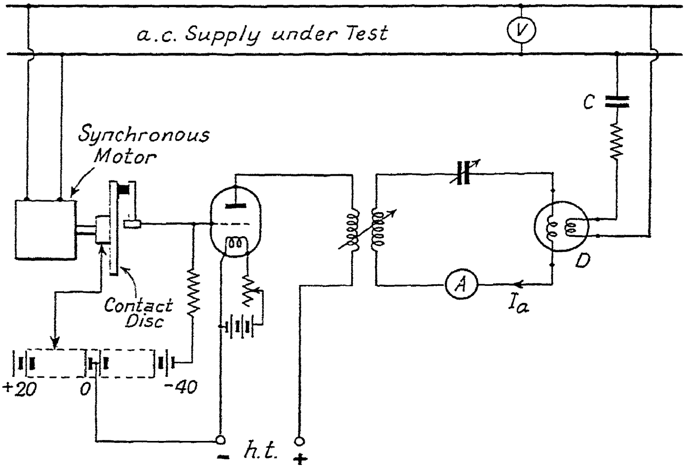


FIG. 331. HARMONIC ANALYSER

The action of the instrument depends upon the fact that a steady deflection of a dynamometer instrument results only from those current waves in the fixed and moving coils which are of the same frequency. Thus, let the current in the moving coil be

$$i = I_0 + I_1 \sin(\theta + \phi_1) + I_2 \sin(2\theta + \phi_2) + I_3 \sin(3\theta + \phi_3) + \dots I_n \sin(n\theta + \phi_n)$$

If the analysing current is given by the expression

$$i_a = I_a \sin (m\theta + \phi_a)$$

(i.e. the analysing current has the frequency of the m^{th} harmonic) then the instantaneous torque

$$T \propto i i_a$$

and the mean torque

$$T_m \propto \frac{1}{2\pi}\int_0^{2\pi} i i_a \, d\theta$$

$$\propto \frac{1}{2\pi}\int_0^{2\pi} I_a \sin (m\theta + \phi_a) [I_0 + I_1 \sin (\theta + \phi_1) + I_2 \sin (2\theta + \phi_2) + \ldots] \, d\theta$$

$$\propto \frac{1}{2\pi}\int_0^{2\pi} I_a \, I_m \sin (m\theta + \phi_a) \sin (m\theta + \phi_m) \, d\theta$$

since all such terms as $\int_0^{2\pi} I_a \, I_r \sin (m\theta + \phi_a) \sin (r\theta + \phi_r) d\theta$ are zero, r being different from m (see page 539).

Evaluating the integral we have

$$T_m \propto \frac{I_a \, I_m}{2} \cos (\phi_m - \phi_a)$$

$$\propto I_a' \, I_m' \cos (\phi_m - \phi_a)$$

or $$D = K \, I_a' \, I_m' \cos (\phi_m - \phi_a)$$

where D is the deflection of the dynamometer and is proportional to T_m. I'_a and I'_m are r.m.s. values of the analysing current and mth harmonic of the current in the moving coil respectively, and K is the calibration constant of the dynamometer.

The phase of the analysing current is altered by rotating the brush arm on the contact disc until $\phi_a = \phi_m$, when the dynamometer deflection will be maximum, given by $D_{max} = K \, I_a' I_m'$.

I_m' can thus be obtained from a knowledge of K and I_a'. I_a' is measured by the thermal type ammeter A.

The analysing current circuit containing the dynamometer fixed coil is tuned to the frequency of the harmonic being measured, chiefly for the purpose of improving the wave-form of the analysing current. The impedance in series with the moving coil consists of a resistor and a capacitor, the latter producing magnification of the harmonics in the current wave in the moving coil (see page 553). Then, if R is the total resistance of the moving coil circuit, L its inductance, and C the series capacitance (all of which, together with ω, must be known) the mth harmonic of the applied voltage wave is given by

$$V_m = I_m' \sqrt{R^2 + \left(m\omega L - \frac{1}{m\omega C}\right)^2}$$

The magnitudes of the various harmonics are finally expressed as percentages of the reading of the voltmeter V, these usually being sufficiently close to the percentages of the fundamental if the harmonics are small.

Analysis of a current wave is carried out by connecting the dynamometer moving coil in parallel with a suitable shunt in the current circuit.

The accuracy obtained with this instrument, when analysing a voltage wave of the order of 100 volts is within $\frac{1}{20}$ of one per cent of the fundamental in the case of the larger harmonics and within $\frac{1}{100}$ of one per cent of the fundamental for the very small harmonics.

A Wave Analyser made by Dawe Instruments, Ltd. measures the component frequencies and amplitudes of wave forms over the frequency range 50 to 16,000 c/s with four input ranges of 0·5, 5, 50 and 500 V maximum.

The circuit uses a local oscillator and balanced modulator which work on the heterodyne principle giving a constant intermediate frequency from the selected component of the input. The subsequent amplifier is sharply tuned to this intermediate frequency and the output is measured by a valve voltmeter. The selectivity can be varied over a wide range with negligible change in the gain of the amplifier, and the use of toroidal inductances and silvered-mica capacitors, with negative feedback, ensures permanent stability of the tuned amplifier.

Muirhead and Co., Ltd. have recently developed the Muirhead-Pametrada wave analyser, designed primarily for analysing wave-forms of complex vibrations, but being applicable also to audio and power frequency wave-forms up to 21 kc/s. The design is quite different from that of the heterodyne type of analyser which produces an output frequency that is almost constant whatever the frequency under test. The Muirhead analyser acts as a tunable filter and feeds to the output meter a signal of the same frequency as that of the selected component. The output can be applied to an oscillograph.

The instrument is described by the makers as of the negative-feedback type with resistance-capacitance tuning. It uses two selective amplifier units connected in cascade and tuning is by varying simultaneously two resistors in the arms of the Wien bridges which provide the negative feedback path for each amplifier.

The Shape of the Current Wave-form when the e.m.f. Wave-form contains Harmonics. The shape of the current wave-form when an e.m.f. whose wave contains harmonics is applied to a circuit is not, in general, the same as that of the impressed e.m.f. wave, but depends upon the resistance, inductance, and capacitance in the circuit. The consideration of the question is simplified if the complex e.m.f. wave is thought of as consisting of a fundamental wave and its harmonic waves, each of which exists separately, and produces its own component of the current wave.

Different forms of circuits will now be considered.

1. RESISTANCE ONLY. Suppose an e.m.f. whose equation is $e = E_1 \sin \omega t + E_3 \sin 3\omega t + E_5 \sin 5\omega t + \ldots$ is impressed upon a circuit of resistance R and whose inductance and capacitance are both negligible. Then the equation of the current is

$$i = \frac{E_1}{R} \sin \omega t + \frac{E_3}{R} \sin 3\omega t + \frac{E_5}{R} \sin 5\omega t + \ldots \qquad . \quad . \quad (293)$$

Thus, the current wave-form is of the same shape as that of the impressed e.m.f.

2. INDUCTANCE ONLY. Suppose the same e.m.f. is impressed upon a circuit of inductance L and having negligible resistance and capacity. The phase of each component of the current will lag 90° behind the e.m.f. component producing it. Also, the reactance of the circuit will not be the same to all the e.m.f. components.

Hence, the components of the current wave are

$$i_1 = \frac{E_1}{\omega L} \sin \left(\omega t - \frac{\pi}{2}\right)$$

$$i_3 = \frac{E_3}{3\omega L} \sin \left(3\omega t - \frac{\pi}{2}\right)$$

$$i_5 = \frac{E_5}{5\omega L} \sin \left(5\omega t - \frac{\pi}{2}\right)$$

and so on.

Thus, the equation of the current wave is

$$i = \frac{E_1}{\omega L} \sin \left(\omega t - \frac{\pi}{2}\right) + \frac{E_3}{3\omega L} \sin \left(3\omega t - \frac{\pi}{2}\right) + \frac{E_5}{5\omega L} \sin \left(5\omega t - \frac{\pi}{2}\right) + \ldots \quad . \quad (294)$$

The wave-shape of the current is thus different from that of the e.m.f. due to the facts that, first, the amplitudes of the harmonics in the current wave are not in a constant ratio to the e.m.f. harmonics to which they are due, but decrease with increasing order of the harmonic; and second, that the phases of the current harmonics are different from those of the e.m.f. harmonics.

The general effect is, however, that inductance in a circuit *reduces* the amplitudes of the harmonics in the current wave and causes the latter to approach more nearly to the sinusoidal wave-shape than the e.m.f. wave.

3. CAPACITANCE ONLY. If an e.m.f. wave

$$e = E_1 \sin \omega t + E_3 \sin 3\omega t + E_5 \sin 5\omega t + \ldots$$

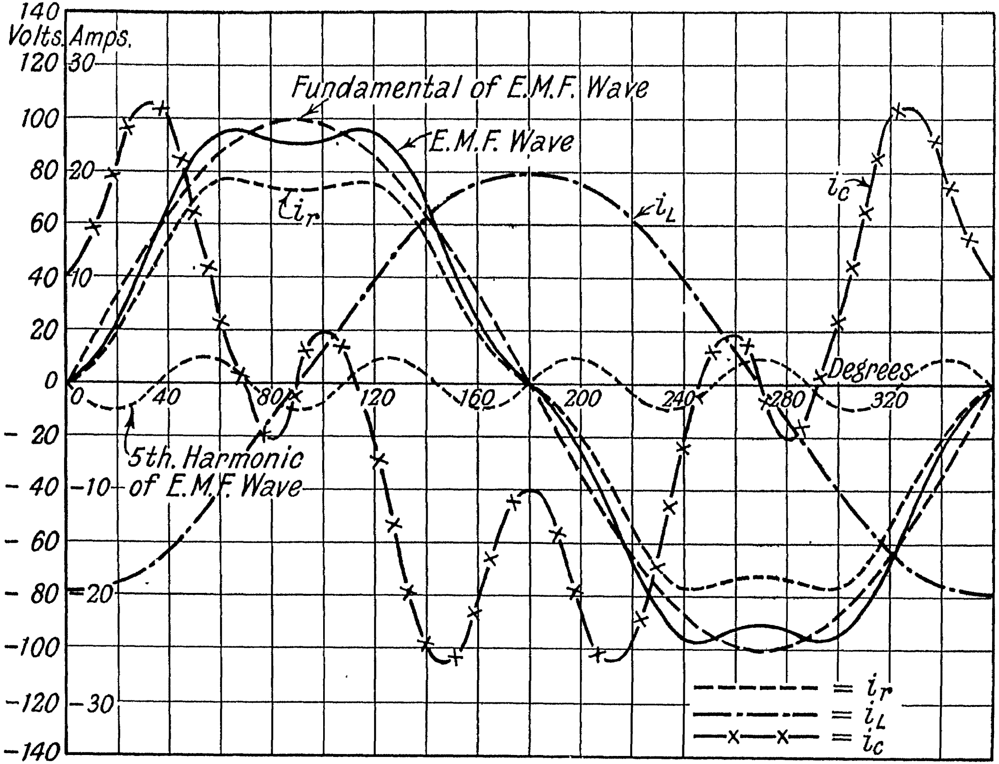

140
Volts.
Amps.
120
30
100
80
20
60
40
10
20
0
– 20
– 40
–10
– 60
– 80
–20
–100
–120
–30
–140
0
40
80
120
160
200
240
280
320
Degrees
Fundamental of E.M.F. Wave
E.M.F. Wave
i_r
i_L
i_c
5th. Harmonic of E.M.F. Wave
= i_r
= i_L
= i_c

Fig. 332

is impressed upon a circuit having capacitance C and whose resistance and inductance are both negligible, the current produced is

$$\frac{e}{\text{The reactance of the capacitor}}$$

The reactance due to the capacitance C is not the same to all the components of the e.m.f. wave, since it is given by the expression

$$\frac{1}{2\pi \times \text{frequency} \times C}$$

and the frequency differs for different harmonics. The current is thus given by

$$i = E_1\omega C \sin\left(\omega t + \frac{\pi}{2}\right) + 3E_3\omega C \sin\left(3\omega t + \frac{\pi}{2}\right) + 5E_5\omega C \sin\left(5\omega t + \frac{\pi}{2}\right) + \ldots \quad . \quad . \quad (295)$$

each component of the current leading the e.m.f. component to which it is due by the angle $\frac{\pi}{2}$.

It can be seen that the effect of capacitance upon the current wave is to increase the amplitudes of the harmonics in it, relative to the fundamental, as compared with the relative amplitudes in the e.m.f. wave.

Thus, the ratio

$$\frac{\text{Amplitude of 3rd harmonic of e.m.f. wave}}{\text{Amplitude of the fundamental of e.m.f. wave}} = \frac{E_3}{E_1}$$

whereas, in the current wave,

$$\frac{\text{Amplitude of 3rd harmonic}}{\text{Amplitude of the fundamental}} = \frac{3E_3\omega C}{E_1\omega C} = \frac{3E_3}{E_1}$$

i.e. the 3rd harmonic is magnified three times in the current wave. Obviously the higher the order of the harmonic the greater the magnification.

Fig. 332 shows a complex voltage wave whose equation is

$$e = 100 \sin \omega t - 10 \sin 5\omega t$$

its fundamental and fifth harmonic components being shown, dotted.

Three current wave-forms are also shown. The first, i_r, is the wave-form of the current when the e.m.f. e is impressed upon a circuit of resistance 5 ohms, the inductance and capacitance of the circuit both being negligible,

$$i_r = \frac{100}{5} \sin \omega t - \frac{10}{5} \sin 5\omega t$$

or

$$i_r = 20 \sin \omega t - 2 \sin 5\omega t$$

This current is in phase with the e.m.f. The second wave-form i_L is that of the current obtained when the e.m.f. is applied to a circuit containing inductance only, the reactance, at the frequency of the fundamental, being 5 ohms.

Then
$$i_L = \frac{100}{5}\sin\left(\omega t - \frac{\pi}{2}\right) - \frac{10}{5 \times 5}\sin\left(5\omega t - \frac{\pi}{2}\right)$$
$$= 20\sin\left(\omega t - \frac{\pi}{2}\right) - 0{\cdot}4\sin\left(5\omega t - \frac{\pi}{2}\right)$$

This current lags by $\frac{\pi}{2}$ behind the e.m.f. wave, and its wave-form is seen to be much more nearly sinusoidal than that of the e.m.f.

The third wave-form, i_C, is that of the current in a circuit having capacitance only, when the e.m.f. e is impressed upon the circuit, the capacitance being such that the reactance of the circuit, at the frequency of the fundamental, is again 5 ohms.

Then,
$$i_C = \frac{100}{5}\sin\left(\omega t + \frac{\pi}{2}\right) - \frac{10}{5/5}\left(\sin 5\omega t + \frac{\pi}{2}\right)$$
or
$$i_C = 20\sin\left(\omega t + \frac{\pi}{2}\right) - 10\sin\left(5\omega t + \frac{\pi}{2}\right)$$

The distortion in the current wave i_C, when capacitance, only, exists in the circuit, is very noticeable.

Resonance with Harmonics. If an e.m.f. of complex wave-form such as
$$e = E_1\sin(\omega t + \phi_1) + E_3\sin(3\omega t + \phi_3) + \ldots \qquad (296)$$
is impressed upon a circuit containing resistance R, inductance L and capacitance C, in series, the equation of the current in the circuit is
$$i = \frac{E_1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\sin(\omega t + \phi_1 - \theta_1)$$
$$+ \frac{E_3}{\sqrt{R^2 + \left(3\omega L - \frac{1}{3\omega C}\right)^2}}\sin(3\omega t + \phi_3 - \theta_3) + \ldots \qquad (297)$$
where $\theta_1, \theta_3, \ldots$, are the phase angles by which the components of the current wave lag behind the e.m.f. components which produce them.

Thus,
$$\tan\theta_1 = \frac{\omega L - \frac{1}{\omega C}}{R}, \quad \tan\theta_3 = \frac{3\omega L - \frac{1}{3\omega C}}{R}$$
and so on.

When the inductive reactance ωL, in any circuit, is equal to the capacitance reactance $\frac{1}{\omega C}$, the condition of "resonance" exists, and the current in the circuit is then given by $\frac{\text{e.m.f.}}{\text{resistance}}$. This current may be very large, and as a result, the voltage across either the inductance or the capacitance may become seriously high, although the voltage across the inductance and capacitance together is zero.

The values of L and C, in the circuit which is at present being considered, may be such that, although ωL is not equal to $\frac{1}{\omega C}$, yet $n\omega L = \frac{1}{n\omega C}$ where n is the order of one of the harmonics contained in the e.m.f. wave; for example, $5\omega L$ might equal 1

Then the amplitude of the 5th harmonic in the current wave would be $\frac{E_5}{R}$, and this harmonic would, also, be in phase with the e.m.f. harmonic producing it. The result is that the current wave contains a disproportionately large fifth harmonic and is therefore considerably distorted.

This resonance effect may be used to ascertain whether any particular harmonic exists in an e.m.f. wave-form or not. A resistance, variable inductance, and variable capacitance, are connected in series, together with an oscillograph element (for the determination of the resulting current wave-form) across the source of the e.m.f. under test. The values of the inductance and capacitance are adjusted so that, at the frequency of the harmonic whose presence is to be detected, resonance is obtained. An oscillograph record of the current which then flows is obtained and examined. The presence of the harmonic which is to be detected will be indicated by a badly-distorted current wave-form.

R.M.S. Value of Complex Waves. The r.m.s. value of any alternating e.m.f. or current is given by

$$E = \left[\frac{1}{\pi}\int_o^\pi (E_1 \sin(\theta + \phi_1) + E_3 \sin(3\theta + \phi_3) + \ldots)^2 d\theta\right]^{\frac{1}{2}} \quad \ldots \quad (298)$$

$$= \left[\frac{1}{\pi}\int_o^\pi \begin{pmatrix} E_1^2 \sin^2(\theta + \phi_1) + E_3^2 \sin^2(3\theta + \phi_3) + \ldots \\ + 2E_1E_3 \sin(\theta + \phi_1)\sin(3\theta + \phi_3) \\ + 2E_1E_5 \sin(\theta + \phi_1)\sin(5\theta + \phi_5) \\ + \ldots \end{pmatrix} d\theta\right]^{\frac{1}{2}}$$

Now the integrals of all such terms as $2E_1E_3 \sin(\theta + \phi_1) \sin(3\theta + \phi_3)$ between the limits π and 0 are zero. Thus,

$$E = \left[\frac{1}{\pi}\int_o^{\pi}(E_1^2 \sin^2(\theta + \phi_1) + E_3^2 \sin^2(3\theta + \phi_3) + \ldots)d\theta\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (299)$$

Now $\int_o^{\pi} \sin^2(\theta + \phi)d\theta = \frac{\pi}{2}$

$$\therefore E = \left[\frac{1}{\pi}\left(E_1^2\frac{\pi}{2} + E_3^2\frac{\pi}{2} + E_5^2 \cdot \frac{\pi}{2} + \ldots\right)\right]^{\frac{1}{2}}$$

or $E = \sqrt{\dfrac{E_1^2 + E_3^2 + E_5^2 + \ldots}{2}}$ . . . . . (300)

Now, E_1, E_3, E_5, . . . , are maximum values of the various components of the complex wave. All these components are sine waves of various frequencies, and therefore the r.m.s values of the components are, in all cases, given by

$$\frac{\text{Maximum value of the component}}{\sqrt{2}}$$

Thus, if E_1', E_3', etc., are the r.m.s. values of the harmonics composing the complex wave, the r.m.s. value of this wave is

$$\sqrt{(E_1')^2 + (E_3')^2 + (E_5')^2 + \ldots} \quad . \quad . \quad . \quad (301)$$

Example. Calculate the r.m.s. value, of the current whose equation is $i = 50 \sin 157t + 20 \sin 471t + 5 \sin 785t$

$$I = \sqrt{\frac{50^2 + 20^2 + 5^2}{2}}$$

$$= \sqrt{\frac{2500 + 400 + 25}{2}} = 38{\cdot}2$$

Power in Circuits in which the e.m.f. and Current Waves are not purely Sinusoidal. It has already been seen that when the e.m.f. impressed upon a circuit is

$$e = E_1 \sin(\omega t + \phi_1) + E_3 \sin(3\omega t + \phi_3) + \ldots$$

the most general expression for the current is

$$i = I_1 \sin(\omega t + \phi_1 - \theta_1) + I_3 \sin(3\omega t + \phi_3 - \theta_3) + \ldots$$

The instantaneous power p in such a circuit is given by

$$\begin{aligned} p = ei = {} & E_1I_1 \sin(\omega t + \phi_1)\sin(\omega t + \phi_1 - \theta_1) \\ & + E_3I_3 \sin(3\omega t + \phi_3)\sin(3\omega t + \phi_3 - \theta_3) \\ & + \ldots . \\ & + E_1I_3 \sin(\omega t + \phi_1)\sin(3\omega t + \phi_3 - \theta_3) \\ & + E_3I_1 \sin(3\omega t + \phi_3)\sin(\omega t + \phi_1 - \theta_1) \\ & + \ldots . \quad . \quad . \quad . \quad . \quad (302) \end{aligned}$$

The mean power in the circuit is

$$\begin{aligned} P &= \frac{1}{\pi}\int_o^{\pi} ei \,.\, d(\omega t) \\ &= \frac{1}{\pi}\int_o^{\pi} [E_1I_1 \sin(\omega t + \phi_1)\sin(\omega t + \phi_1 - \theta_1) \\ &\qquad + E_3I_3 \sin(3\omega t + \phi_3)\sin(3\omega t + \phi_3 - \theta_3) \\ &\qquad\qquad + \ldots . \\ &\qquad + E_1I_3 \sin(\omega t + \phi_1)\sin(3\omega t + \phi_3 - \theta_3) \\ &\qquad + E_3I_1 \sin(3\omega t + \phi_3)\sin(\omega t + \phi_1 - \theta_1) \\ &\qquad\qquad + \ldots .]\, d(\omega t) \qquad . \quad . \quad . \quad (303) \end{aligned}$$

The integrals between 0 and π of all such terms as $E_1I_3 \sin(\omega t + \phi_1)\sin(3\omega t + \phi_3 - \theta_3)$, where the frequencies of the two sine waves which are multiplied together are different, are zero.

Thus,

$$\begin{aligned} P &= \frac{1}{\pi}\int_o^{\pi} [E_1I_1 \sin(\omega t + \phi_1)\sin(\omega t + \phi_1 - \theta_1) \\ &\qquad + E_3I_3 \sin(3\omega t + \phi_3)\sin(3\omega t + \phi_3 - \theta_3) \\ &\qquad\qquad + \ldots .]\, d(\omega t) \\ &= \frac{1}{\pi}\left[\frac{\pi}{2}(E_1I_1\cos\theta_1 + E_3I_3\cos\theta_3 + \ldots .)\right] \\ &= \frac{1}{2}(E_1I_1\cos\theta_1 + E_3I_3\cos\theta_3 + \ldots .) \end{aligned}$$

or $P = E_1'I_1'\cos\theta_1 + E_3'I_3'\cos\theta_3 + \ldots .$ . . . (304)

where E_1', E_3', . . . , and I_1', I_3', ... are the r.m.s. values of the component harmonics of the voltage and current.

Thus, the mean power in the circuit is the sum of the mean powers due to the various components of the current and voltage, since $E_1'I_1 \cos\theta_1$ is the mean power due to the fundamentals of the two waves, $E_3'I_3' \cos\theta_3$ that due to the 3rd harmonic, and so on.

Power Factor. The power factor of a circuit in which the voltage and current waves are non-sinusoidal requires special definition.

Consider the usual definition of power factor as the cosine of the angle of phase-difference between the voltage and current waves. If these waves are complex—and possibly differently-shaped—this angle of phase difference between them is somewhat indefinite. This difficulty is overcome by defining the power factor of the circuit as the cosine of the angle of phase-difference between two "equivalent sine waves" having respectively r.m.s. values equal to those of the voltage and current in the circuit. Let ϕ_e be the phase difference between the two equivalent sine waves, then the power factor is

$$\cos\phi_e = \frac{P}{EI}$$

where P is the mean power in the circuit, and E and I are the r.m.s. values of the equivalent sine waves of voltage and current. P is the value of the power which would be measured by a wattmeter connected in the circuit, while E and I are the values of voltage and current which would be measured by alternating current instruments, which give the equivalent sine wave values.

Determination of Wave-form. There are two general methods of determining the wave-form of an alternating voltage or current, namely, contact or point-by-point methods, and by the use of oscillographs.

The former methods are useful only when conditions in the circuit under test are steady, i.e. when each succeeding wave of voltage is exactly the same, both in amplitude and form, as the preceding wave. Oscillograph methods are not subject to this limitation, and are, therefore, much more generally used than the point-by-point methods.

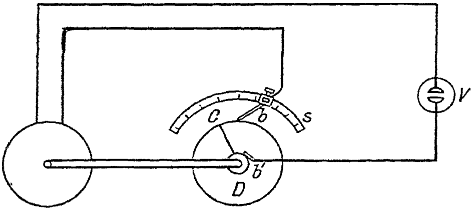


FIG. 333. JOUBERT CONTACT METHOD

JOUBERT'S CONTACT METHOD. This method was used by Joubert (Ref. (14)) for the determination of the wave-form of an alternator. An electrostatic voltmeter, or a valve voltmeter with a reservoir capacitor, is connected, instantaneously, to the terminals of the alternator or supply circuit, once during every cycle of the voltage wave, the connection of the instrument to the alternator being made at the same point in the cycle in each case. The instrument reading will give the ordinate of the voltage wave at this one point in the cycle. The connection must exist in each cycle only for a time which is a very small fraction of the periodic time of the cycle.

The contact device is then moved through a known angle, so that connections to the voltmeter are next made at some other—known—point in the cycle. This process is continued until the wave-form can be traced from measurements of the ordinates at sufficient points in the cycle.

Fig. 333 shows how these instantaneous contacts are made. D is a disc of hard insulating material, either mounted on the shaft of the alternator or synchronously driven. It carries a metal ring near its centre upon which a brush b' rests. A light metal rod is also carried by the disc. It is permanently connected to the ring at the centre and projects slightly beyond the rim of the disc at *C*, rotating with the disc. Another brush *b*, placed as shown, makes contact with the rod at *C*, instantaneously, once for every revolution of the disc and connects the voltmeter *V* to the alternator terminals, thus giving the voltage at one point in the cycle depending upon the position of the brush *b*. This brush is carried on a scale graduated in "electrical degrees," so that the point of the cycle corresponding to any setting can be determined.

The Joubert contact method has been extended to the separation of harmonics in complex wave-forms and to the determination of the peak value and wave-form of a high voltage of the order of 1,000 kV. Descriptions of these applications are given in the papers mentioned in Refs. (37) and (38).

The *Hospitalier Ondograph* is an improvement on the above method. This is a piece of apparatus in which the wave-form of the e.m.f. is recorded upon a revolving drum which is driven by a small synchronous motor supplied from the source of e.m.f. under test. A contact maker is also driven by this motor and, by means of gearing, arrangements are made for the brushes making the contacts to be continuously displaced relative to the e.m.f. wave. Thus the contact-point on this wave is slowly varied, and the complete wave traced on a sheet of paper, carried on the drum, by a pen whose position, at any point on the cycle, is dependent upon the e.m.f. corresponding to that point. A complete description of the ondograph will be found in Hospitalier's original paper (Ref. (15)).

Oscillographs. Oscillographs are used for the observation and measurement of transient and other phenomena, as well as for the determination of alternating current and voltage wave-forms. There are three types of oscillograph in use—

(*a*) Electromagnetic.
(*b*) Electrostatic.
(*c*) Cathode-ray.

They all must consist of a moving system, which is caused to deflect when the current or voltage under test is applied to the instrument, and an optical system, for the indication or recording of the wave.

The deflection of the moving system must, at any instant, be strictly proportional to the applied voltage or to the current at that instant. Thus the moving system must possess very little inertia in order that it may respond instantly to changes of current or voltage. The natural period of the system must be much smaller than that of the alternating wave under test, and the damping must be critical. Eddy current and hysteresis effects in the instrument should be negligible, and its inductance and capacitance must be very small.

(*a*) ELECTROMAGNETIC OSCILLOGRAPH. Two forms of electromagnetic oscillograph have been developed—the moving-magnet type, by Blondel, and the moving-coil type, by Duddell. The Duddell instrument, which has the advantages of negligible inductance and freedom from hysteresis errors, was commonly used for years but has now been displaced, for many purposes, by the cathode-ray oscillograph. Electromagnetic oscillographs are best suited to work at low voltages and comparatively low frequencies. They are, of course, vibration galvanometers having an especially low natural period of vibration.

Duddell Oscillograph. The principle of this instrument is illustrated by Fig. 334. A single loop of thin phosphor-bronze strip forms the vibrator, or moving system. This is situated in the field of a powerful magnet—either a permanent magnet or an electromagnet—having specially-shaped pole-pieces to obtain an intense field between them. If high sensitivity is required an electromagnet is used, the magnet being worked with a high degree of saturation in order that small changes in magnetizing current shall have a negligible effect upon the strength of the magnetic field.

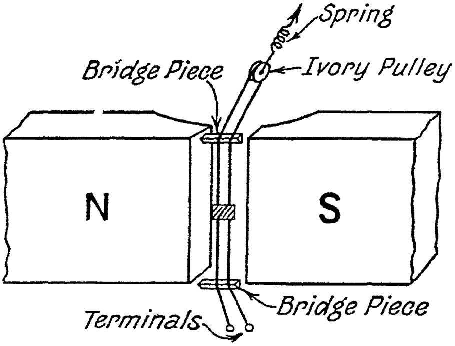


FIG. 334. DUDDELL OSCILLOGRAPH

The vibrator loop is formed by passing the phosphor-bronze strip round a small ivory pulley, to which a spring and tension-adjusting device are attached. The loop passes over two ivory bridge-pieces which confine the vibrating portion of the loop to the section which is situated in the magnetic field. A small, light mirror is cemented to the loop midway between the bridge pieces. A piece of soft iron, cut away to clear the mirror, is usually fitted between the two sides of the loop, so that each side has its own air gap. The clearances between the sides of the loop and the pole pieces are very small—of the order of 0·01 cm.

If a current of I amperes passes through the loop which is situated in a magnetic field of strength H, forces of $\frac{HIl}{10}$ dynes act on each side, causing one side to move inwards and the other outwards, thus causing the loop as a whole to deflect, as shown in Fig. 335. l is the length of loop between the bridge pieces. If the current I is alternating, the loop is caused to vibrate. In this type of oscillograph the undamped natural period of the loop is about $\frac{1}{10,000}$ second.

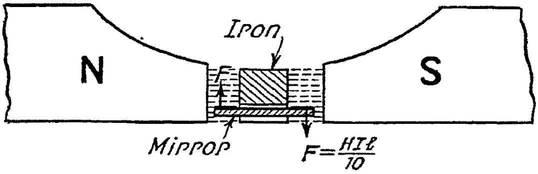


FIG. 335

(It may be as low as $\frac{1}{17,000}$ sec.) Damping is obtained by immersing the moving system in oil, each side of the loop vibrating in a small oil-filled chamber formed by the pole-pieces and the central soft iron piece. Wave-forms of frequencies up to about 300 cycles per second can be faithfully recorded by such an instrument.

These oscillographs are often fitted with several vibrators, side by side. (The Cambridge Instrument Co. make both six- and twelve-element oscillographs of this type.) A fixed mirror, between them, is used to give a zero line on the film upon which the wave-forms are recorded. The connections of two vibrators for obtaining current and voltage wave-forms are shown in Fig. 336.

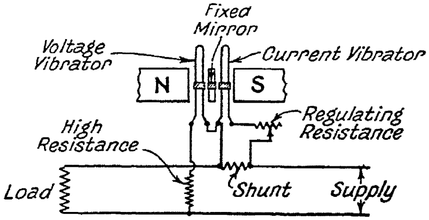


FIG. 336. CONNECTIONS OF DUDDELL OSCILLOGRAPH

The resistances of the loops are usually about 4 or 5 ohms, and currents of 1 milliamp can be detected, the safe working current of the vibrators being about 100 milliamps. It is thus necessary to shunt the current vibrator and to connect a high resistance in series with the voltage vibrator, as shown in the figure.

A beam of light cast upon the mirror of one of these loops will be reflected, and the reflected beam will move backwards and forwards in a horizontal plane (assuming the loops to be in a vertical position). In recording the wave-shape, the photographic film is passed at constant speed—the actual speed depending upon the frequency of the waves being recorded—in a direction perpendicular to that of the movement of the reflected beam of light, the result of these two perpendicular motions being the wave-shape required. If time measurements are to be made upon the "oscillogram" so obtained, a current of known frequency is passed through one of the loops, giving a wave on the oscillogram which can then be used as a time scale.

The oscillograph can be calibrated by observing the deflection produced by a known direct current in the vibrator loop.

If a wave-form is to be observed, but not recorded, the optical arrangements are as shown in Fig. 337. The reflected beam of light from the vibrator mirror falls on a plane mirror and is reflected from thence upon a screen as shown. A cam, driven by a small synchronous motor, which is supplied from the source whose wave-form is required, is responsible for rocking this mirror, and so giving a forward motion of the reflected beam on the screen. This forward motion is continued for about $1\frac{1}{2}$ cycles of the wave, and the plane mirror is returned, by the cam, to its initial position during the next half-cycle, during which period the light is cut off by a rotating shutter attached to the motor shaft. Persistence of vision gives the effect of a continuously-existing wave ($1\frac{1}{2}$ cycles long) on the screen.

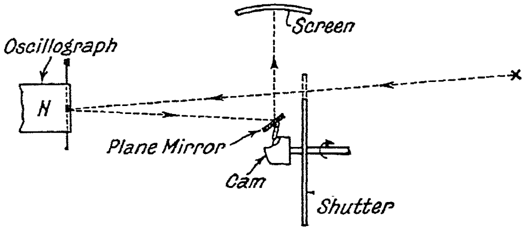


FIG. 337. OPTICAL ARRANGEMENTS FOR OBSERVING FORMS WITH A DUDDELL OSCILLOGRAPH

Theory of the Duddell Oscillograph. Since the instrument is, essentially, a moving-coil vibration galvanometer, its equation of motion is

$$a\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + c\theta = Gi$$

as shown when the theory of the vibration galvanometer was discussed (see page 259).

In considering the vibration galvanometer the current i was assumed to be purely sinusoidal, its equation being $i = I_{max} \cos \omega t$. In the case of the oscillograph, however, this assumption cannot be made. The current i must be assumed to contain harmonics and to be given by the expression

$$i = \sum_{n=1}^{n=a} I_n \sin (n\omega t \pm \phi_n) \quad . \quad . \quad . \quad . \quad (305)$$

according to Fourier's Theorem. n is the order of the various harmonics and varies from 1 to the order of the highest harmonic present in the wave.

The general expression for the deflection θ was found, when discussing the theory of the vibration galvanometer, to contain a transient term, and a term corresponding to a steady vibratory motion. In the oscillograph the damping is critical ($b^2 = 4ac$) and the transient term rapidly falls to zero. The expression representing the motion of the vibrator is

$$\theta = G \sum_{n=1}^{n=a} \frac{I_n}{c + n^2\omega^2 a} \sin (n\omega t \pm \phi_n - \beta) \quad . \quad (306)$$

where $\beta = \tan^{-1} \dfrac{2n\omega\sqrt{ac}}{c - an^2\omega^2}$

This expression follows from the vibration galvanometer theory. Owing to the term $an^2\omega^2$ in the denominator, the higher harmonics have a disproportionately small effect upon the deflection, as compared with the lower harmonics. The harmonics also, in the expression for the deflection θ, have not the same relative phases as those in the current wave in the loop, and thus the oscillograph does not give a true reproduction of the current wave down to the harmonics of a high order. It can, however, be made to give a reproduction which is sufficiently accurate for practical purposes.

Obviously, if the damping and inertia are both zero—i.e. if the constants a and b are both zero—the deflectional equation becomes

$$c\theta = G \sum_{n=1}^{n=a} I_n \sin (n\omega t \pm \phi_n) \quad . \quad . \quad . \quad (307)$$

and the deflection is directly proportional to the current at any instant. Under these conditions the amplitude of the deflection produced by the nth harmonic is

$$\theta_n = \frac{GI_n}{c}$$

When inertia and damping are present the amplitude of the deflection produced by the nth harmonic is

$$\theta_n' = \frac{GI_n}{c + n^2\omega^2 a}$$

The ratio of the actual amplitude of the deflection produced by the nth harmonic, to the amplitude produced under ideal conditions, is thus,

$$\frac{\theta_n'}{\theta_n} = \frac{\dfrac{GI_n}{c + n^2\omega^2 a}}{\dfrac{GI_n}{c}} = \frac{c}{c + n^2\omega^2 a} = \frac{1}{1 + \dfrac{n^2\omega^2 a}{c}}$$

Now, from equation (180), page 263, the periodic time of the damped natural oscillations of the moving system is that value of t which makes $\dfrac{\sqrt{4ac - b^2}}{2a} . t$ equal to 2π. Let this value of t be T', then $T' = \dfrac{4\pi a}{\sqrt{4ac - b^2}}$. If the damping is zero (i.e. $b = 0$) the free period T_0 of the moving system, is given by

$$T_0 = 2\pi\sqrt{\frac{a}{c}}$$

Let T be the period of the fundamental of the wave whose form is to be determined by the oscillograph, then

$$\omega = 2\pi f = \frac{2\pi}{T}$$

or

$$T = \frac{2\pi}{\omega}$$

Substituting T_0 and T in the expression for the ratio $\dfrac{\theta_n'}{\theta_n}$ we have

$$\frac{\theta_n'}{\theta_n} = \frac{1}{1 + n^2 \dfrac{4\pi^2}{T^2} \cdot \dfrac{T_o^2}{4\pi^2}} = \frac{1}{1 + n^2\left(\dfrac{T_o}{T}\right)^2}$$

Calling this ratio R_n, we have

$$R_n = \frac{1}{1 + n^2\left(\dfrac{T_o}{T}\right)^2} \text{ or } \frac{T_o}{T} = \frac{1}{n}\sqrt{\frac{1}{R_n} - 1} \quad . \quad . \quad (308)$$

Example. A Duddell oscillograph has a moving system whose natural period of oscillation is $\dfrac{1}{10{,}000}$ sec. The wave-form of a complex wave, the frequency of whose fundamental is 50 cycles per second, is to be determined. What are the percentage errors in the reproduction of the 3rd. 5th, 7th, and 11th harmonics?

$$T_o = \frac{1}{10{,}000} \text{ sec.} \quad T = \frac{1}{50} \text{ sec.}$$

Then $$R_3 = \frac{1}{1 + 3^2\left(\dfrac{\dfrac{1}{10{,}000}}{\dfrac{1}{50}}\right)^2} = \frac{1}{1 + 9\left(\dfrac{50}{10{,}000}\right)^2} = 0{\cdot}9998$$

$$R_5 = \frac{1}{1 + 25\left(\frac{50}{10{,}000}\right)^2} = 0{\cdot}9994$$

$$R_7 = \frac{1}{1 + 49\left(\frac{50}{10{,}000}\right)^2} = 0{\cdot}9988$$

$$R_{11} = \frac{1}{1 + 121\left(\frac{50}{10{,}000}\right)^2} = 0{\cdot}9970$$

Thus the percentage errors in reproduction are—

In the 3rd harmonic . .	0·02 per cent
,, 5th ,, . .	0·06 ,,
,, 7th ,, . .	0·12 ,,
,, 11th ,, . .	0·30 ,,

Phase Displacement of Harmonics during Reproduction. It has been seen (equation (306)) that a phase displacement β occurs, in the expression for the deflection of the oscillograph, in the case of any particular harmonic.

This angle β is given by

$$\beta = \tan^{-1}\frac{2n\omega\sqrt{ac}}{c - an^2\omega^2}$$

Substituting $\frac{2\pi}{T}$ for ω and $\frac{T_o}{2\pi}$ for $\sqrt{\frac{a}{c}}$ we have, after simplifying,

$$\beta = \tan^{-1}\frac{2n}{\frac{T}{T_o} - n^2 \cdot \frac{T_o}{T}} \quad . \quad . \quad . \quad . \quad (309)$$

Thus, in the above example, the phase displacement in the reproduction of the seventh harmonic, relative to the actual phase of this harmonic, is

$$\tan^{-1} . \frac{14}{\left(\frac{1}{50}\Big/\frac{1}{10{,}000}\right) - 49 \cdot \left(\frac{1}{10{,}000}\Big/\frac{1}{50}\right)}$$
$$= \tan^{-1} 0{\cdot}0699$$
$$= 4^\circ$$

This angle must be measured on the scale of degrees corresponding to the seventh harmonic. Thus 360° of the fundamental wave corresponds to 7 × 360 degrees of the seventh harmonic.

(*b*) ELECTROSTATIC OSCILLOGRAPH. This instrument, due to Ho and Koto (Ref. (21)), is really an electrometer of special design. It is best suited to low-frequency high-voltage work (above 2,000 volts) and has the advantages of consuming no energy, and of requiring only a very small current. With the development of the cathode-ray oscillograph the electrostatic type has largely fallen out of use and will not, therefore, be described, although it was fully discussed in earlier editions of this book.

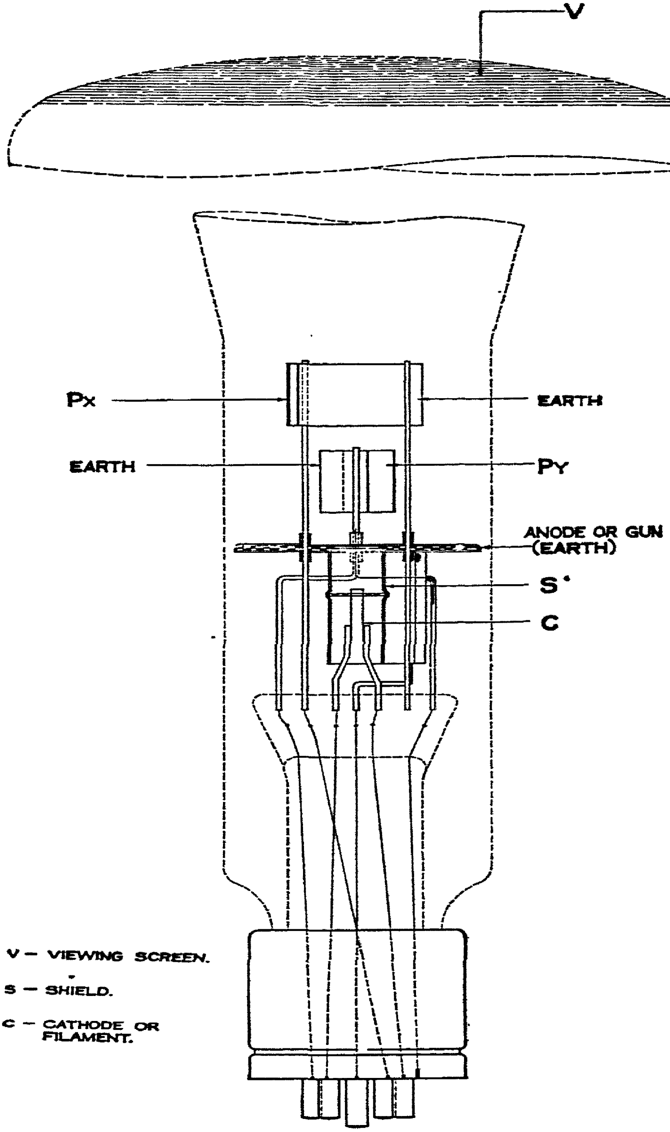
(*c*) THE CATHODE-RAY OSCILLOGRAPH. The cathode-ray oscillograph differs fundamentally from the previous instruments, as the moving metallic conductors are absent. They are replaced by a beam of electrons, which itself represents a current, and can be deflected by an electrostatic or an electromagnetic field. The inertia of this beam is very considerably less than that of a metallic conductor. Consequently the c.r.o. can be used at frequencies much above the highest which can be observed on the Duddell oscillograph.

The c.r. tube has two main parts; the electron "gun," and the deflecting arrangements (see Fig. 338, which shows the internal arrangement of the tube).

The *electron gun* is a device for producing and focusing a concentrated beam of fast electrons. The source of the electrons in the Crookes tube, from which the cathode-ray tube was eventually derived, is a piece of metal at room temperature. The field required to tear the electrons out of the metal is large, and high voltages must therefore be used. Modern c.r. tubes use a heated cathode coated with oxides of the rare-earth elements, which emit electrons very readily. The emitted electrons are accelerated by an electric field set up between the cathode and an anode placed near to it. The gun is so proportioned that the majority of the electrons form a concentrated beam. Electrons which are outside this beam are usually intercepted by a diaphragm.

The focusing arrangements may be either electrostatic or electromagnetic. In the former case, an electrostatic field is set up between two conductors in such a way that electrons passing through the field are deflected towards the axis. Such a device is termed an "electron lens" and the study of the electron paths through such a lens is termed "electron optics." The design of electron lenses is more difficult than that of optical lenses, and the performance of the electron lenses which are available is much inferior to that of optical lenses. In consequence, the electron beam must not deviate far from the direction of the axis or aberrations of the image will be produced. The last electrode of the lens system is normally at earth potential, and preceding electrodes and the cathode are made successively more negative. If this were not done the screen of the tube would have to be maintained at a positive potential equal to that of the last electrode of the lens. This leads to difficulties when an earthed body approaches the screen, since the potential of the latter is momentarily reduced by the capacitive effect and the electrons are repelled. This shows itself in practice as a defocusing of the beam when the tube face is touched.

Electromagnetic focusing depends upon the fact that an electron



(*Standard Telephones & Cables, Ltd.*)

FIG. 338. CATHODE-RAY OSCILLOGRAPH

which enters a constant magnetic field perpendicular to its path is deflected and moves in a circular path. The narrow axial beam of electrons produced by the electron gun is caused to enter a magnetic field parallel to the axis of the tube. Electrons which are travelling parallel to this axis are unaffected by the magnetic field while electrons which have a component of velocity away from the axis move in a spiral path which eventually returns to the axis. Provided that the axial velocity of all the electrons starting from a single point (the diaphragm) is the same, they will all, no matter what their radial velocity, return to the axis together at another single point. If it is arranged that this second point lies on the tube of the screen, focusing of the beam is achieved. Magnetic focusing is generally used with magnetic deflection, and although it is widely used in television it finds little application in c.r.o's.

One other method of focusing is of historical interest, the so-called "gas-focusing." Gas-focused tubes were not evacuated completely, and the electron beam passing through the residual gas produced a large concentration of ions. These, being much heavier than the electrons, and moving more slowly, formed a more or less persistent core of positive charge which held together the electron beam by electrostatic attraction. The great disadvantage of this means of focusing is that it fails when the electron beam is rapidly deflected, since the ion-core has not time to be formed in each new position.

The *deflecting arrangements* may again be either electrostatic or electromagnetic. Electrostatic deflection is accomplished quite easily by causing the focused electron beam to pass between two plates between which a potential difference exists. The electrons are attracted to the positive plate, and repelled from the negative. The deflecting plates usually open out towards the screen, so that they do not intercept the deflected beam. Two sets of plates are fitted, producing deflections at right-angles. The plates which produce a deflection to left or right of the screen, as normally mounted, are termed the X-plates, while those which produce a vertical deflection are termed the Y-plates. The deflecting plates are supplied with voltages which are balanced about earth, so that no further acceleration or deceleration of the electron beam leaving the electron lens is produced. Without this precaution the beam would be defocused by the deflecting voltage applied to the plates.

Electromagnetic deflection is achieved by producing a magnetic field perpendicular to the axis of the tube, by means of coils outside the tube. The electrons entering this field travel in circular paths, and emerge from the field with their paths deflected in a plane perpendicular to the deflecting field. Two fields at right angles are used to produce deflections in the X-direction (horizontally) and in the Y-direction (vertically). A greater deflection of the beam can be produced magnetically than electrostatically, and this enables a shorter tube to be used. The power required is considerably greater, however, and the inductance of the deflecting coils leads to difficulty when deflections are very rapid. In consequence the majority of c.r.o's use electrostatic deflection.

The beam of electrons, after deflection, travels along the c.r. tube until it impinges on the "screen." This is a coating of fluorescent materials on the end face of the tube, which shows a luminous spot at the point where the electron beam strikes it.

Theory of Electrostatic Deflection. If the mass of an electron is m, its charge q, the potential difference between the cathode and last anode V, the potential gradient between the deflecting plates E, and L_1 and L_2 are respectively the axial length of the deflecting plates, and the distance from the centre of the deflecting plates to the screen, then the electron beam will be travelling at a velocity given by

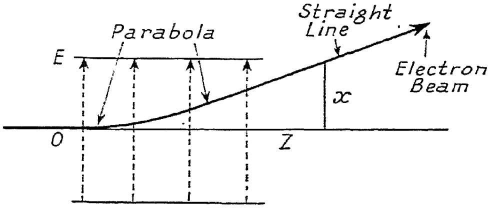


FIG. 339

$$qV = \frac{1}{2} mv^2 \quad . \quad . \quad . \quad . \quad (310)$$

Here v is the velocity of the beam, and the two sides of the equation give respectively the energy acquired by an electron in moving through the p.d. V, and the kinetic energy of the electron.

Referring to the Fig. 339, an electron entering the field between the plates will be accelerated with an acceleration f given by

$$mf = qE \quad . \quad . \quad . \quad . \quad . \quad . \quad (311)$$

The radial velocity of the electron will then be

$$v' = \int_0^t f \, . \, dt = \frac{qE}{m} \, . \, t \quad . \quad . \quad . \quad (312)$$

and its displacement from the axis

$$x = \int_0^t v' \, . \, dt = \frac{qE}{2m} \, . \, t^2 \quad . \quad . \quad . \quad (313)$$

Since the distance which the electron has moved along the axis is

$$z = vt = \sqrt{\frac{2qV}{m}} \, . \, t$$

it follows that the path of the electron is a parabola. It is a property of the parabola that the tangent at any point (x, z) intersects the z-axis at the point $\frac{1}{2}z$. Thus if the tangent to the emergent beam is produced back it meets the axis at the mid-point of the plates.

The angle α which the tangent to the emergent beam makes with the axis is given by

$$\tan \alpha = \frac{v'}{v} = \frac{qE}{m} \, . \, t \sqrt{\frac{m}{2qV}}$$

where t is the *time* taken for the electron to traverse the length L_1 of the plates, i.e.

$$t = \frac{L_1}{v} = L_1 \sqrt{\frac{m}{2qV}}$$

Thus

$$\tan \alpha = \frac{qEL_1m}{m \, . \, 2qV} = \frac{EL_1}{2V}$$

The radial distance of the beam from the axis when it arrives at the screen will be

$$y = L_2 \tan \alpha = \frac{EL_1L_2}{2V}$$

This simplified analysis has assumed that E is constant, i.e. it has neglected fringing of the field at the edge of the plates, and the non-uniformity of E which is produced when the plates are not parallel. In practice, the tube is calibrated with a steady potential applied between the plates, and the sensitivity in volts per cm. is thus obtained under working conditions.

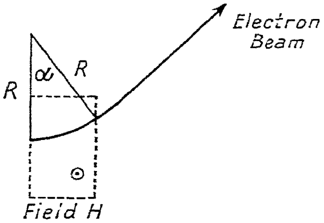


FIG. 340

Theory of Magnetic Deflection. An electron with charge q travelling with velocity v in the field H will experience a force qvH dynes, perpendicular to v and to H, in the sense given by the L.H. rule. It is evident that the speed of the electron will be unaffected, since the force acting upon it has no component in the direction of v. The electron thus has a constant velocity, and a constant acceleration perpendicular to its path. It follows that this path must be a circle. To find the radius of the circular path, we equate the force on the electron to the centrifugal force corresponding to its motion

$$qvH = \frac{mv^2}{R}$$

$$R = \frac{mv}{q \, . \, H}$$

Thus if the length of the circular path is S, the angle α will be

$$\alpha = \frac{S}{R} = \frac{SqH}{mv}$$
$$= \frac{SqH}{m}\sqrt{\frac{m}{2qV}},$$

using the value of v already found on page 571. Note that $S \fallingdotseq L_1$.

The tangent to the emergent beam will not intersect the axis exactly at the mid-point of the field, so that L_2 should not strictly be measured from the mid-point. For small values of α, however, the error is negligible, and we may write for the deflection

$y' = L_2 \tan \alpha = L_2\alpha$ approximately, when α is small, so that

$$y' = \sqrt{\frac{q}{2m}}\,\frac{HSL_2}{\sqrt{V}}$$

The value of $\frac{q}{m}$, the ratio of the charge (in e.m.u.) to the mass of an electron, is $1{\cdot}77 \times 10^7$, so that $y' \fallingdotseq 0{\cdot}3\,\frac{HSL_2}{\sqrt{V}}$ if V is in volts. As before, the treatment is only approximate, since it neglects the variations of H at the edges of the field, and as before the tube is calibrated with a known steady voltage.

Electromagnetic deflection can sometimes be used with advantage when a deflection proportional to a current is required, but the deflecting coils require an appreciable power, and their inductance may interfere with the circuit in which the current is to be measured. It is usual practice to obtain a voltage proportional to the current, and to apply this, with amplification if necessary, to a tube with E.S. deflection.

Recording with a C.R. Tube. Three different techniques are used for recording the information from a c.r. tube, according to the particular circumstances.

One method is to use a moving film on to which the image of the tube face is projected by a lens. The electron beam is deflected only in one plane, perpendicular to the direction of motion of the film, and the latter is either a continuous roll, or, for higher speeds, is wound upon a drum. If a drum is used, arrangements must generally be made to extinguish the beam until the record is required, and again when the length of film on the drum has been exposed. Alternatively, the shutter may be opened for one revolution of the drum at the appropriate time. If the beam remains at rest before and after the event to be recorded, however, the requirements are less critical, and it is only necessary to ensure that the shutter is opened in time to record the event, and not long enough for the record to be blurred by excess of light.

A second method is to project the face of the tube upon a stationary film, and to traverse the electron beam at a constant speed from left to right, at the same time as it is deflected vertically by the quantity to be measured. The horizontal movement, or "scan" is started at the beginning of the period which is to be recorded, and the beam moves only once across the screen. It is usual to extinguish the beam except during the period of the scan, since otherwise the intense spot stationary at the side of the screen "burns" the latter, i.e. renders it insensitive at that spot. Arrangements must be made to start the scan at the correct moment, but the shutter of the camera is left open until the trace has been recorded. This method of "single-stroke" working can be used for visual observation if the material of the screen has some delay, i.e. if the screen continues to give out light at the points where it was excited for some time afterwards.

The third method of recording can be used only with cyclic phenomena, and it consists in scanning the electron beam horizontally for an integral number of cycles of the phenomenon to be observed, and then returning it very quickly to the left-hand side of the screen, where it starts on a second scan. Thus a stationary picture of one or more waves is presented. This picture can be observed, and the arrangement forms the commonest application of the c.r. tube, namely the c.r. oscilloscope. Photographic records can be obtained by exposing a film for a suitable length of time to the stationary picture.

The third method has the advantage that a very faint, and consequently well-focused trace can be used, with a relatively long exposure, but the method is naturally limited to repetitive phenomena. Recording by the first method on continuous chart avoids the difficulties of synchronization, but at the expense of producing great lengths of unwanted record. The maximum chart speed for continuous feed is about 100 in./sec. With drum recording, chart speeds up to 1,000 in./sec., can be obtained, and frequencies up to about 10 kc/s can be recorded. With single-stroke working the highest writing speed which (to the author's knowledge) has been achieved is 20,000 km per second (or 20,000 mm. per microsecond) (Ref. (58)) though this is not necessarily the limit.

Synchronization of the record is quite easily carried out, either for drum recording or single-stroke recording, if the event can be controlled. In this case the electron beam is started by a pulse which also starts the event. If the event cannot be controlled, it is possible to use the signal which it produces to start the electron beam. The difficulty is then to keep down the unavoidable delay in starting the beam, which causes the beginning of the event to go unrecorded. The techniques involved are complicated, and resemble those used in radar.

Time-base Circuits. It was seen above that it is often necessary to traverse the electron beam at a constant speed from left to right, either once or continually at definite intervals. This is accomplished by the time-base circuit. Only circuits suitable for E.S. deflection will be considered here.

All time-base circuits depend fundamentally upon the charging of a capacitor. The voltage between the plates of a capacitor is

$$V = \frac{Q}{C}$$

so that the rate of change of voltage is

$$\frac{dv}{dt} = \frac{1}{C} \cdot \frac{dQ}{dt} = \frac{I}{C} \qquad (314)$$

Thus if the current flowing into a capacitor is constant, the voltage across it increases uniformly with time. The design of time-base circuits is directed towards obtaining a current which remains constant whatever the charge on the capacitor.

If the time-base is to be repetitive, as is usual, arrangements must also be made to return the beam rapidly to the left-hand side of the screen as soon as it reaches the right-hand side. The frequency of the time-base will depend on the rate of charging of the capacitor, and this must be variable over a wide range, in order to present input wave-forms of various frequencies. In addition it is necessary to synchronize the time-base with the input, to ensure that each sweep of the time-base begins at exactly the same part of the cycle of the input wave. Without this precaution the presentation of the wave-form on the screen of the c.r.o. would move erratically from side to side, or drift gradually to left or right as the time-base frequency altered.

The simplest possible time-base is obtained by charging a capacitor through a high resistance from a high-voltage supply. The current through the resistance depends on the voltage across it, and if the maximum voltage across the capacitor is small compared with the supply voltage, this current will remain nearly constant during the charging. The rate of rise of the deflecting voltage will thus be constant and a linear time base will result.

A method of producing a linear time base due to N. V. Kipping, employs a neon lamp with a variable capacitor across its terminals, the full connections being as shown in Fig. 341.

A neon lamp has the property of "striking" (i.e. commencing to pass a current) only when the voltage applied to it reaches a certain value and of "failing" (i.e. ceasing to pass a current) at a considerably lower voltage. Thus, referring to the figure, the capacitor C builds up to a voltage given by $e = E\,(1 - \varepsilon^{-\frac{t}{RC}})$, where E is the voltage of the battery supplying the lamp (200 volts) and R is the resistance in series with the capacitor and lamp, before the lamp strikes. The time taken for it to build up to the striking voltage of the lamp obviously depends upon the value of C (for a given value of R). When the lamp strikes the current through R increases, and the voltage across the terminals of the lamp falls below the failing value, after which the charging of C recommences. The neon lamp therefore "blinks" and a voltage which rises gradually and then falls rapidly is thus applied to one pair of deflecting plates. This produces an approximately linear time base, as shown in Fig. 342. The frequency of the blink can be altered by adjustment of C. Such a simple circuit is not, however, very satisfactory.

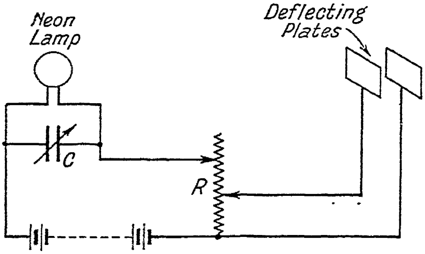


FIG. 341. CIRCUIT FOR OBTAINING THE TIME BASE

Fig. 343 gives a diagram of connections of a circuit utilizing a gas-filled triode which is typical of a number of such circuits devised for the production of a linear time base. The time base, or *sweep*, circuit itself is enclosed within the frame (shown dotted), the terminals for connection to the cathode-ray oscillograph being T_1, T_2, and T_3.

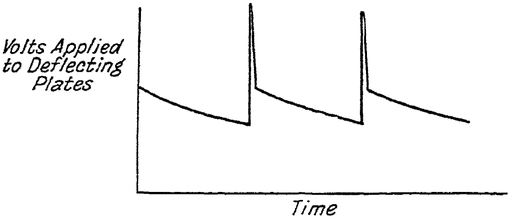


FIG. 342

P_e is a pentode and G a gas-filled triode (the gas may be mercury vapour, neon, etc.). P_1 and P_2 are potential dividers. R_1 (about 50,000 ohms) is a resistance in the grid circuit of the triode and R_2 (about 500 ohms) is a limiting resistance in its anode circuit. R_3 (about 2 megohms) and C_2 (about 2 microfarads) are for the purpose of isolating d.c. potentials so that these do not produce a steady deflection of the cathode ray.

The action of the circuit is as follows. When the d.c. supply is first switched on the capacitor C_4 (or C_3) acts instantaneously as a short circuit and the full d.c. voltage is applied to the pentode. C_4 charges (at almost constant current due to the shape of the anode current/anode volts characteristic of the pentode) and the voltage across it rises until it is sufficient for the triode G to discharge. The voltage required for discharge depends upon the grid bias of G and is therefore controlled by the potential divider P_2.

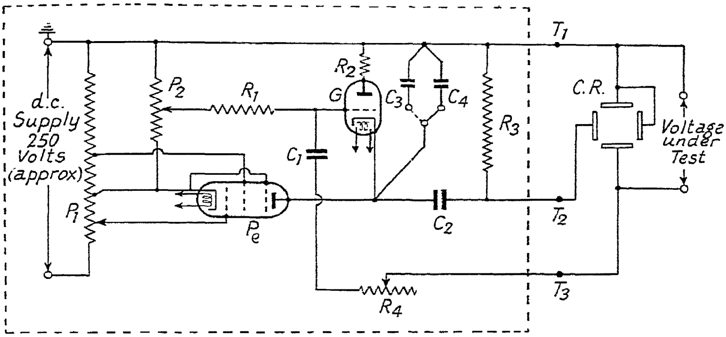


FIG. 343

Again, the charging current of C_4 depends upon the grid bias of the pentode and is controlled by potential divider P_1. Now, as already shown, the rate of sweep on the oscillograph depends upon the rate of increase of voltage across the time-base pair of plates, i.e. upon the rate of increase of the voltage across C_4 and, since i is constant, $\frac{dv}{dt}$ (and therefore the rate of sweep) is constant, giving a linear time base.

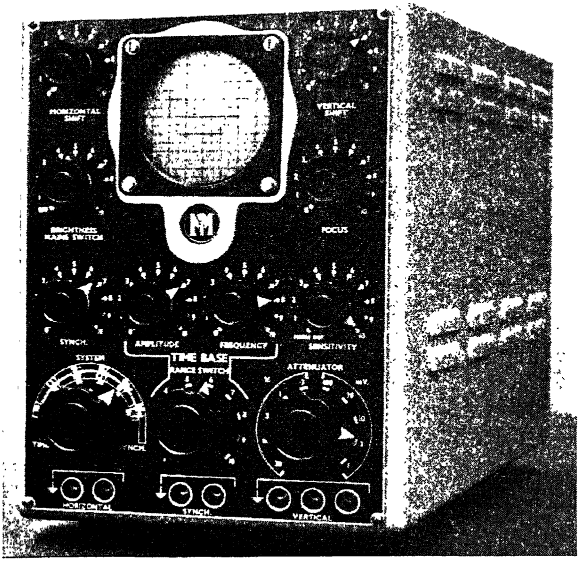
The rate of sweep depends upon the setting of P_1, and the amplitude of sweep upon the setting of P_2.

The capacitors C_3 and C_4 are usually of the order of 1 microfarad and 0·1 microfarad respectively.

An important feature of the circuit is the synchronizing branch circuit consisting of capacitor C_1 (about 0·1 microfarad) and R_4 (about 2 megohms). This enables the sweep frequency to be "locked" to some sub-multiple of the frequency of the voltage under test by applying to the grid circuit of the triode a small alternating voltage of this frequency. The result is to maintain the trace of the wave stationary upon the oscillograph screen.

Such a circuit gives almost constant velocity of sweep (i.e. linear time base) with almost instantaneous return, for frequencies from (say) 0·5 per second to about 10,000 per second.

Fig. 344 shows the Mullard C.R. Oscillograph which is typical. It contains a two-stage, push-pull amplifier, using pentode valves



(*Mullard Electronic Products, Ltd.*)

FIG. 344. MULLARD CATHODE RAY OSCILLOGRAPH

and giving a gain of 7,000 or 1 mV (r.m.s.) per cm. deflection. The time-base frequency range is from 0·25 to 16,000 c/s.

The deflection sensitivity is

Vertical plates: 7 V (r.m.s.)/cm.
21 V (d.c.)/cm.
Horizontal plates: 14 V (r.m.s.)/cm.
42 V (d.c.)/cm.

Synchronizing from the input voltage and from 50 c/s mains is provided internally and it is also possible to synchronize the time base using a mechanical contact breaker.

Another feature is the possibility of single-stroke operation for non-recurrent phenomena or for photographic recording.

High Voltage C.R. Oscillographs. E. L. White (Ref. (59)) has described the use of h.v. oscillographic tubes for the study of fast transient phenomena, particularly in the work of the Electrical Research Association on high-voltage surges and on spark breakdown in small gaps. The paper shows examples of single-sweep traces recorded on a sealed-off tube at 10 kV (Fig. 345). One of the traces is of a 1,000 megacycles/sec. oscillation.

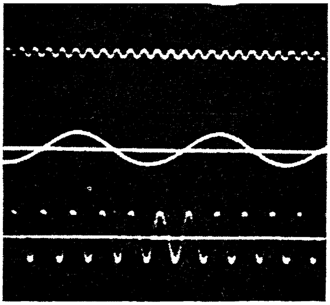


FIG. 345

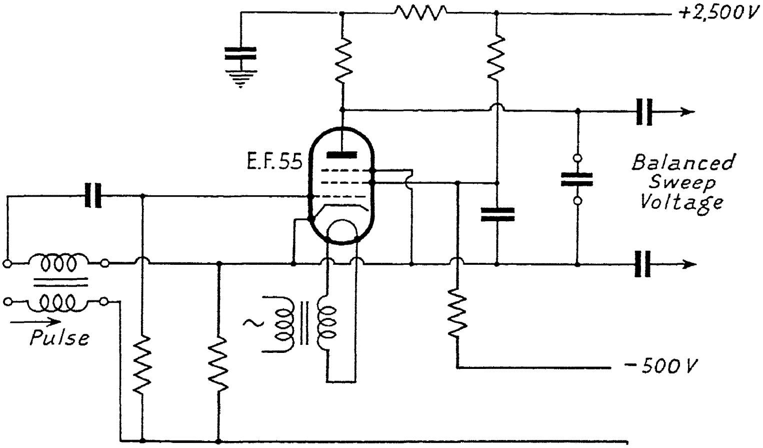


FIG. 346

The connections of the E.R.A. high-speed sweep circuit are given in Fig. 346. This circuit, described fully in Ref. 57, obtains the drive from a pulse-transformer. The generator valve is an EF55 having its base modified to give greater clearance around the anode pin. The generator produces a maximum rate of charge of sweep voltage of 4×10^{10} volt/sec.

The works mentioned in Refs. (7), (23), (24), (25) and (53) give, also, very full information regarding cathode ray oscillographs and their applications, while Ref. (52) is concerned especially with the various forms of time base.

BIBLIOGRAPHY AND REFERENCES

(1) *Alternating Currents*, A. T. Dover.

(2) *Alternating Current Machinery*, J. R. Barr and R. D. Archibald.

(3) *Theory and Calculation of Electric Currents*, J. L. La Cour and O. S.

(4) *Alternating Currents*, Vol. I, A. Russell.

(5) *Practical Mathematics for Technical Students*, Part II, T. S. Usherwood and C. J. A. Trimble.

(6) *Electrical Measurements*, F. A. Laws.

(7) *Oscillographs*, J. T. Irwin.

(8) *Dictionary of Applied Physics*, Vol. II.

(9) *Electrician* (1898), Vol. XXVIII, p. 362.

(10) S. P. Thompson, *Electrician* (1905), Vol. LV, p. 79.

(11) "Methode der Zeriegung in Sinuswellen," C. Runge, *E.T.Z.*, Vol. XXVI, 1905), p. 247.

(12) "An Electric Harmonic Analyser," J. D. Cockcroft, R. T. Coe, J. A. Tyacke, and Miles Walker, *Jour. I.E.E.*, Vol. XLIII, p. 69.

(13) "Harmonic Analysis by Selected Ordinates," A. E. Clayton, *Jour. I.E.E.*, Vol. LIX, p. 491.

(14) J. Joubert, *Journal de Physique*, Vol. IX (1880), p. 297.

(15) "The Slow Registration of Rapid Phenomena by Stroboscopic Methods," E. Hospitalier, *Jour. I.E.E.*, Vol. XXXIII, p. 80.

(16) "The Calculation of Harmonics in Rectified Currents," E. L. E. Wheatcroft, *Jour. I.E.E.*, Vol. LXIX, p. 100.

(17) "Note on the Cyclogram for the Determination of Wave-form," W. Cramp, *Jour. I.E.E.*, Vol. LXIX, p. 81.

(18) "A Portable Electric Harmonic Analyser," R. T. Coe, *Jour. I.E.E.*, Vol. LXVII, p. 1249.

(19) A. Blondel, *La Lumière Electrique* (1891), Vol. XLI, p. 401; *Comptes Rendus* (1893), Vol. CXVI, pp. 502, 748; *L'Eclairage Electrique* (28th October, 1902).

(20) W. Duddell, *Electrician*, Vol. XXXIX, p. 636.

(21) "An Electrostatic Oscillograph," H. Ho and S. Koto, *Proc. Physical Soc., London*, Vol. XXVI (1913), p. 16.

(22) F. Braun, *Wied. Ann.* Vol. LX, (1897), p. 552.

(23) "The Use of the Cathode-ray Tube as a Wattmeter and Phase-difference Measurer for High-frequency Electric Currents," J. A. Fleming, *Jour. I.E.E.*, Vol. LXIII, p. 1045.

(24) "The Cathode-ray Oscillograph," A. B. Wood, *Jour. I.E.E.*, Vol. LXIII, p. 1046.

(25) "Measurements in Electrical Engineering by Means of Cathode Rays," J. T. MacGregor-Morris and R. Mines, *Jour. I.E.E.*, Vol. LXIII, p. 1056.

(26) "The Cathode-ray Oscillograph," D. A. Keys, *Journal Franklin Institute*, 196 (November, 1923), p. 577.

(27) A. Dufour, *Comptes Rendus*, Vol. CLVIII, (1914), p. 1339.

(28) J. B. Johnson, *Physical Review*, Vol. XVII, (1921), p. 420.

(29) "A Low-voltage Cathode-ray Oscillograph," N. V. Kipping, *Wireless World and Radio Review* (5th Sept., 1923), p. 759.

(30) "Wave-form Examination with the Cathode-ray Oscillograph," *Electrical Communication*, Vol. III, (July, 1924) p. 69.

(31) W. D. Coolidge, *Phys. Review*, Vol. II, (1913), p. 409.

(32) "The Use of the Oscillograph in the Testing of Heavy Electrical Plant," B. G. Churcher, *Metropolitan-Vickers Gazette* (December, 1927, and January, 1928).

(33) "The Measurement of the Alternating Components in the Electromotive Force of Commutating Machinery," B. G. Churcher, *Metropolitan-Vickers Gazette* (January, 1929).

(34) "Harmonic Analysis Diagrams," R. Beattie, *Electrician* (9th and 16th June, 1911).

(35) "An Extension of Fischer-Hinnen's Method of Harmonic Analysis," R. Beattie, *Electrician* (8th September, 1911).

(36) "A Simple Graphical Method for the Harmonic Analysis of a Cyclic Function," J. R. Ashworth, *Electrician* (15th September, 1911).

(37) "A Device for Separating Harmonics of Complex Waves," D. C. Gall, *Jour. of Sc. Inst.*, August, 1932.

(38) "The Use of Commutators for Measurement Purposes," C. Dannatt, *Metropolitan-Vickers Gazette*, August, 1933.

(39) "Cathode-Ray Tubes and Their Applications," J. M. Stinchfield, *Trans. A.I.E.E.*, Vol. 53, p. 1608.

(40) "Accuracy of Measurements made with Hot-filament Cathode-ray Tubes of the Gas-focused Type," J. T. MacGregor-Morris and H. Wright, *Jour. I.E.E.*, Vol. LXXI, p. 57.

(41) "The Technique of the High-speed Cathode-ray Oscillograph," F. P. Burch and R. V. Whelpton, *Jour. I.E.E.*, Vol. LXXI, p. 380.

(42) "Recent Developments in Cathode-ray Oscillographs," A. B. Wood, *Jour. I.E.E.*, Vol. LXXI, p. 41.

(43) "The Resonance Method of Wave-form Analysis," C. F. J. Morgan, *Jour. I.E.E.*, Vol. LXXI, p. 819.

(44) "A Circular Time-base giving Radial Deflections for Use with the Cathode-ray Oscillograph," Staff of the Radio Research Station (Slough), *Jour. I.E.E.*, Vol. LXXI, p. 82.

(45) "The Design and Operation of a High-speed Cathode-ray Oscillograph," J. L. Miller and J. E. L. Robinson, *Jour. I.E.E.*, Vol. LXXIV, p. 511.

(46) *Theory of Alternating Current Wave Forms*, P. Kemp.

(47) "A Cathode Ray Oscillograph for the Direct Measurement of High-voltage Transients," A. K. Nuttall, *Jour. I.E.E.*, Vol. LXXVIII, p. 229.

(48) "A Cathode Ray Oscillograph Equipment embodying a High-voltage, Gas-filled, Sealed Glass Oscillograph Tube," S. P. Smith, C. E. Szeghö, and E. Bradshaw, *Jour. I.E.E.*, Vol. LXXVI, p. 656.

(49) "Experimental Verification of the Theory of the Cathode Ray Oscillograph, and the Influence of Screen Potential," J. T. MacGregor-Morris and V. A. Hughes, *Jour. I.E.E.*, Vol. LXXIX, p. 454.

(50) "Recurrent-surge Oscillographs, and their Application to Short-time, Transient Phenomena," K. J. R. Wilkinson, *Jour. I.E.E.*, Vol. LXXXIII p. 663.

(51) "An Electrostatic Analyser for Complex Waves of Small Amplitude," J. C. Prescott, *Jour. I.E.E.*, Vol. LXXXV, p. 302.

(52) *Time Bases*, O. S. Puckle.

(53) *The Cathode-ray Oscillograph in Industry*, W. Wilson.

(54) *Electrical Measurements and the Calculation of the Errors Involved*, Part II, D. Karo.

(55) "A New Election Microscope with Continuously Variable Magnification," J. B. le Poole, *Philips Tech. Rev.*, Vol. 9, No. 2, pp. 33–64, 1947.

(56) "An Experimental Electron Microscope for 400 Kilovolts," A. C. van Dorsten, W. J. Oosterkamp and J. B. la Poole, *Philips Tech. Rev.*, Vol. 9, No. 7, pp. 193–224, 1947.

(57) "A New Oscillograph for the recording of very fast electrical transients," R. E. Baur and W. Nethercot, *E.R.A. Report, Ref. U/T*115.

(58) "The Avoidance of a Possible Error in the Recording by Sealed-off Cathode-ray Oscillograph of Transients," H. W. Baxter, *Jour. Sci. Insts.*, Vol. 21, No. 9, p. 159, Sept., 1944.

(59) "High Voltage Oscillographic Tubes," E. L. White, *The Electronics Forum*, No. 26, June, 1952, p. 3.

CHAPTER XVI

TRANSIENT PHENOMENA

PHENOMENA which are not simple periodic functions of time, and which usually exist for only a short time, are referred to as "transients" or "transient phenomena." We have already had several examples of such phenomena in previous chapters.

In this chapter, transients produced by the sudden opening or closing of various circuits, will be considered.

Initial Conditions when a Steady Unidirectional Voltage is Impressed upon Circuits. When such a voltage is applied to a closed circuit the current which will *eventually* flow in the circuit will, of course, be given by $\frac{\text{Applied voltage}}{\text{Resistance of the circuit}}$. At the instant of switching the voltage on to the circuit, however, the current is zero. The initial, or transient, conditions with which we are now concerned are those which exist while the current is changing from its initial value to its final steady value. These conditions depend upon the resistance, inductance, and capacitance of the circuit under consideration, and various possible cases will now be discussed.

Resistance Only. If a steady voltage E, from a battery (say), is applied to a circuit, of resistance R, whose inductance and capacitance are negligibly small, the current in the circuit rises to the value $\frac{E}{R}$ instantaneously, without any transient conditions.

Resistance and Inductance in Series. Let E be the impressed steady voltage, and R and L the resistance and inductance of the circuit; then, when the voltage is switched on to the circuit, we have

$$E = Ri + L\frac{di}{dt} \qquad . \quad . \quad . \quad . \quad (315)$$

where i is the instantaneous value of the current in the circuit at any time t. The voltage-drop in the resistance of the circuit, together with the back e.m.f. of self-induction, is equal and opposite to the impressed voltage at all instants.

The instantaneous current i can be obtained as follows—

$$E - Ri = L\frac{di}{dt}$$

or

$$\frac{dt}{L} = \frac{di}{E - Ri}$$

Integrating each side we have

$$\int \frac{dt}{L} = \int \frac{di}{E - Ri}$$

$$\frac{t}{L} = \frac{-1}{R} \log_\varepsilon (E - Ri) + A$$

where A is a constant.

$$-\frac{Rt}{L} = \log_\varepsilon (E - Ri) - AR$$

$$= \log_\varepsilon (E - Ri) + \log_\varepsilon B$$

where B is another constant such that $\log_\varepsilon B = - AR$.

Hence $$\varepsilon^{-\frac{Rt}{L}} = B\,(E - Ri)$$

B can be found from the conditions at the instant of switching. Thus, when $t = 0$, $i = 0$, so that

$$\varepsilon^0 = B\,(E) \text{ or } B = \frac{1}{E}$$

Substituting this value for B we have

$$\varepsilon^{-\frac{Rt}{L}} = \frac{1}{E}\,(E - Ri)$$

so that $$i = \frac{E}{R}\,(1 - \varepsilon^{-\frac{Rt}{L}}) \qquad . \quad . \quad . \quad . \quad (316)$$

This means that the current rises according to an exponential law, the current-time graph being as shown in Fig. 347, curve (1). The maximum, or final, value is attained when $t = \infty$.

Then

$$i = \frac{E}{R}\,(1 - \varepsilon^{-\infty}) = \frac{E}{R}$$

Thus, if a circuit possesses inductance the current only reaches its steady value—as given by Ohm's law—after infinite time, although it may rise to within a very small percentage of this value in quite a short period of time.

Example. Calculate the value of the current after 1 second, when a voltage of 100 volts is applied to a circuit having a resistance of 20 ohms and an inductance of 100 millihenries.

$$i = \frac{100}{20}\,(1 - \varepsilon^{-\frac{20}{0\cdot 1}})$$

$$= 5\left(1 - \frac{1}{\varepsilon^{200}}\right) = 5\left(1 - \frac{1\cdot 38}{10^{87}}\right) \doteqdot 5$$

In this example, the ratio $\frac{R}{L}$ is large, and the ratio $\frac{L}{R}$—which is called the "time-constant" of the circuit—is thus small. The current therefore very rapidly attains a value which, for all practical purposes, is the same as its final steady value of 5 amp.

If the resistance had been 0·2 ohms instead of 20 ohms, the time-constant would have been 0·5 instead of 0·005, and the current, after 1 second, is given by

$$\qquad\qquad {}^{-\frac{0\cdot2}{0\cdot1}}\Big)$$

$$= 432\cdot3 \text{ amp.}$$

which is appreciably less than its final value of 500 amp.

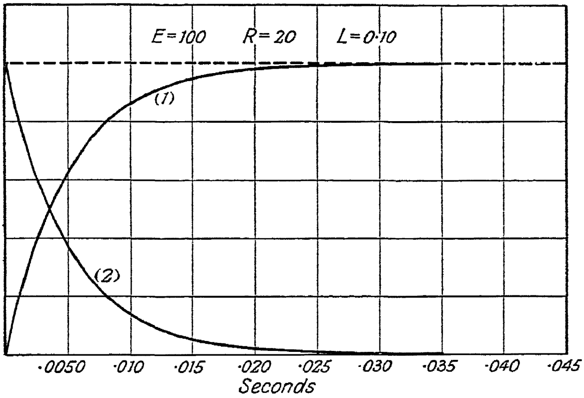


FIG. 347. CURVES OF RISE AND DECAY OF CURRENT IN AN INDUCTIVE CIRCUIT

Obviously, in all such circuits, after a time $\frac{L}{R}$ seconds the current is given by

$$i = \frac{E}{R}(1 - \varepsilon^{-1}) = 0\cdot6322\frac{E}{R}$$

Transient Effect when the Impressed Voltage is Removed. If the impressed voltage is suddenly removed from the circuit (without opening the circuit) when a current $\frac{E}{R}$ amp. is flowing in it (a considerable time having passed since the instant of switching on) the current does not disappear instantaneously, just as it did not rise to its final steady value instantaneously. The energy which was absorbed by the magnetic field of the inductive part of the circuit during the initial rise of the current to its maximum value, and which has been stored in the field while the steady current has been flowing, is now discharged again, and maintains a current for some time after the removal of the impressed voltage.

The current variation with time, after removing the voltage, is obtained as below. The voltage equation of the circuit is now

$$L\frac{di}{dt} + Ri = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (317)$$

$$\therefore \qquad -\frac{dt}{L} = \frac{di}{Ri}$$

Integrating, we have

$$-\int\frac{dt}{L} = \int\frac{di}{Ri}$$

$$-\frac{t}{L} = \frac{1}{R}\log_\varepsilon Ri + A'$$

where A' is a constant.

$$\therefore\ \varepsilon^{\frac{-Rt}{L}} = B'Ri$$

where B' is a constant such that $\log B' = A'R$.

If $$i = \frac{E}{R} \text{ when } t = 0,$$

$$\varepsilon^0 = B'E \text{ or } B' = \frac{1}{E}$$

Hence, the law of die-away of the current is

$$i = \frac{E}{R}\varepsilon^{-\frac{Rt}{L}} \quad . \quad . \quad . \quad . \quad . \quad (318)$$

This curve is shown in Fig. 347, curve (2).

Circuit containing Resistance, Inductance, and Capacitance. Suppose a steady voltage E is suddenly switched on to a circuit containing resistance R, inductance L, and capacitance C. The voltage equation of the circuit when current of instantaneous value i begins to flow is

$$Ri + L\frac{di}{dt} + \frac{\int idt}{C} = E \quad . \quad . \quad . \quad . \quad . \quad (319)$$

The expression $\int idt$ represents the quantity of electricity which has been given to the capacitor during the first t seconds after switching on the voltage, t being variable. Hence $\frac{\int idt}{C}$ represents the voltage across the capacitor terminals after t seconds.

Differentiating both sides of the above equation with respect to t we have

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

or, rearranging, $$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \quad . \quad . \quad . \quad . \quad (320)$$

It has been shown already that the solution of an equation of this form is

$$i = A\varepsilon^{m_1t} + B\varepsilon^{m_2t}$$

where A and B are constants, and m_1 and m_2 are the roots of the auxiliary equation $Lm^2 + Rm + \frac{1}{C} = 0$.

Thus, $$m_1 = \frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

and $$m_2 = \frac{-R - \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

The final form of the expression for the current i depends upon whether the term under the square-root sign is positive, zero, or negative—i.e. upon whether R^2 is greater than, equal to, or less than $\frac{4L}{C}$. There are, therefore, three cases to be considered.

Case 1. $R^2 > \frac{4L}{C}$.

In this case, m_1 and m_2 are both real quantities. To find A and B, consider the initial conditions of the circuit. Suppose $i = 0$ when $t = 0$ and that the capacitor is uncharged (i.e. $\int idt = 0$). Then, from the original voltage equation, when $t = 0$,

$$L\frac{di}{dt} = E \text{ or } \frac{di}{dt} = \frac{E}{L}$$

From the current equation we have

$$\begin{aligned} 0 &= A\varepsilon^0 + Be^0 \\ &= A + B \end{aligned}$$

$$\therefore \quad A = -B$$

also $$\frac{di}{dt} = Am_1\varepsilon^0 + Bm_2\varepsilon^0 = \frac{E}{L}$$

$$\therefore \quad Am_1 - Am_2 = \frac{E}{L}$$

or
$$A = \frac{E}{L\,(m_1 - m_2)}$$

and
$$B = \frac{-E}{L\,(m_1 - m_2)}$$

Hence
$$i = \frac{E}{L\,(m_1 - m_2)}\,(\varepsilon^{m_1 t} - \varepsilon^{m_2 t}) \text{ amp.}$$

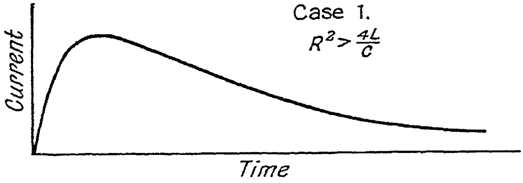


FIG. 348

Substituting for m_1 and m_2 in terms of R, L, and C, we have

$$= \frac{E}{\sqrt{R^2 - \frac{4L}{C}}}\left[\varepsilon^{\left(\frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L}\right)t} - \varepsilon^{\left(\frac{-R - \sqrt{R^2 - \frac{4L}{C}}}{2L}\right)t}\right] \quad . \quad (321)$$

Case 2. $R^2 = \frac{4L}{C}$.

The most general solution of the equation for the current i is now

$$i = (A + Bt)\,\varepsilon^{-\frac{Rt}{2L}}* \quad . \quad . \quad . \quad (322)$$

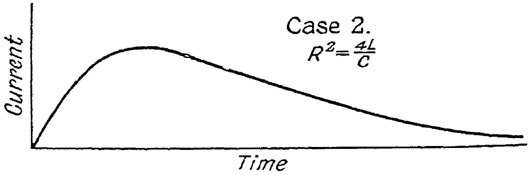


FIG. 349

If, initially, $i = 0$ and $\int idt = 0$, when $t = 0$, then, when $t = 0$,

$$\frac{di}{dt} = \frac{E}{L}$$

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* See Piaggio, *Differential Equations*, p. 32.

Thus, $A = 0$

and $\frac{di}{dt} = B = \frac{E}{L}$

$$\therefore \quad i = \frac{E}{L} t \varepsilon^{-\frac{Rt}{2L}} \quad \text{. . . . . . (323)}$$

Case 3. $R^2 < \frac{4L}{C}$.

The roots m_1 and m_2 are then imaginary, and can be written

$$m_1 = -k_1 + jk_2$$
$$m_2 = -k_1 - jk_2$$

where $j = \sqrt{-1},\ k_1 = \frac{R}{2L}$

$$\text{and } k_2 = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L}$$

Hence, the equation for i is

$$i = A\varepsilon^{(-k_1 + jk_2)t} + B\varepsilon^{(-k_1 - jk_2)t}$$

or $$i = \varepsilon^{-k_1 t}[A\varepsilon^{jk_2 t} + B\varepsilon^{-jk_2 t}] \quad \text{. . . . . . (324)}$$

Since $\varepsilon^{jpx} = \cos px + j \sin px$

and $\varepsilon^{-jpx} = \cos px - j \sin px$

we can write

$$i = \varepsilon^{-k_1 t}[A (\cos k_2 t + j \sin k_2 t) + B (\cos k_2 t - j \sin k_2 t)]$$
$$= \varepsilon^{-k_1 t}[M \cos k_2 t + N \sin k_2 t]$$

where $M = A + B$ and $N = j\ (A - B)$

$$= \varepsilon^{-k_1 t}\ [P \sin\ (k_2 t + \alpha)]$$

where $P = \sqrt{M^2 + N^2}$ and $\alpha = \tan^{-1}\frac{M}{N}$

$$\therefore \quad i = \varepsilon^{-\frac{R}{2L}t}\left[P \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t + \tan^{-1}\frac{M}{N}\right)\right] \quad \text{. (325)}$$

If $i = 0$ and $\int idt = 0$ when $t = 0$

Then $\frac{di}{dt} = \frac{E}{L}$ when $t = 0$.

From the final equation for i, when $t = 0$, $0 = P \sin\left(\tan^{-1}\frac{M}{N}\right)$.

Therefore, either $P = 0$ or $\tan^{-1}\frac{M}{N} = 0$.

Now P cannot be zero, since this would cause the current to be zero for all values of t—which it is not. Thus, $\tan^{-1}\frac{M}{N} = 0$, which gives

$$i = P\varepsilon^{-\frac{R}{2L}t} \sin \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t \qquad (326)$$

Now, when $t = 0$, by differentiation.

$$\frac{di}{dt} = P\frac{\sqrt{\frac{4L}{C} - R^2}}{2L} = \frac{E}{L}$$

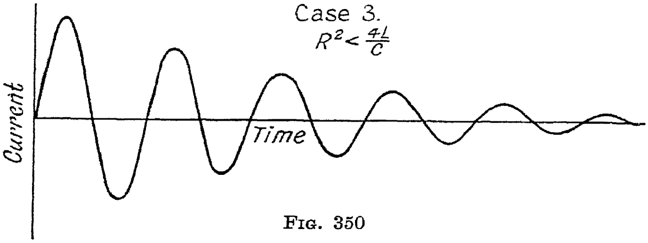

FIG. 350

Hence, $P = \frac{2E}{\sqrt{\frac{4L}{C} - R^2}}$

Thus, finally,

$$i = \frac{2E}{\sqrt{\frac{4L}{C} - R^2}}\varepsilon^{-\frac{R}{2L}t} \sin \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t$$

or

$$i = \frac{E}{\sqrt{\frac{L}{C} - \frac{R^2}{4}}}\varepsilon^{-\frac{R}{2L}t} \sin \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right) t \qquad (327)$$

The general shapes of the current-time graphs for these three cases are as shown in Figs. 348, 349, and 350. The current, in the first two cases, is non-oscillatory, and in the third case oscillatory. In all three cases the final value of the current will, of course, be zero, since there is a capacitor in the circuit and the applied voltage is unidirectional.

Circuit containing Resistance and Inductance, when the Impressed Voltage is Alternating. Let R and L be the resistance and inductance of the circuit and let the impressed voltage be given by $e = E_{max} \sin \omega t$.

The voltage equation for the circuit is then

$$e = Ri + L\frac{di}{dt} = E_{max}\sin\omega t \quad . \quad . \quad . \quad . \ (328)$$

or
$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E_{max}}{L}\sin\omega t^{*}$$

Hence,
$$i\,\varepsilon^{\int\frac{R}{L}dt} = \int\left(\frac{E_{max}}{L}\sin\omega t\right)\varepsilon^{\int\frac{R}{L}dt} + A$$

or
$$i\,\varepsilon^{\frac{Rt}{L}} = \frac{E_{max}}{L}\int\varepsilon^{\frac{Rt}{L}}\sin\omega t + A$$

$$\therefore \qquad i = \frac{E_{max}}{L}\,.\,\varepsilon^{-\frac{Rt}{L}}\int\varepsilon^{\frac{Rt}{L}}\sin\omega t + A\varepsilon^{-\frac{Rt}{L}} \quad . \ (329)$$

The term $\int\varepsilon^{\frac{Rt}{L}}\sin\omega t = \varepsilon^{\frac{Rt}{L}}\dfrac{\left[\frac{R}{L}\sin\omega t - \omega\cos\omega t\right]}{\frac{R^2}{L^2}+\omega^2}$

$$\therefore \qquad i = \frac{E_{max}}{L\left[\frac{R^2}{L^2}+\omega^2\right]}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right) + A\varepsilon^{-\frac{Rt}{L}}$$

$$= \frac{E_{max}}{L\left[\frac{R^2}{L^2}+\omega^2\right]}\,\frac{(R\sin\omega t - \omega L\cos\omega t)}{L} + A\varepsilon^{-\frac{Rt}{L}}$$

$$= \frac{E_{max}}{R^2+\omega^2L^2}\,(R\sin\omega t - \omega L\cos\omega t) + A\varepsilon^{-\frac{Rt}{L}}$$

$$= \frac{E_{max}}{\sqrt{R^2+\omega^2L^2}}\left[\frac{R}{\sqrt{R^2+\omega^2L^2}}\sin\omega t - \frac{\omega L}{\sqrt{R^2+\omega^2L^2}}\cos\omega t\right] + A\varepsilon^{-\frac{Rt}{L}}$$

or
$$i = I_{max}\sin(\omega t - \alpha) + A\varepsilon^{-\frac{Rt}{L}} \quad . \quad . \quad . \quad . \ (330)$$

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* This differential equation is of the form

$$\frac{dy}{dx} + Py = M$$

which is solved by first multiplying each side by $\varepsilon^{\int Pdx}$ and then integrating each side with respect to x, when we obtain the equation

$$y\varepsilon^{\int Pdx} = \int M\varepsilon^{\int Pdx} + A$$

where A is a constant.

where $\alpha = \tan^{-1}\dfrac{\omega L}{R}$ and I_{max} is the maximum value of the current under steady conditions and equals $\dfrac{E_{max}}{\sqrt{R^2 + \omega^2 L^2}}$.

The first term of the above expression for i obviously represents the current which flows under steady conditions (i.e. after the voltage has been switched on for some time). The second term represents a transient current which disappears comparatively rapidly, but which exists for a short time after closing the switch. The value of the constant A depends upon the position on the voltage wave at which the switch is closed—i.e. upon the value of e at that instant.

If the switch could be closed when e had the value $E_{max}\sin\alpha$, then the current at the instant of switching would be given by

$$i = I_{max}\sin(\alpha - \alpha) + A\varepsilon^{-\frac{Rt}{L}}$$

But i is zero at the instant of switching, and therefore we have

$$0 = 0 + A\varepsilon^0$$

Hence, A is zero. In this case, therefore, there would be no transient current, the voltage and current being in their correct (i.e. their final) relative phases from the instant of switching. In all other cases, however, the transient term will not be zero.

Thus, suppose that the switch is closed when the voltage is at some point on its wave corresponding to a time t' seconds after its zero value. Then, counting time—in the expression for the current—from the instant when the voltage is zero, when $i = 0$, we have, when $t = t'$ (at the instant of closing the switch),

$$0 = I_{max}\sin(\omega t' - \alpha) + A\varepsilon^{-\frac{Rt'}{L}}$$

Hence, $$A = -I_{max}\sin(\omega t' - \alpha)\,.\,\varepsilon^{\frac{Rt'}{L}}$$

The general equation of the current is therefore

$$i = I_{max}\sin(\omega t - \alpha) - I_{max}\,\varepsilon^{\frac{Rt'}{L}}\,.\,\varepsilon^{-\frac{Rt}{L}}\sin(\omega t' - \alpha)$$

or $$i = I_{max}\sin(\omega t - \alpha) - I_{max}\,\varepsilon^{\frac{R}{L}(t'-t)}\sin(\omega t' - \alpha) \qquad (331)$$

The general shape of the current-time graph is as shown in Fig. 351, from which it can be seen that the effect of the transient term is to produce dissymmetry in the first few cycles of the current wave; this dissymmetry rapidly dies away so that the wave approaches very closely to its normal sinusoidal form after a few cycles.

Circuit containing Resistance and Capacitance Only. If the same alternating voltage $e = E_{max} \sin \omega t$ is impressed upon a circuit having resistance R and capacitance C, but no inductance, the voltage equation is

$$e = Ri + \frac{\int i dt}{C} = E_{max} \sin \omega t \quad . \quad . \quad (332)$$

By differentiation we have

$$\omega E_{max} \cos \omega t = R\frac{di}{dt} + \frac{i}{C}$$

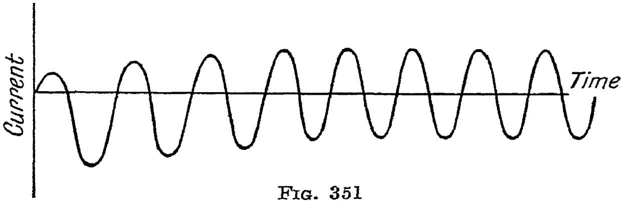


FIG. 351

This is an equation of the same form as that obtained when resistance and inductance were contained in the circuit, as in the previous paragraph.

It can be solved in the same way, giving

$$i = \frac{E_{max}}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \sin(\omega t + \alpha) + A\varepsilon^{-\frac{t}{RC}}$$

or
$$i = I_{max} \sin(\omega t + \alpha) + A\varepsilon^{-\frac{t}{RC}}. \quad . \quad . \quad . \quad (333)$$

Circuit containing Resistance, Inductance, and Capacitance in Series, with an Alternating Impressed Voltage. If, as before, the impressed voltage is given by $e = E_{max} \sin \omega t$ and the resistance, inductance, and capacitance of the circuit are R, L, and C, the voltage equation is

$$e = Ri + L\frac{di}{dt} + \frac{\int i dt}{C} = E_{max} \sin \omega t \quad . \quad . \quad (334)$$

Differentiating, we have

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \omega E_{max} \cos \omega t$$

The solution of this equation will give an expression for i which is the sum of a particular integral and the complementary function, the latter being obtained by solving the equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

The solution of this equation has already been shown to be

$$i = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t}$$

where A and B are constants and m_1 and m_2 are the roots of the equation

$$Lm^2 + Rm + \frac{1}{C} = 0 \quad \text{(see page 587)}$$

As regards the particular integral, by analogy with the differential equation for the motion of the moving system of a vibration galvanometer, given on page 261, we have the solution

$$i = \frac{\omega E_{max}}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}} \cos(\omega t - \beta) \qquad . \qquad . \quad (335)$$

where $\beta = \tan^{-1}\dfrac{R\omega}{\frac{1}{C} - L\omega^2} = \tan^{-1}\dfrac{R}{\frac{1}{\omega C} - \omega L}$

This may be written as

$$i = \frac{E_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \alpha)$$

where $\tan \alpha = \dfrac{\frac{1}{\omega C} - \omega L}{R}$, i.e. $\alpha = 90 - \beta$,

or, otherwise

$$i = \frac{E_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t - \gamma)$$

where $\gamma = \tan^{-1}\dfrac{\omega L - \frac{1}{\omega C}}{R} = -\alpha$.

The complete equation for the current i is therefore

$$i = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t} + \frac{E_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t - \gamma)$$

or $$i = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t} + I_{max} \sin(\omega t - \gamma) \qquad . \qquad . \quad (336)$$

The last term obviously represents the current in the circuit when the transient current—represented by the first two terms—has died away.

Now,
$$m_1 = \frac{-R+\sqrt{R^2-\frac{4L}{C}}}{2L}$$

and
$$m_2 = \frac{-R-\sqrt{R^2-\frac{4L}{C}}}{2L}$$

so that the final form of the equation for the current depends, as before, upon whether R^2 is greater than, equal to, or less than, $\frac{4L}{C}$. These three cases will now be considered in turn.

Case 1. $R^2 > \frac{4L}{C}$.

Both m_1 and m_2 are now real values, and

$$i = A\varepsilon^{m_1 t} + B\varepsilon^{m_2 t} + I_{max} \sin (\omega t - \gamma)$$

To evaluate the constants A and B, the initial conditions in the circuit must be known. Suppose that the switch is closed when the voltage has a value corresponding to time t' seconds on its wave (i.e. at the closing of the switch $e = E_{max} \sin \omega t'$). Let time be counted from the instant when the voltage is zero. Then, the current is zero at the instant of closing the switch, so that $i = 0$ when $t = t'$. Suppose that the capacitor is discharged when the switch is closed, so that $\int idt = 0$ when $t = t'$.

We then have

$$i = A\varepsilon^{m_1 t'} + B\varepsilon^{m_2 t'} + I_{max} \sin (\omega t' - \gamma) = 0$$

Integrating the expression for i, we have also

$$\int idt = \frac{A\varepsilon^{m_1 t'}}{m_1} + \frac{B\varepsilon^{m_2 t'}}{m_2} - \frac{I_{max}}{\omega} \cos (\omega t' - \gamma) = 0$$

Substituting for $A\varepsilon^{m_1 t'}$ (from the first of these equations) in the second equation, we have

$$\frac{-B\varepsilon^{m_2 t'} - I_{max} \sin (\omega t' - \gamma)}{m_1} + \frac{B\varepsilon^{m_2 t'}}{m_2} - \frac{I_{max}}{\omega} \cos (\omega t' - \gamma) = 0$$

From which

$$B = \frac{m_1 m_2 \frac{I_{max}}{\omega} \cos (\omega t' - \gamma) + m_2 I_{max} \sin (\omega t' - \gamma)}{\varepsilon^{m_2 t'} (m_1 - m_2)}$$

$$= \frac{m_2 I_{max} \varepsilon^{-m_2 t'}}{m_1 - m_2} \left[\frac{m_1}{\omega} \cos (\omega t' - \gamma) + \sin (\omega t' - \gamma) \right]$$

or $B = K_1 I_{max} \varepsilon^{-m_2 t'} [\sin (\omega t' - \gamma + \theta)]$ . . . . (337)

where $K_1 = \dfrac{m_2\sqrt{m_1^2 + \omega^2}}{\omega\,(m_1 - m_2)}$ and $\theta = \tan^{-1}\dfrac{m_1}{\omega}$.

Similarly,

$$A = K_2 I_{max}\, \varepsilon^{-m_1 t'} \sin\left(\omega t' - \gamma + \frac{\theta}{2}\right) \quad . \quad . \quad . \quad . \quad (338)$$

where $K_2 = -2K_1 \cos\dfrac{\theta}{2}$.

Case 2. $R^2 = \dfrac{4L}{C}$.

In this case, $m_1 = m_2 = -\dfrac{R}{2L}$, and the most general expression for the transient, when these roots are equal, is

$$\varepsilon^{-\frac{R}{2L}t}(A + Bt)$$

Thus the equation for i is now

$$i = \varepsilon^{-\frac{R}{2L}t'}(A + Bt) + I_{max}\sin(\omega t - \gamma) \quad . \quad . \quad . \quad (339)$$

If the switch is closed when $e = E_{max}\sin\omega t'$ (i.e. $t = t'$ when the switch is closed), and if the capacitor is then uncharged, we have at this instant

$$t = t',\ i = 0,\ \int i dt = 0$$

$$\therefore 0 = \varepsilon^{-\frac{R}{2L}t}[A + Bt'] + I_{max}\sin(\omega t' - \gamma) = i$$

and, by integrating the expression for i,

$$0 = \frac{-2L}{R}\varepsilon^{-\frac{Rt'}{2L}}\left[A + Bt' + \frac{2L}{R}B\right] - \frac{I_{max}}{\omega}\cos(\omega t' - \gamma) = \int idt$$

From these two equations A and B may be evaluated as before.

Case 3. $R^2 < \dfrac{4L}{C}$.

In this case, the roots m_1 and m_2 are imaginary. If $m_1 = -k_1 + jk_2$ and $m_2 = -k_1 - jk_2$ $\left(\text{where } k_1 = \dfrac{R}{2L} \text{ and } k_2 = \dfrac{\sqrt{\dfrac{4L}{C} - R^2}}{2L}\right)$ the transient terms may be combined to give the expression

$$\varepsilon^{-\frac{R}{2L}t}\left[F\sin\left(\frac{\sqrt{\dfrac{4L}{C} - R^2}}{2L}t + \beta\right)\right] \quad \text{(see page 262)}$$

where $F = \sqrt{C^2 + D^2}$

and $C = A + B,\ D = j\,(A - B)$

$$\beta = \tan^{-1}\frac{C}{D}$$

Hence, the whole expression for the current takes the form

$$i = F\varepsilon^{-\frac{Rt}{2L}} \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L}\, t + \beta\right) + I_{max} \sin\,(\omega t - \gamma) \quad . \quad (340)$$

If, as in the previous cases, the switch is closed when

$$e = E_{max} \sin \omega t'$$

then $i = 0$ when $t = t'$. If the capacitor is uncharged at this instant, $\int idt = 0$ when $t = t'$, we have

$$0 = F\varepsilon^{-\frac{R}{2L}t'} \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t' + \beta\right) + I_{max} \sin\,(\omega t' - \gamma) = i$$

Also, by integration, when $t = t'$

$$0 = \frac{F\varepsilon^{-\frac{R}{2L}t'}}{\frac{R^2}{4L^2} + \frac{\frac{4L}{C} - R^2}{4L^2}} \left[-\frac{R}{2L} \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L}\, t' + \beta\right) - \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} \cos\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L}\, t' + \beta\right)\right] - \frac{I_{max}}{\omega} \cos\,(\omega t' - \gamma) = \int idt$$

$$\text{or } 0 = \frac{F\varepsilon^{-\frac{R}{2L}}}{\sqrt{\frac{1}{LC}}} \times$$

$$\left[\frac{-\frac{R}{2L} \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L} t' + \beta\right) - \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} \cos\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L}\, t' + \beta\right)}{\sqrt{\frac{1}{LC}}}\right]$$

$$- \frac{I_{max}}{\omega} \cos\,(\omega t' - \gamma)$$

$$0 = -F\sqrt{LC}\varepsilon^{-\frac{R}{2L}t}\sin\left(\frac{\sqrt{\frac{4L}{C}-R^2}}{2L}t' + \beta + \phi\right) - \frac{I_{max}}{\omega}\cos(\omega t' - \gamma)$$

where $\phi = \tan^{-1}\dfrac{\sqrt{\frac{4L}{C} - R^2}}{R}$

Substitute $\dfrac{\sqrt{\frac{4L}{C} - R^2}}{2L} = k_2$ for convenience in simplification.

From these two equations for i and $\int idt$ we have

$$F\sqrt{LC}\,\varepsilon^{-\frac{R}{2L}t'}\sin(k_2t' + \beta + \phi) = -\frac{I_{max}}{\omega}\cos(\omega t' - \gamma)$$

and $$F\varepsilon^{-\frac{R}{2L}t'}\sin(k_2t' + \beta) = -I_{max}\sin(\omega t' - \gamma)$$

By division,

$$\sqrt{LC}\,\frac{\sin(k_2t' + \beta + \phi)}{\sin(k_2t' + \beta)} = \cot(\omega t' - \gamma)$$

or $$\frac{\sin(k_2t' + \beta)\cos\phi + \cos(k_2t' + \beta)\sin\phi}{\sin(k_2t' + \beta)} = \frac{\cot(\omega t' - \gamma)}{\sqrt{LC}}$$

$$\therefore \cos\phi + \cot(k_2t' + \beta)\sin\phi = \frac{\cot(\omega t' - \gamma)}{\sqrt{LC}}$$

$$\therefore \quad \cot(k_2t' + \beta) = \frac{\cot(\omega t' - \gamma)}{\sqrt{LC}\sin\phi} - \cot\phi$$

Hence, $$k_2t' + \beta = \cot^{-1}\left[\frac{\cot(\omega t' - \gamma)}{\sqrt{LC}\sin\phi} - \cot\phi\right]$$

or $$\beta = \cot^{-1}\left[\frac{\cot(\omega t' - \gamma)}{\sqrt{LC}\sin\phi} - \cot\phi\right] - k_2t'$$

Thus, $$F = \frac{-\varepsilon^{\frac{R}{2L}t'}I_{max}\sin(\omega t' - \gamma)}{\sin(k_2t' + \beta)}$$

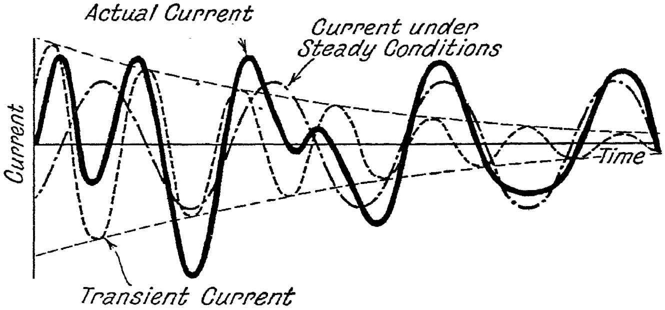
or $$F = \frac{-\varepsilon^{\frac{R}{2L}t'}I_{max}\sin(\omega t' - \gamma)}{\sin\left[\cot^{-1}\left\{\frac{\cot(\omega t' - \gamma)}{\sqrt{LC}\sin\phi} - \cot\phi\right\}\right]} \quad . \quad (341)$$

The final expression for i may be obtained by substituting these values of F and β in the expression

$$i = F\varepsilon^{-\frac{Rt}{2L}} \sin\left(\frac{\sqrt{\frac{4L}{C} - R^2}}{2L}\, t + \beta\right) + I_{max} \sin(\omega t - \gamma)$$

The general shape of the current wave, immediately after closing the switch which applies the voltage to the circuit, is shown in Fig. 352. As the transient current dies away, the wave approaches the wave obtained under steady conditions, as shown.

The Calculation of Transients by the use of Laplace Transformations. As already seen in this chapter, the relationship between the voltage and current in an electric circuit is, in general, a differential equation the solution of which has two parts—the particular integral, representing the steady state to which the quantities tend after a long time, and the complementary function, representing the transient phenomena which die away.



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The steady-state solution can be found by Ohm's law, using resistance only in d.c. circuits and impedance in a.c. circuits.

Laplace transformations, which are fully discussed by J. C. Jaeger (Ref. (4)), may be used to provide a convenient method of calculating both parts of the solution—steady state and transient —when the voltage or the current in a circuit is suddenly changed. The notes on the method, given here, are intended merely as an introduction. The works mentioned in Refs. (4) and (5) should be referred to by readers wishing to make a fuller study of the subject.

Note 1. The Laplace transform is defined by the equation

$$\overline{V}(p) = \int_0^\infty \varepsilon^{-pt} \,.\, V(t)dt$$

where $\overline{V}(p)$ is the transform of $V(t)$.

Note 2. Table XVI gives a list of the transforms which are most useful for the calculation of electrical transients.

TABLE XVI

Voltage	Transform
Zero up to time $t = 0$, then a constant V	V/p
,, ,, ,, ,, $V\varepsilon^{-\alpha t}$	$V/(p + \alpha)$
,, ,, ,, ,, $Vt \,.\, \varepsilon^{-\alpha t}$	$V/(p + \alpha)^2$
,, ,, ,, ,, $V \sin \omega t$	$V\omega/(p^2 + \omega^2)$
,, ,, ,, ,, $V \cos \omega t$	$Vp/(p^2 + \omega^2)$
,, ,, ,, ,, $V\varepsilon^{-\alpha t} \,.\, \sin \omega t$	$V\omega/[(p + \alpha)^2 + \omega^2]$
,, ,, ,, ,, $V\varepsilon^{-\alpha t} \,.\, \cos \omega t$	$V(p + \alpha)/[(p + \alpha)^2 + \omega^2]$

The quantities in the first column (which are written for voltage but could be written similarly for current) are zero up to time $t = 0$, so that their value for a negative value of t is *not* that obtained by inserting this negative value in the expression given.

Note 3. The transform of any voltage V (not necessarily constant) is written $\bar{V}$ and the transform of current $\bar{I}$. The "generalized impedance" is written $\bar{Z}$ and this is obtained by writing p in place of $j\omega$ in the complex expression for the impedance of the circuit, e.g. for a circuit containing resistance, inductance and capacitance

$$\frac{1}{pC}$$

Note 4. To obtain the solution to any given problem, when the voltage or current is suddenly changed from zero to some other (constant or alternating) value all that is needed is to solve the subsidiary equation $\bar{I} = \frac{\bar{V}}{\bar{Z}}$, and then to consult Table XVI to find the corresponding value of I (or V).

Example 1. A constant voltage of 50 V is suddenly applied to a circuit containing a resistance of 10 Ω and inductance of 5 H in series. What will be the current?

The applied voltage is 50 V and, from Table XVI, $\bar{V} = \frac{50}{p}$. The generalized impedance is

$$\bar{Z} = R + pL$$

Thus
$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{50}{p(R + pL)}$$

Splitting this expression into partial fractions we have

$$\bar{I} = \frac{50}{Rp} - \frac{50L}{R(R + pL)}$$

and, from lines 1 and 2 of Table XVI it follows that

$$I = \frac{50}{R} - \frac{50}{R}\varepsilon^{-\frac{R}{L}t}$$
$$= 5(1 - \varepsilon^{-2t})$$

Thus I is zero up to time $t = 0$, and, for positive values of t

$$I = 5(1 - \varepsilon^{-2t})$$

If the given voltage, or current, is not zero up to time $t = 0$, we adopt a simple artifice. For example, if the voltage of 50 V had been applied up to $t = 0$ and then removed, we should consider this applied voltage to be made up of a voltage of 50 V applied continuously and a voltage of $-$ 50 V applied suddenly at $t = 0$. The current produced by the first voltage is, by Ohm's law,

$$I = \frac{V}{R} = \frac{50}{10} = 5$$

while the current produced by the voltage $-$ 50 is, as above,

$$I = -5(1 - \varepsilon^{-2t})$$

The total current flowing is, therefore,

$$I = 5 - 5(1 - \varepsilon^{-2t}) = 5\varepsilon^{-2t}$$

Example 2. A constant voltage V is suddenly applied to a circuit containing resistance R, inductance L and capacitance C. What is the current which will flow?

The applied voltage V is transformed to $\bar{V} = \frac{V}{p}$.

The generalized impedance is

$$\bar{Z} = R + pL + \frac{1}{pc}$$

Thus

$$\bar{I} = \frac{\frac{V}{p}}{R + pL + \frac{1}{pc}}$$

which can be written

$$\bar{I} = \frac{V}{L\left[p^2 + \left(\frac{R}{L}\right)p + \frac{1}{LC}\right]}$$

By completing the square we can write

$$\bar{I} = \frac{V}{L[(p + \mu)^2 + n^2]}$$

Where

$$\mu = \frac{R}{2L} \text{ and } n^2 = \frac{1}{LC} - \mu^2$$
$$= \frac{1}{LC} - \frac{R^2}{4L^2}$$

From line 6 of Table XVI, the expression for the current is thus

$$I = \frac{V}{nL}\varepsilon^{-\mu t} . \sin nt$$

This is correct if $n^2 > 0$. If $n^2 = 0$

$$I = \frac{V}{L}t\,\varepsilon^{-\mu t} \quad \text{(see Ref. (4))}$$

and if $n^2 = -k^2 < 0$

$$I = \frac{V}{kL} \varepsilon^{-\mu t} . \sinh kt$$

These expressions—although of slightly different form—will be found to agree with those obtained for these circuit conditions earlier in the chapter.

BIBLIOGRAPHY AND REFERENCES

(1) *Theory and Calculation of Transient Electric Phenomena and Oscillations*, C. P. Steinmetz.
(2) *Theory and Practice of Alternating Currents*, A. T. Dover.
(3) *Electrical Engineering*, T. F. Wall.
(4) *An Introduction to the Laplace Transformation*, J. C. Jaeger.
(5) *Transformation Calculus and Electrical Transients*, S. Goldman.
(6) *Transients in Linear Systems*, M. F. Gardner and J. L. Barnes.
(7) "Mathematical Methods Applicable to Linear Phenomena," S. Whitehead, *Jour. Sci. Inst.*, vol. 21, No. 5, May, 1944.

CHAPTER XVII

MEASURING INSTRUMENTS

IN this, and the succeeding chapters, we shall be concerned with the principles and construction of the various measuring instruments which are used for the measurement of current, voltage, power, and energy. Certain features are common to all such instruments and these will now be discussed.

Classification. In the broadest sense, instruments may be divided into two classes—

(*a*) Absolute instruments.
(*b*) Secondary instruments.

(*a*) ABSOLUTE INSTRUMENTS give the value of the electrical quantity to be measured in terms of the constants of the instrument and of its deflection, no comparison with another instrument being necessary. For example, the tangent galvanometer gives the value of the current to be measured in terms of the tangent of the angle of deflection produced by the current, and of the radius and number of turns of the galvanometer, and of the horizontal component of the earth's magnetic field. No calibration of the instrument is thus necessary. Another example of an absolute instrument is the Rayleigh current balance, which has already been discussed in an earlier chapter.

(*b*) SECONDARY INSTRUMENTS are so constructed that the value of current, voltage, or other quantity to be measured can only be determined from the deflection of the instrument, provided the latter has been calibrated by comparison with either an absolute instrument or one which has already been calibrated. The deflection obtained is meaningless until such a calibration has been made.

This class of instrument is in most general use, absolute instruments being seldom used except in standards laboratories and similar institutions, and it is principally, therefore, with secondary instruments that we shall deal in this and the following chapters.

Effects Utilized in Measuring Instruments. Secondary instruments may be classified according to the various effects of electric current or voltage upon which their operation depends. The effects utilized are—

(*a*) Magnetic effect.
(*b*) Heating effect.
(*c*) Chemical effect.
(*d*) Electrostatic effect.
(*e*) Electromagnetic induction effect.

Table XVII shows in what kinds of instruments these various effects are most generally used in practice.

TABLE XVII

Effect	Instruments Utilizing the Effect
Magnetic effect . .	Ammeters, voltmeters, wattmeters, integrating meters, and most other electrical instruments.
Heating effect . . .	Ammeters and voltmeters.
Chemical effect . .	Integrating meters (D.C. ampere-hour meters).
Electrostatic effect . .	Voltmeters (indirectly, ammeters and wattmeters).
Electromagnetic induction effect . . .	Alternating current ammeters, voltmeters, wattmeters, and integrating meters.

Of these effects the first and last have been most commonly used.

The use of the term "integrating meter" in the above table may perhaps require explanation. An integrating instrument is one which measures the total amount, either of quantity of electricity, or of electrical energy, supplied to a circuit over a period of time. Its readings give, therefore, either the number of ampere-hours, or of watt-hours, supplied.

The introduction of the term "integrating meter" leads to another method of classifying secondary instruments, namely, as either—

(*a*) Indicating.
(*b*) Recording.
(*c*) Integrating.

Ammeters, voltmeters, and wattmeters, belong to the first of these three classes. As stated above, ampere-hour and watt-hour meters belong to the third class. Recording instruments give a continuous record of the variations of some electrical quantity, such as current or power, by means of a path traced out by a pen (attached to the moving system of the instrument) on a sheet of paper carried by a revolving drum.

Indicating Instruments. In most indicating instruments it is essential that three distinct forces shall act upon their moving system in order that they shall indicate satisfactorily. These forces are—

(i) A deflecting force.
(ii) A controlling force.
(iii) A damping force.

Thus, the deflecting, or operating, force causes the moving system of the instrument to move from its "zero" position (i.e. its position when the circuit in which it is connected, is disconnected from the supply). The magnitude of this movement would be somewhat indefinite (in most cases) unless some controlling force existed which limits the movement and ensures that the magnitude of the deflection is always the same for a given value of the quantity to be measured. A "damping" force is also necessary in order to bring the moving system to rest in its deflected position quickly. Without such damping, owing to the inertia of the moving parts, the pointer of the instrument would oscillate about its final deflected position for some time before coming to rest, and this would cause waste of time in taking readings, as well as preventing any but very slow variations of the quantity to be measured, from being observed.

Curves showing the effect of damping upon the variation of position, with time, of the moving system of an instrument, are given in Fig. 353. Oscillation about the final position is shown in the under-damped curve. If the instrument is over-damped, the moving system rises slowly from zero to its final deflection. When the instrument rises quickly to its deflected position without oscillation the damping is said to be "critical," and the instrument "dead-beat." In practice it is found that the best results are obtained when the damping is slightly less than the critical value.

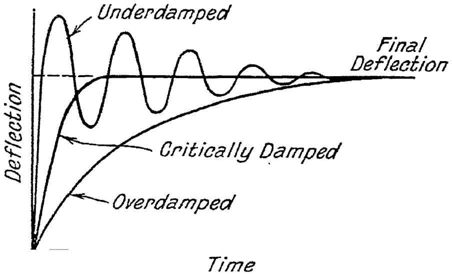


Fig. 353. Damping Curves

The damping force must operate only while the moving system of the instrument is actually moving. The final deflection of the instrument must not be affected by the damping.

The deflection of the moving system of an indicating instrument will rise from zero to such a value that the controlling torque at this deflected position is equal (and opposite) to the deflecting torque. The deflecting torque produced by any given value of the quantity to be measured is constant and thus the controlling torque must increase in magnitude with the deflection until it balances the constant deflecting torque.

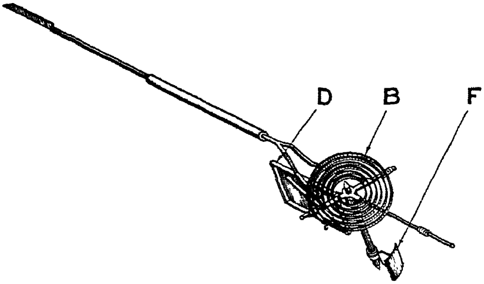
The operating or deflecting torque is produced by utilizing one or other of the effects already mentioned. The actual method of production of this torque depends upon the type of instrument and will be discussed later.

The controlling torque in indicating instruments is almost always obtained by a spring or, much less commonly, by gravity.

Spring Control. A hair-spring—usually of phosph
attached to the moving system, is used in indicating instruments

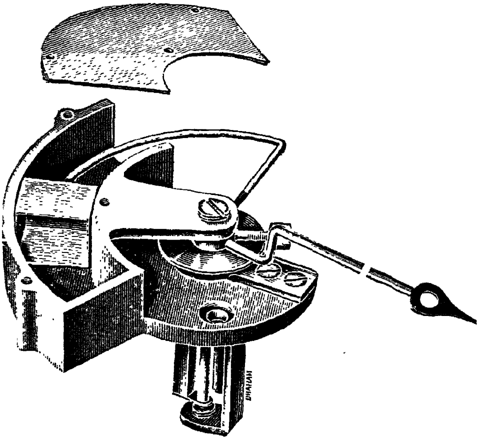
for control purposes, the arrangement being as shown in Fig. 354—in which B is the control spring, F the moving iron, and D a damping vane—and in Fig. 355.

To give a controlling torque which is directly proportional to the angle of deflection of the moving system the number of turns on the spring should be fairly large, so that the deformation per unit length is small. The stress in the spring must be limited to such a value that there is no permanent set.



(*Elliott Bros. (London), Ltd.*)

FIG. 354. MOVING SYSTEM OF A SPRING-CONTROLLED INSTRUMENT



(*Nalder Bros. and Thompson, Ltd.*)

FIG. 355. MOVING SYSTEM OF A MOVING-IRON INSTRUMENT SHOWING THE AIR-DAMPING CHAMBER

Suppose that a spiral spring is made up of a total length of L in. of strip whose cross-section is rectangular, the radial thickness being t and the depth b, both in inches. Let E be Young's Modulus (in pounds per square inch) for the material of the spring. Then, if θ radians be the deflection of the moving system to which one end of the spring is attached, the law connecting the controlling torque with θ is

$$T = \frac{Ebt^3}{12L} \cdot \theta \qquad . \quad (342)$$

Thus, controlling torque $\propto \theta$

$\propto$ instrument deflection

Another formula which is of use in connection with the design of springs is

$$\frac{L}{t/2} = \frac{E}{S_{max}} \times \theta \qquad . \quad . \quad . \quad . \quad (342a)$$

where S_{max} is the maximum stress in the spring.

In order to avoid overstressing the spring, the ratio $\frac{L}{t}$ should be about 2,000 if the material is phosphor-bronze.

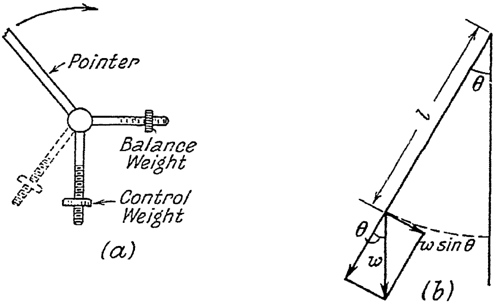


FIG. 356. GRAVITY CONTROL

Instead of a spiral spring, one in the form of a long helix is sometimes used. In some instruments the control is exerted by a suspension consisting of a single strip of phosphor-bronze.

The requirements of materials used for springs are that they shall be non-magnetic; not subject to appreciable fatigue; and, in cases where the spring is to be used as a lead to the moving system (as it often is), it is also necessary that its resistance shall be low and that its temperature coefficient shall be small.

Silicon bronze, hard-rolled silver or copper, platinum-silver, platinum-iridium, and German silver have all been used, in addition to phosphor-bronze, as spring materials, but, either on account of fatigue, or of high temperature coefficient, they are all inferior to the latter.

Fatigue in springs, the presence of which may be detected by the fact that the moving system does not return to zero after deflection, but remains slightly deflected, may be avoided to a great extent by proper annealing and ageing during manufacture.

Two springs, coiled in opposite directions, are sometimes fitted so that the effect of variations of temperature upon the length of the spring may not produce an erroneous deflection. When the instrument deflects, one spring is extended while the other is compressed.

Gravity Control. In gravity-controlled instruments, a small weight is attached to the moving system in such a way that it produces a restoring, or controlling, torque when the system is deflected. This is illustrated in Fig. 356. The controlling torque, when the deflection is θ, is $Wl \sin \theta$ where W is the control weight and l its distance from the axis of rotation of the moving system and it is, therefore, proportional only to the sine of the angle of deflection, instead of, as with spring control, being directly proportional to the angle of deflection. The controlling torque can be varied quite simply by adjustment of the position of the control weight upon the arm which carries it. Gravity-controlled instruments must obviously be used in a vertical position in order that the control may operate. They must also be level, or their zero position will be affected. For these reasons, control by gravity is not so well suited to indicating instruments generally, and particularly to portable instruments, as the spring control.

Comparison of Spring and Gravity Control. As against the disadvantages mentioned above, gravity control has the advantages, when compared with spring control, of cheapness, and of independence upon temperature, and freedom from deterioration with time.

Consider an instrument in which the deflecting torque T_D is directly proportional to the current (say) to be measured.

Thus, if I is the current

$$T_D = kI \quad . \quad . \quad . \quad . \quad . \quad (343)$$

If the instrument is spring-controlled, the controlling torque being T_C, then when the deflection is θ,

$$T_C = k_s\theta \quad . \quad . \quad . \quad . \quad . \quad (344)$$

Also $$T_C = T_D$$

or $$k_s\theta = kI$$

$$\therefore \qquad \theta = \frac{k}{k_s} . I \quad . \quad . \quad . \quad . \quad (345)$$

Thus the deflection is proportional to the current throughout the scale.

If the instrument is gravity-controlled,

$$T_C = k_g \sin \theta \quad . \quad . \quad . \quad . \quad (346)$$

and $$T_C = T_D = kI$$

$$\therefore \qquad k_g \sin \theta = kI$$

$$\sin \theta = \frac{k}{k_g} . I$$

$$\theta = \sin^{-1}\left(\frac{k}{k_g} . I\right) \quad . \quad . \quad . \quad (347)$$

For example, if the maximum deflection, produced by a current I_{max}, is 90°, then when the current is $\frac{I_{max}}{2}$, the deflection if the instrument is spring controlled, is 45°.

If gravity control is used, we have

$$\sin 90^\circ = \frac{k}{k_g} . I_{max} = 1$$

or

$$I_{max} = \frac{k_g}{k}$$

∴ when

$$I = \frac{I_{max}}{2}$$

$$\theta = \sin^{-1} . \frac{\frac{I_{max}}{2}}{I_{max}} = \sin^{-1} . 0{\cdot}5$$

or

$$\theta = 30^\circ$$

Thus, a gravity-controlled instrument would have a scale which was "cramped" at its lower end, instead of being uniformly divided, if the deflecting torque is directly proportional to the quantity to be measured. In practice, however, gravity control would not usually be used in such an instrument.

Balancing of Moving Parts. In order that the deflection of a spring-controlled instrument shall be independent of its position, and also that the wear on the bearings shall be uniform, it is essential that the centre of gravity of the moving system shall lie on the axis of rotation. This condition is attained by the use of balance weights carried on fairly short arms attached to the moving system. In the gravity-controlled system shown in Fig. 356, both the "control" weight and the "balance" weight take a part in balancing the weight of the pointer and the rest of the moving system.

In determining suitable weights and distances from the axis of rotation for control and balancing purposes, attention must be paid to their effects upon the weight and inertia of the moving system. If large weights at short radii are used, the weight of the moving system is made large, whereas small weights at a large radii add considerably to the inertia of the system and necessitate a large damping torque. A compromise is usually effected.

Torque and Weight Ratio. In order to reduce the load on the bearings and to reduce the friction torque, which is proportional to the pressure on the bearing surface, the weight of the moving parts should be made as small as possible.

Expressing the deflecting torque in terms of the force which, acting at a radius of 1 cm., would produce full-scale deflection, the ratio of this torque to the weight of the moving system should be, if possible, not less than 0·1. The torque/weight ratio is influenced by whether the axis of the moving system is vertical or horizontal.

DAMPING. There are three systems of damping in general use. These are as follows—

(*a*) Air friction damping.
(*b*) Fluid friction damping.
(*c*) Eddy current damping.

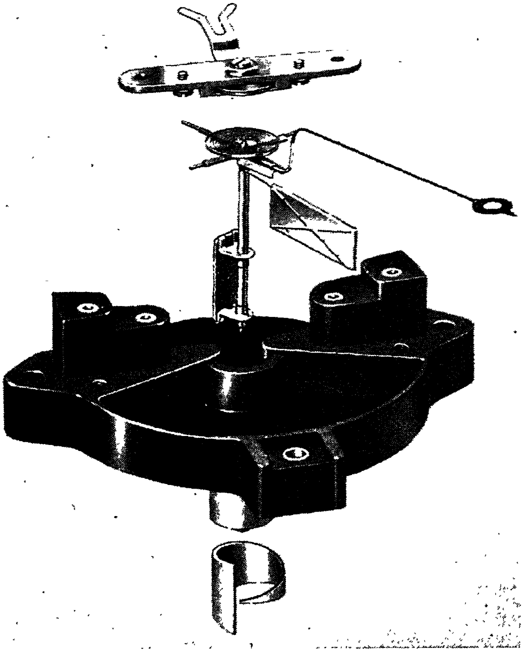
The system used varies with the type of instrument. Eddy current damping is perhaps the most efficient form, and is used in cases where the introduction of a permanent magnet—necessary for the induction of the eddy currents—will not produce errors due to distortion of an existing magnetic field which is being used for operating purposes. It is a very convenient form of damping when either a permanent magnet or some metal part, such as a disc, already forms part of the operating system of the instrument. For these reasons this form of damping is used in "hot-wire," moving-coil, and induction instruments.

The disadvantages of fluid friction damping are that it can only be used in instruments which are used in a vertical position and also that, owing to "creeping" of the oil used for the purpose, it is difficult to keep the instrument clean. An advantage of this method is, however, that the damping oil can be used for insulation purposes, as well as for damping, in some forms of instrument in which the whole moving system is immersed in the oil. When the system is so immersed the up-thrust of the oil on the system reduces the load on the bearings or suspension.

Air Friction Damping. Two methods of damping by air friction are illustrated in Figs. 355 and 357. In one case a light aluminium piston is attached to the moving system and moves in an air chamber closed at one end, as shown in Fig. 355. The cross-section of this chamber may be either circular or rectangular. The clearance between the piston and the sides of the chamber should be small (a few thousandths of an inch) and uniform. If the piston is moving rapidly into the chamber the air in the closed space is compressed and the pressure opposes the motion of the piston (and therefore of the whole moving system). If the piston is moving out of the chamber, rapidly, the pressure in the closed space falls, and the pressure on the open side of the piston is greater than that on the opposite side. Motion is thus again opposed. With this damping system care must be taken to ensure that the arm carrying the piston is not bent or the piston will touch the sides of the chamber during its movement. The solid friction which thus occurs may result in a serious error in the deflection. When once bent, it is often difficult to straighten the piston arm so that it does not touch the sides of the chamber at any point during the deflection. The second method utilizes a vane, mounted on the spindle of the moving system. This vane is of thin aluminium sheet and moves in a closed, sector-shaped box, as shown in

Fig. 357 illustrating the air damping system (and the other working parts) of an Evershed and Vignoles moving-iron instrument.

Fluid Friction Damping. In this method of damping, no very careful fitting, as in the previous method, is necessary. A light vane, attached to the spindle of the moving system, dips into a pot of damping oil and should be completely submerged by the oil. Fig. 358 illustrates the method.



(*Evershed & Vignoles, Ltd.*)

FIG. 357. AIR DAMPER

The frictional drag on the disc in the first system is always in the direction opposing motion, and increases with the speed of rotation of the disc. There is no friction force when the disc is stationary. The suspending stem of the disc should be cylindrical and of small diameter where it penetrates the oil surface, so that surface tension effects may be negligible. In the second system (Fig. 358 (*b*)), increased damping (as compared with the previous system) is obtained by the use of vanes, in vertical planes, carried on a spindle and immersed in oil, as shown.

The principle requirements of oil for damping purposes are that it shall not evaporate quickly, shall not have any corrosive action upon metals, and that its viscosity shall not change appreciably with temperature. It should also be a good insulator.

Eddy Current Damping. When a sheet of conducting material moves in a magnetic field so as to cut through lines of force, eddy currents are set up in it and a force exists between these currents and the magnetic field, which is always in the direction opposing the motion. This force is proportional to the magnitude of the current, and to the strength of field. The former is proportional to the velocity of movement of the conductor, and thus, if the magnetic field is constant, the damping force is proportional to the velocity of the moving system and is zero when there is no movement of the system. The theory is discussed in Chapter XIV.

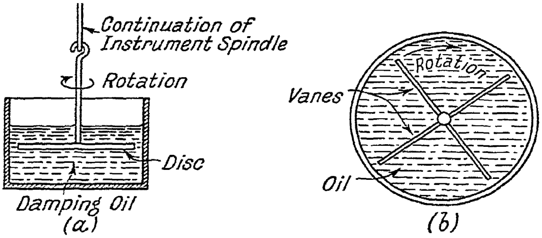


FIG. 358. FLUID FRICTION DAMPING

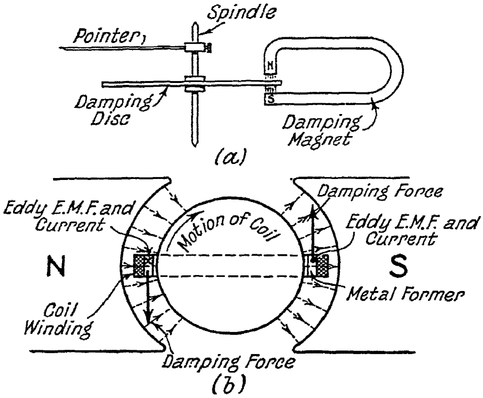
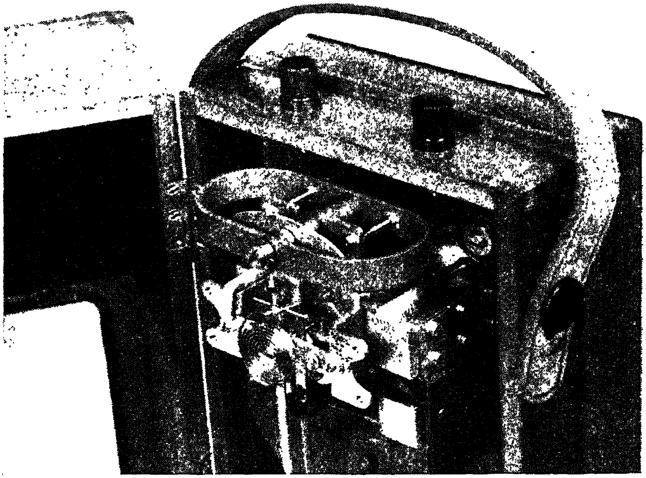


FIG. 359. EDDY CURRENT DAMPING

Fig. 359 shows two methods of applying this method of damping. In diagram (*a*) a thin disc of conducting, but non-magnetic material

—usually copper or aluminium—is mounted on the spindle which carries the pointer of the instrument. When the spindle rotates, the edge of the disc cuts through the lines of force in the gap of a permanent magnet, and eddy currents, with consequent damping, are produced. An arrangement similar to this is often used in hot-wire instruments.

Fig. 359 (*b*) shows the essential parts of a permanent-magnet, moving-coil, instrument. The coil is wound on a light metal former in which eddy currents are induced when the coil moves in the permanent-magnet field. The directions of the eddy currents, and of the damping forces produced as a result of them, are shown in the figure.



(*Evershed & Vignoles, Ltd.*)

FIG. 360

Fig. 360 shows an adjustable magnetic damping device used in the latest type of Evershed and Vignoles recorder.

CONSTRUCTIONAL DETAILS OF INDICATING INSTRUMENTS.

(1) *Methods of Supporting Moving Systems.* The two commonest methods of supporting the moving system of an instrument are—

(*a*) By pivoting. (*b*) By thread suspension.

When the system is pivoted, the ends of the spindle are conical, and should be of hardened steel. These ends fit into conical holes in jewels (preferably sapphire), which form the bearings, on the fixed part of the instrument.

To reduce friction at the pivots the contact area should be small, but the pressure per unit area must be carefully considered, since, if the pivots are sharply pointed, this may exceed the crushing strength of the material of the pivot. Although the weight of the moving system may be only a few grammes, if the area of the point of the pivot is very small, the pressure per square inch may be many tons. For example, this pressure may be 80 tons per sq. in. or even more.

In mounting the moving part, a small amount of play should be allowed so that the pivots are not forced hard on to the jewels.

The question of the wear on instrument pivots and bearings is so important—particularly in electricity meters, in which the rotating system is almost continuously in movement—that fundamental studies extending over many years have been made especially by G. F. Shotter and by the Electrical Research Association. Refs. (29), (30), (32) deal very fully with the matter. It has been shown that the relation between the direction of pressure and the optic axis is very important in the steel-sapphire combination. Tests have confirmed that, with a set frictional limit as basis, jewels having an active surface at an angle of 90° to the optic axis have a much longer life than those cut at other angles.

Glass has been shown inferior to sapphire as a bearing material (except, perhaps, when the direction of the load may vary) but German work (Ref. (30)) during World War II has indicated that hardened spinel bearings may be almost as good.

Thread suspension is advantageous when the operating forces are small compared with the weight of the moving part, since bearing friction is avoided. Such suspensions are, however, delicate, and protection from vibration and shock is necessary. Phosphor-bronze strip is most commonly used for these suspensions.

(ii) *Permanent Magnets.* In most cases when permanent magnets are used in instruments it is essential that their strength shall not vary with time.

Such magnets may be of very hard steel, containing a small percentage of tungsten or of cobalt and chromium together. The coercive force of cobalt-chromium steel is high, and thus magnets made of this steel are not subject to self-demagnetization to the same extent as tungsten steel magnets.

Within the last two decades great advances have been made in the development of magnetic materials having properties suiting them to the construction of permanent magnets. In particular, Alnico (i the United States) or Alcomax (in England), alloys of iron, nickel and aluminium, have especially good properties with coercive forces which may be as high as 700.

R. M. Bozorth (Ref. (20)) gives data for modern permanent-magnet materials. The energy product $(BH)_{max}$ (see Chap. I, p. 45) ranges from 300,000 for tungsten steel to as much as 4,500,000 for Alnico (or Alcomax) having a composition of 24 Co, 14 Ni, 8 Al, 3 Cu. The coercive force H_c for this material is given as 575 and the residual induction B_{rem} as 12,500.

During manufacture, permanent magnets are artificially aged by being placed in a weak alternating magnetic field, or by heating. This reduces their strength somewhat, but ensures permanence of the magnetism remaining.

(iii) *Pointers and Scales.* The shape and size of pointer used depends upon the type of instrument. In all cases, however, the weight and inertia of the pointer must be reduced as far as possible, both to reduce the load on the bearings of the moving system and to avoid the high degree of damping which would be necessary if the moving system had considerable inertia.

For the sake of lightness, aluminium strip or tube is used for the pointer, a truss construction being used in some cases for rigidity.

In some instruments where precision in reading, at close range, is aimed at, a strip of mirror is fitted on the plate which bears the scale. The end of the pointer is flattened so that, when viewed from above, it appears as a narrow strip or edge. The eye of the observer must be moved until the end of the pointer and its image in the mirror are coincident, before a reading is taken. This avoids error due to parallax.

The moving system of most indicating instruments rotates through an angle of about 90° (for full-scale deflection), although some makers have designed instruments having angles of deflection of 120° or even greater. The length of scale in many instruments is about 6 in. Thus, for an accuracy, in the reading, of (say) $\frac{1}{2}$ per cent, it is necessary to observe the position of the pointer to within 0·03 in. at full-scale deflection and to within smaller fractions of an inch when the deflection is less than the full-scale value. Hence the necessity for a clearly-marked and carefully divided scale and for a sharply-pointed pointer. In the case of good-class instruments the card upon which the scale is to be marked is blank when first fitted. The main divisions of the scale are then marked in by comparison of the instrument with a sub-standard one, after which the scale is completed by means of some dividing instrument. Stiff card, mounted on a metal sheet, is generally used for the scale. An excellent review of the various types of pointers and scales in use in electrical measuring instruments, with comments on their relative effectiveness and on possible future developments, with many illustrations, has been given by L. B. S. Golds (Ref. (14)).

(iv) *Cases.* These may be of hard wood or brass, but are most often of cast iron or pressed steel. The steel cover is an advantage for magnetic screening in the case of instruments which are affected by external magnetic fields. The base which carries the operating portion of the instrument is, also, often of steel for the same reason. The cover should be fitted so as to exclude dust and moisture from the instrument. When steel covers are used, the moving system of the instrument should be mounted in a position as far away from the case as possible, in order to avoid errors due to hysteresis and eddy current effects in the case.

Recording Electrical Quantities. The value of an electrical quantity which varies with time may be recorded,

(*a*) to preserve information which could be obtained at any instant from indicating instruments—as when the voltage or power of a supply circuit is recorded, or,

(*b*) to give information which could not be obtained readily from indicating instruments. Examples are wave-forms and transient phenomena or the phase relationships in different parts of a circuit.

It is often necessary to record an electrical quantity which is being used to study some other physical quantity such as strain or pressure. The speed of response of the recording instrument is then important whereas a recorder which merely replaces an indicating instrument is not usually required to have a quick response. The recording instruments considered below are given in (roughly) ascending order of response speed.

Recorders. These instruments generally resemble the corresponding indicating instruments—voltmeters, wattmeters, etc., but have a light arm, carrying a pen, in place of the pointer. The pen rests lightly on a chart which is moved, at a slow and uniform speed, in a direction perpendicular to that of the deflection of the pen. This chart is unwound from a drum on to another similar drum by clockwork. The path traced out by the inked pen gives a continuous record of the variations of deflection of the instrument.

Owing to the friction of the pen on the chart and to the necessarily greater weight of the moving system, the design of an indicating instrument must be somewhat modified if it is to be used for recording purposes.

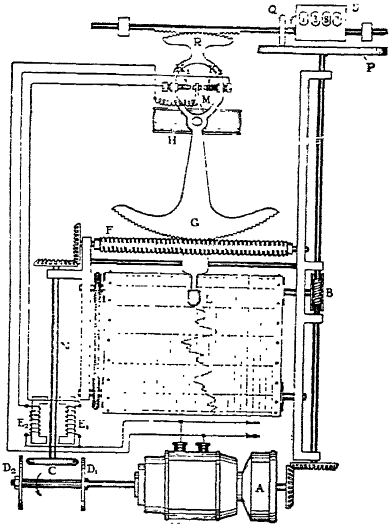
In order that friction shall not introduce serious errors, the operating torque must be increased in proportion to the increased friction torque. The controlling torque must, therefore, also be increased, and increased damping is necessary on account of the greater inertia of the moving system. It may be necessary also to increase the size of the bearings on account of the greater load on them.

The line diagram in Fig. 361 shows the construction of a relay-operated recording instrument by Messrs. Everett-Edgcumbe. The tongue M is moved by any variation in the quantity to be measured. This closes the circuit of one of the electro-magnets E_1, E_2, causing the shaft N to deflect. The friction wheel C engages with one of the discs D_1, D_2, and causes the worm F to rotate. This moves both the toothed quadrant G and the pen L.

The chart is driven by the small motor, whose speed is maintained constant by the centrifugal governor A.

Instruments of this kind can be divided into two classes; those in which the pen-arm is vertical, and those in which it is horizontal.

The former has the advantage that the time-lines on the chart are straight, whereas in the second they are segments of circles. The pen, however, is generally heavier in the instruments with a vertical pen, since the ink must be carried in the pen at the end of the arm, or in a trough into which the pen dips. Instruments with a horizontal pen generally carry the ink in a well at the axis about which the pen arm rotates, and ink flows along the arm to the writing point by capillary action.



(Everett-Edgcumbe & Co., Ltd.)
FIG. 361. RECORDING INSTRUMENT

Instruments which are intended to record information normally shown on indicating instruments have a response time, from zero to full scale, of several seconds. For the purposes for which the recorders are used this time is unimportant, and it is not generally quoted.

Milliammeter Recorders with Quick Response. A pen recorder of normal pattern having a moving-coil movement driving a horizontal pen can be made to have a response time, from zero to full-scale, down to about 0·1 sec. by careful design. The moving parts must be as light as possible and the driving power must be increased as the response time is reduced. Such an instrument is made by Messrs. Everett-Edgcumbe. The four-inch-wide chart runs normally at 1 inch per second but can run at up to 4 inches per second. The response time is 0·1 second and the instrument consumes 1 watt for full-scale deflection.

By still further reducing the weight of the pen and by considerably reducing its movement, the response of such an instrument can be improved further. The recorder made by Kelvin and Hughes (Industrial), Ltd. has a pen movement of 7·5 mm. on each side of the centre zero. Several models of different coil resistance are made and one example has a uniform frequency response up to 90 c/s while it consumes 1 watt for full deflection to either side.

Milliammeter Recorder driven by Amplifier. The power required to drive a recorder having a milliammeter movement with a fast response is often greater than is available from the circuit providing the quantity (whether voltage or current) which is to be recorded. In such cases the recorder may be driven by an electronic amplifier which requires a very small input power.

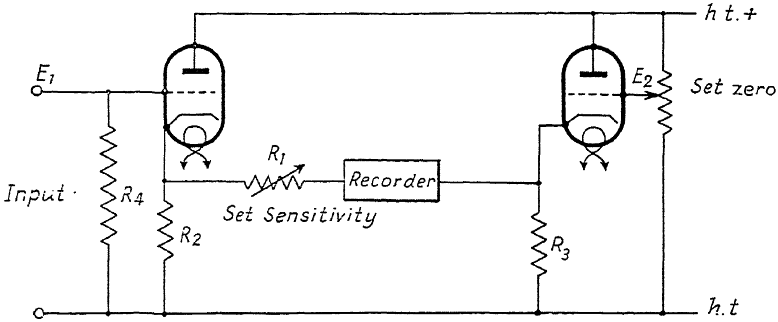


FIG. 362

An amplifier which can be used for this purpose is shown in Fig. 362. The circuit shows a differential cathode follower. The resistance of the recorder and the series resistance R_1 is small compared with R_2 or R_3. If the input voltage E_1 is the same as the voltage E_2 applied to the grid of the second valve and if the characteristics of the valves are the same, no current will flow through the recorder. If the applied voltage is then increased the current through the first valve increases and the voltage of its cathode rises. This causes a current to flow through the recorder, and raises the voltage of the cathode of the second valve, which in turn decreases the current through that valve. Thus the final effect of raising the input voltage E_1 is to increase the current through the first valve, decrease the current through the second, and cause an equalizing current to flow through the recorder.

The current flowing through the recorder for a given change of the input voltage is dependent mainly on the circuit resistances, and is very little affected by changes of the valve characteristics or of the h.t. voltage. It can be adjusted by varying the value of R_1.

Since the resistance R_4 may be high—of the order 0·5 MΩ—the power required to drive the amplifier is very small.

The principle of the Tinsley (H. Tinsley and Co., Brit. Patent 537784 and others) d.c. amplifier, which is free from drift or instability and is operated entirely from a.c. mains, is illustrated in Fig. 363 (see also Ref. (23)). A thyratron T is used and its anode is supplied with an alternating voltage. Its grid is supplied in antiphase through capacitor C so that the thyratron is maintained below the critical firing potential. A vacuum photocell P shunts the grid to the cathode.

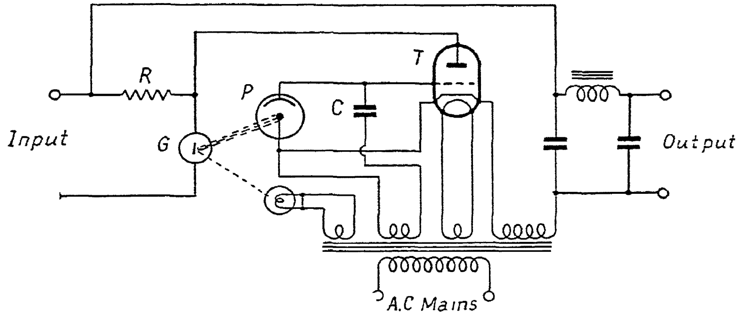


FIG. 363

The voltage to be measured is applied to a reflecting galvanometer G and the spot of light is deflected over the photocell, this causing the latter to act as a variable resistance across C. The phase of the potential applied to the grid is thus changed and the thyratron fires for part of each positive cycle. The resulting current pulses are fed through the feed-back resistance R so as to oppose the applied voltage. Equilibrium is reached when the product of the output current and the resistance R equals the input voltage (less a small quantity required to move the galvanometer the necessary 3 or 4 mm.). If R is very small the output current is very large and must be exactly proportional to the applied voltage provided the galvanometer movement is small.

The uni-directional pulses are fed to the recorder through a smoothing circuit.

The maximum output current of the amplifier is 20 mA into a load of 2,500 Ω (max.) for all inputs.

Servo-operated Pen Recorders. If a recorder must be driven by an amplifier, it is inherently more accurate to use the amplifier in a servo-system. An example of such an instrument is that manufactured by Messrs. Evershed and Vignoles. The operation of the recorder can be understood from Fig. 364.

The moving coil is wound in two equal halves, which are supplied with currents in opposition by a balanced amplifier. There is no control spring, and when the currents in the two halves of the moving coil are equal, the pen will remain at rest. The pen-arm carries a contact which bears lightly on a small potentiometer. The input to the amplifier is taken from the centre-tap of a resistance connected between the input terminal and the pen-arm contact.

If the voltage picked up by the pen-arm contact is equal and opposite to the input voltage, there is no input to the amplifier, the currents in each half of the moving-coil are equal, and the pen remains at rest. If the input voltage is increased, the currents in the two parts of the moving coil became different, and a torque is set up which drives the pen in a direction to increase the negative voltage picked up by the pen-arm contact. When this becomes equal and opposite to the applied voltage there is again no input to the amplifier, and the pen comes to rest.

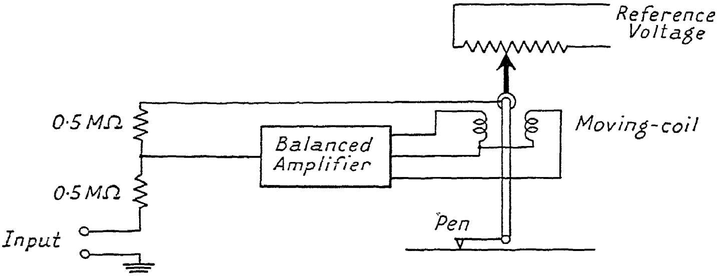


FIG. 364

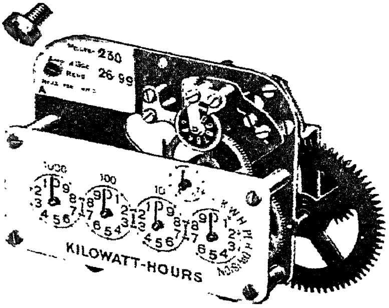
The recorder is thus essentially a self-resetting potentiometer, and the accuracy is unaffected by the amplifier characteristics. The instrument has a chart width of $2\frac{7}{8}$ in., and has a robust vertical pen arm. The response is uniform up to about 10 c/s for an amplitude of $1\frac{1}{2}$ in. peak-to-peak.

Integrating Instruments. These instruments measure, and register, either the total quantity of electricity, in ampere-hours, or the total amount of energy, in kilowatt-hours, supplied to a circuit in a given time. They give no direct indication as to the rate at which the energy is being supplied. Their registrations are independent of the rate at which a given quantity of energy is supplied, provided that the current flowing is sufficient to cause the instrument to operate.

Ampere-hour meters are used on direct-current circuits where the supply voltage is constant. The number of watt-hours supplied to such a circuit is obtained by multiplying the measured number of ampere-hours by the supply voltage. The registering dials (or scales) are usually marked in kilowatt-hours, which means that the instrument only gives correct readings when connected in a circuit whose voltage is that for which the instrument was calibrated. Obviously, when such instruments are used, the constancy of the voltage is depended upon. This is justifiable in most cases, since, by law, the voltage in public supply systems must be maintained within 4 per cent of the nominal value.

These meters have the advantage of simplicity, cheapness, and of low power consumption.

Watt-hour meters measure the watt-hours supplied to a circuit, directly, the operating torque being due, in part, to a current proportional to the supply voltage.



(*Ferranti, Ltd.*)

FIG. 365. METER REGISTERING MECHANISM

With the exception of those of the electrolytic type—in which the quantity of electricity supplied is indicated by the level of liquid in a graduated tube—integrating meters register by means of a train of gear wheels and dials similar to that shown in Fig. 365.

Meters of the "motor" type are most generally used, and in these instruments the train of wheels is driven from the spindle of the rotating system of the instrument. This spindle has a worm cut on it, and this engages with a pinion and thus drives the wheel-train. The spindles of the wheels in the train carry hands which move over the dials (five or six in number) which register units, tens, hundreds, and so on.

The essential parts of such meters are—

(*a*) An operating system which produces a torque proportional to the current or power, and causes the rotating system to rotate.

(*b*) A braking device—usually a permanent magnet—which produces a braking torque proportional to the speed of rotation, and thus causes the rotating system to run at a steady speed such that the braking torque is equal to the operating torque.

(*c*) A device for registering the number of revolutions of the rotating system, this usually taking the form of a train of wheels operating the hands of dials.

BIBLIOGRAPHY AND REFERENCES

(1) *Electrical Measuring Instruments,* Vol. I, C. V. Drysdale and A. C. Jolley.

(2) *Electrical Measuring Instruments,* D. J. Bolton.

(3) "Electrical Measuring Instruments other than Integrating Meters," C. V. Drysdale, *Jour. I.E.E.*, Vol. LXVI, p. 596.

(4) *Industrial Electrical Measuring Instruments,* K. Edgcumbe and F. E. J. Ockenden.

(5) "Electrical Measuring Instruments," *Bureau of Standards, Circular,* No. 20.

(6) "Some Recent Advances in Alternating-current Measuring Instruments," K. Edgcumbe and F. E. J. Ockenden, *Jour. I.E.E.*, Vol. LXV, p. 553.

(7) "Permanent Magnets in Theory and Practice," S. Evershed, *Jour. I.E.E.*, Vol. LXIII, p. 725.

(8) "Precautions in the Use of Standard Instruments," W. H. Lawes, *Jour. I.E.E.*, Vol. LXVII, p. 541.

(9) "An Investigation of Problems relating to the Use of Pivots and Jewels in Instruments and Meters," V. Stott, *Jour. I.E.E.*, Vol. LXIX, p. 751.

(10) "Future Progress in Electrical Measuring Instruments," C. V. Drysdale, *Jour. I.E.E.*, Vol. LXXII, p. 365.

(11) "A Permanent-Magnet Device for enabling a More Uniform Division of the Scale of Alternating-Current Indicating Instruments to be readily obtained," H. E. M. Barlow, *Jour. I.E.E.*, Vol. LXXVII, p. 618.

(12) "Indicating Ammeters, Voltmeters, Wattmeters, Frequency and Power-factor Meters," *British Standard* 89: 1937.

(13) "Graphic (Recording or Chart-Recording) Ammeters, Voltmeters, Wattmeters, Power-factor Meters and Frequency Meters," *British Standard* 90: 1940.

(14) "A Survey of Modern Methods of Presentation of Instrument Readings and Recordings," L. B. S. Golds, *Jour. I.E.E.*, Vol. 98, Pt. II, p. 671, 1951.

(15) "Performance Limits in Electrical Instruments," A. H. M. Arnold, *Jour. I.E.E.*, Vol. 98, Pt. II, p. 701, 1951.

(16) "The Properties of Insulating Materials used in Instruments," C. G. Garton, *Jour. I.E.E.*, Vol. 98, Pt. II, p. 728.

(17) "New Principles in Electrical Instruments," D. C. Gall, *Jour. I.E.E.*, Vol. 98, Pt. II, p. 665.

(18) "Radiation Measuring Instruments for the Infra-red to Ultra-violet Waveband," A. C. Menzies, *Jour. I.E.E.*, Vol. 98, Pt. II, p. 771.

(19) "Radiation Measuring Instruments for X-rays to Cosmic Rays," D. Taylor and W. Abson, *Jour. I.E.E.*, Vol. 98, Pt. II, p. 760.

(20) *Ferromagnetism,* R. M. Bozorth.

(21) "Alnico; Properties and Equipment for Magnetization and Test," B. M. Smith, *Gen. Elec. Rev.*, Vol. 45, No. 4, April, 1942.

(22) "Permanent Magnets," A. Edwards, *Elec. Rev.*, June 30, 1944.

(23) "A Direct Current Amplifier and its Application to Industrial Measurements and Control," D. C. Gall, *Jour. I.E.E.*, Vol. 89, Pt. II, p. 434.

(24) "Meter and Instrument Jewels and Pivots," (Second Interim Report), *E.R.A. Report, Ref. T/T*39, G. F. Shotter.

(25) "British Achievements in Electrical Measuring Instruments," S. Whitehead, *Proc. Joint Engineering Conference,* 1951, p. 473.

(26) "Discussion on Practical Considerations in Instrument Design," *Jour. I.E.E.*, Vol. 95, Pt. II, p. 681.

(27) "Symposium of Papers on Electrical Instrument Design," *Proc. I.E.E.* Vol. 98, Pt. II, pp. 653–792.

(28) "Symposium of Papers on Electrical Meteorological Instruments," *Proc. I.E.E.*, Vol. 98, Pt. II, pp. 429–84.

(29) "A Simplified Method for Checking the Orientation of the Optic Axis of Meter Jewels," S. F. Knight, *Proc. I.E.E.*, Vol. 97, Pt. II, p. 275.

(30) "Synthetic Sapphire and Spinel Production in Germany," M. H. Barnes, *Field Information Agency Technical—Final Report, No.* 655.

(31) "The German Electrical Indicating Instrument Industry, its Technique and Materials," G. N. Harding, *BIOS Final Report No.* 1716, Item No. 9, (H.M. Stationery Office).

(32) "Meter and Instrument Jewels and Pivots," G. F. Shotter, *Jour. I.E.E.*, Vol. 93, Pt. II, p. 15.

CHAPTER XVIII

AMMETERS AND VOLTMETERS

AMMETERS and voltmeters are classed together because there is no essential difference in principle between them. Except in the case of electrostatic instruments, a voltmeter carries a current which is proportional to the voltage to be measured, and this current produces the operating torque. In an ammeter this torque is produced by the current to be measured, or by a definite fraction of it. Thus, the only real difference between the two instruments is in the magnitude of the current producing the operating torque.

An ammeter is usually of *low resistance*, so that its connection in series with the circuit in which the current is to be measured does not appreciably alter this current. A voltmeter, on the other hand, is connected across the voltage to be measured, and must therefore have a *high resistance* so that the current taken by it may be small. A low range ammeter—i.e. one which gives full scale deflection for a very small current—may thus be used as a voltmeter if a high resistance is connected in series with it. The current which flows through it when it, together with its series resistance, is connected across the voltage to be measured, must be within its range when used as an ammeter.

Example. A milliammeter, whose resistance is 5 ohms, gives full-scale deflection for a current of 15 milliamp. Calculate the resistance which must be connected in series with it in order that it may be used as a voltmeter for voltages up to 100 volts.

Let R be the required series resistance. The current flowing through the instrument when 100 volts are applied to the instrument and resistance in series, must be 0·015 amp.

$$\therefore \qquad \frac{100}{R+5} = 0{\cdot}015$$

$$\text{or} \qquad R + 5 = \frac{100}{0{\cdot}015} = 6666 \text{ ohms}$$

$$\therefore R = 6661 \text{ ohms}$$

Power Loss. If R_A is the resistance of an ammeter in which a current I flows, the power loss in the instrument is I^2R_A watts.

Again, if R_V is the resistance of a voltmeter to which a voltage E is applied, the power loss in the instrument is $\dfrac{E^2}{R_V}$.

Obviously, in order that the power loss in the instruments shall be small, R_A must be small and R_V large.

Operation of Ammeters and Voltmeters on Alternating-current Circuits. If an ammeter or voltmeter, whose operating torque is proportional to the current passing through it, is used on an alternating current circuit, the torque acting on the moving system is alternating. This means that the moving system tends to oscillate about its zero position. If the current through it alternates rapidly, the moving system of the instrument cannot, in general, follow these alternations, and no deflection will be observed.

It is thus essential that the torque, in an instrument to be used on an alternating current circuit, shall be proportional to the *square* of the current passing through the instrument. The deflection will then be proportional to the *mean value of the square of the current* (assuming that spring control is used), and the instrument can thus be used for the measurement of r.m.s. values of current or voltage.

Types of Instruments. The following types of ammeters and voltmeters are in common use—

(*a*) Moving iron.
(*b*) Moving coil—
    (i) permanent-magnet form,
    (ii) dynamometer form.
(*c*) Hot wire.
(*d*) Electrostatic (voltmeters only).
(*e*) Induction.

Of these the permanent-magnet, moving-coil type can be used for direct-current measurements only, and the induction type for alternating current measurements only. The other types can be used with either direct or alternating current.

The moving-iron and moving-coil types both depend, for their action, upon the magnetic effect of current. The former is the most generally used form of indicating instrument, as well as the cheapest. It can be used for either direct- or alternating-current measurements and, if properly designed, is very accurate. The moving-coil permanent-magnet instrument is the most accurate type for direct-current measurements, and instruments of this type are frequently constructed to have sub-standard accuracy.

Hot-wire instruments have the advantage that their calibration is the same for both d.c. and a.c. They are particularly suited to alternating-current measurements, since their deflection depends directly upon the heating effect of the alternating current, i.e. upon the r.m.s. value of the current. Their readings are thus independent of the frequency or wave-form of the current, and of any stray magnetic fields which may exist in their vicinity.

As voltmeters, electrostatic instruments have the advantage that their power consumption is exceedingly small. They can be made to cover a large range of voltage, and can be constructed to have sub-standard accuracy. Their main disadvantage is that the electrostatic principle is only directly applicable to voltage measurements.

The induction principle is more generally used for watt-hour meters than for ammeters and voltmeters owing to the comparatively high cost, and inaccuracy, of induction instruments of the latter types.

Errors in Ammeters and Voltmeters. There are certain errors which occur in most types of instruments, while other errors occur only in those of a particular type. These latter errors will be dealt with later, together with the instruments in which they occur.

Of the errors common to most types of instrument, friction and temperature errors are perhaps the most important. To reduce the effect of friction torque, and consequently the error produced by it, the weight of the moving system must be made as small as possible compared with the operating forces, i.e. the ratio of torque to weight must be large (about $\frac{1}{10}$ for full deflection).

A vertical spindle is generally to be preferred to a horizontal one from the point of view of a small friction torque.

The most serious error produced by the heat generated in the instrument, or by changes in room temperature, is that due to a change in the resistance of the working coil. Such a change of resistance is of little importance in ammeters, but in voltmeters, in which the working current should be directly proportional to the applied voltage, it is essential that the resistance of the instrument shall remain as nearly constant as possible.

Thus, the power loss in the instrument should be small, and resistance coils which are likely to produce appreciable heating should be mounted, if possible, in such a position that they are well ventilated.

To eliminate temperature errors, the working coil is wound with copper wire and is of comparatively low resistance. A high "swamping" resistance, of material whose temperature coefficient is small, is connected in series with the coil, so that, although the resistance of the coil may change considerably, the change in total resistance is small.

Other errors resulting from heating may be caused by expansion of the control spring, or of other parts of the instrument, although such errors are usually small. Lack of balance in the moving system and changes in the strength of permanent magnets (if used) are other possible sources of error which are common to several types of instrument.

Permissible Errors in Ammeters and Voltmeters. British Standard Specification No. 89 (1937) grades indicating instruments on the basis of their limits of error. These grades are—

Sub-standard. First Grade.

The limits of error over the effective range, expressed as a percentage of the maximum scale value, may be summarized from the specification as follows—

Voltmeters and ammeters: From 0·2 per cent for a single-range voltmeter of the permanent-magnet, moving-coil type up to 0·6 per cent for a multi-range, self contained ammeter of the same type. From 0·3 per cent for a single-range voltmeter of the electrodynamic type up to 0·5 per cent for a double range ammeter (not exceeding 10 A by series-parallel connection, one range being double the other self-contained). From 0·5 per cent to 0·6 per cent for other types.

Wattmeters: From 0·25 per cent for a special electro-dynamic wattmeter (single element for d.c. or single-phase or three-phase balanced load) to 0·6 per cent for an electro-dynamic wattmeter with a multi-voltage range and current range between 0·25 and 20 A. For a double-element wattmeter of the last-mentioned type the limit is 1·2 per cent.

FIRST GRADE. The limits of error permitted for first-grade ammeters, voltmeters and single-element wattmeters are from 1 per cent up to 3·5 per cent, the latter being for thermo-couple, hot-wire and rectifier instruments.

TABLE XVIII

Type of Instrument	Pattern	Accuracy of which the Instrument is Capable	Ammeter Ranges for which this Accuracy may be Expected	Voltmeter Ranges for which this Accuracy may be Expected
Permanent-magnet moving coil (D.C. only)	Laboratory	Sub-standard	Up to 500 amp.	Up to 750 volts
	Combined ammeter and voltmeter testing set	Sub-standard or 1st Grade		
Moving iron	Laboratory	Sub-standard d.c. and a.c. up to 60 ~	0·5 to 10 amp.	Above 75 volts
	Testing sets	1st Grade d.c. and a.c. up to 60 ~		
	Switchboard	1st Grade d.c. and a.c. up to 60 ~		
Electro-dynamic ammeters and voltmeters	Laboratory and testing sets	Sub-standard d.c. and a.c. up to 60 ~	0·5 to 10 amp.	Above 75 volts
		1st Grade	Above 10 amp.	Above 300 volts (single range) or 600 volts (multi-range)
Hot-wire		1st Grade	D.c. and a.c. up to 100 ~ (shunted type) up to 10^6 ~ unshunted, max. current < 3 amp.	D.c. and a.c. up to 5,000 ~
Thermo-couple	With self-contained thermo-couples	1st Grade	D.c. and a.c. up to 10^6 ~ 5 mA to 1 amp.	
Induction ammeters, voltmeters and wattmeters		1st Grade	(At standard or marked frequency and temperature)	
Electrostatic voltmeters		1st Grade		1,000 to 7,000 volts d.c. and a.c. up to 10,000 ~
Rectifier voltmeters and milliammeters		1st Grade	1 mA to 50 mA for full scale deflection	Not less than 10 volts for full scale deflection
			Frequency 25 to 10,000 ~. Indications affected by wave-form	

The limits of error, and capabilities of different types, of recording instruments are given in B.S. 90 : 1940.

Moving-iron Instruments. There are two general types of such instruments, namely, the *attraction* type and the *repulsion* type. In all moving-iron instruments the current to be measured (or a current proportional to the voltage to be measured) is passed through a coil of wire, the number of turns on which depends upon the current passing through it. A certain number of ampere-turns is required for the operation of the instrument, and this number can be made up by having a few turns and a large current, or *vice versa.*

In the attraction form of instrument, a small piece of iron is drawn into the core of the coil when the current flows. In the repulsion form there are two rods, or pieces, of iron inside the coil—one fixed and one movable. These are similarly magnetized when the current flows through the coil, and repulsion of the moving iron from the fixed one ensues. The force of repulsion is obviously roughly proportional to the square of the current in the coil, since each iron is magnetized thereby.

Whatever the direction of the current in the coil of the instrument, the magnetization of the moving iron is always such that attraction takes place in the attraction form and repulsion in the repulsion form. They are thus "unpolarized" instruments (i.e. instruments which are independent of the direction in which current passes through them).

The principle of a moving-iron instrument of the attraction type is illustrated in Fig. 366, and of the repulsion type in Fig. 367.

In the instruments shown, gravity control is used. In the past, this method of control was very generally used for moving-iron instruments, but now spring control is used almost universally. The method of damping moving-iron instruments is by air friction, two different forms of damping chambers being shown in the figures.

In the attraction type of instrument the moving iron is eccentrically pivoted, and consists of thin discs of soft iron. This iron tends to move, when the current flows, from the weaker magnetic field outside the coil into the stronger field inside it. The shape of the disc is such that a suitably divided scale is obtained.

In the repulsion type, the shapes of the irons vary in different makes of instrument. They may be rods or the fixed iron may consist of a tongue-shaped piece of sheet iron bent into a cylindrical form, the moving iron being another piece of sheet iron, bent and mounted so as to move parallel to the fixed iron and towards its narrower end. Such an arrangement is shown in Fig. 368. These shapes of the irons give a more uniform scale than is obtained with plain rods. Figs. 369, 370 and 371 show the construction of moving-iron instruments by different makers. The Record Electrical Co., Ltd. make a "Cirscale" moving-iron instrument (see also p. 643) of the repulsion type, with two separately pivoted repulsion movements, to give a 270° scale.

THEORY OF ATTRACTION TYPE MOVING-IRON INSTRUMENTS. Referring to Fig. 372, suppose the axis of the soft iron disc, when in the zero position, makes an angle ϕ with the direction perpendicular to the field of the coil of the instrument, as shown. Let θ be its deflection (into the core of the coil) when a current I flows in the coil and produces a field strength H, this field being assumed uniform and in a direction parallel to the axis of the coil. The magnetization

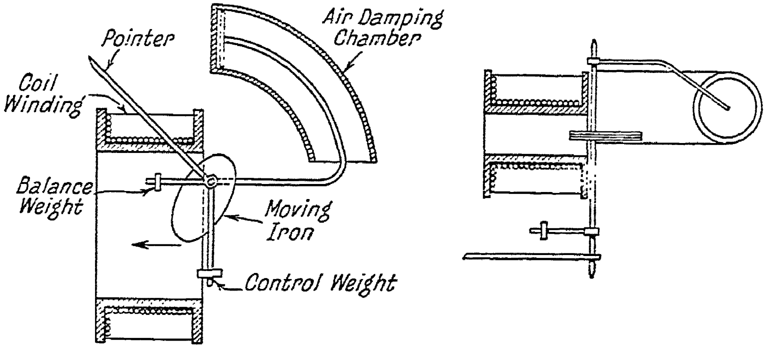


FIG. 366. ATTRACTION TYPE OF MOVING-IRON INSTRUMENT

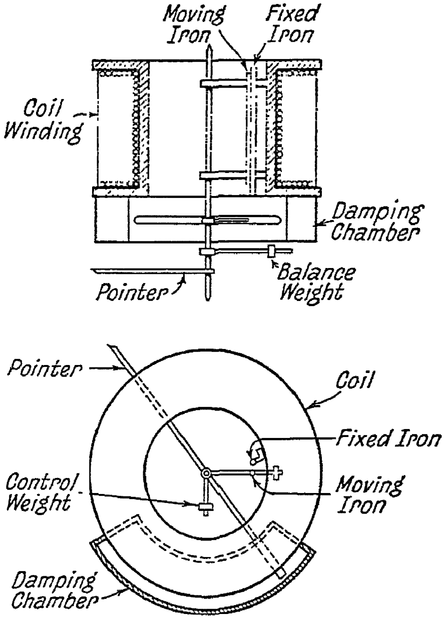


FIG. 367. REPULSION TYPE OF MOVING-IRON INSTRUMENT

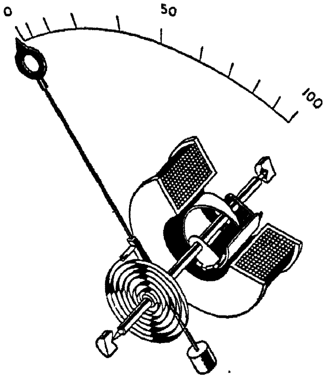
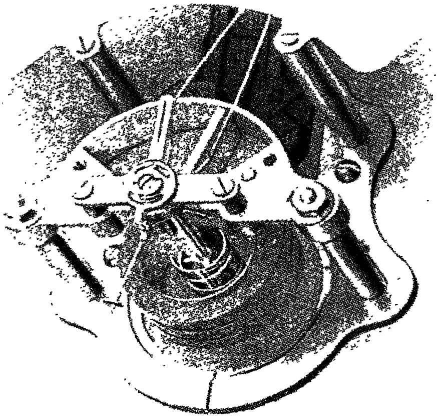


FIG. 368

of the disc in this position is proportional to the component of H in the direction of its axis, i.e. to $H \cos \{90 - (\theta + \phi)\}$ or to $H \sin (\theta + \phi)$. The force pulling the disc into the coil is thus proportional to $H^2 \sin (\theta + \phi)$. If constant permeability of the iron of the disc is assumed, this force F is proportional to $I^2 \sin (\theta + \phi)$ for all values of I (and of θ). If it acts at a distance l from the pivot, the deflecting torque is obviously given by



(*Sangamo Weston, Ltd.*)

FIG. 369. MOVING-IRON INSTRUMENT

$$T_D = Fl \cos (\theta + \phi) \quad . \quad . \quad . \quad . \quad (348)$$

Thus
$$T_D \propto H^2 \sin (\theta + \phi) \cos (\theta + \phi)$$
$$\propto I^2 \sin 2(\theta + \phi)$$

since l will be constant.

This may be written

$$T_D = kI^2 \sin 2(\theta + \phi) \quad . \quad . \quad . \quad . \quad (349)$$

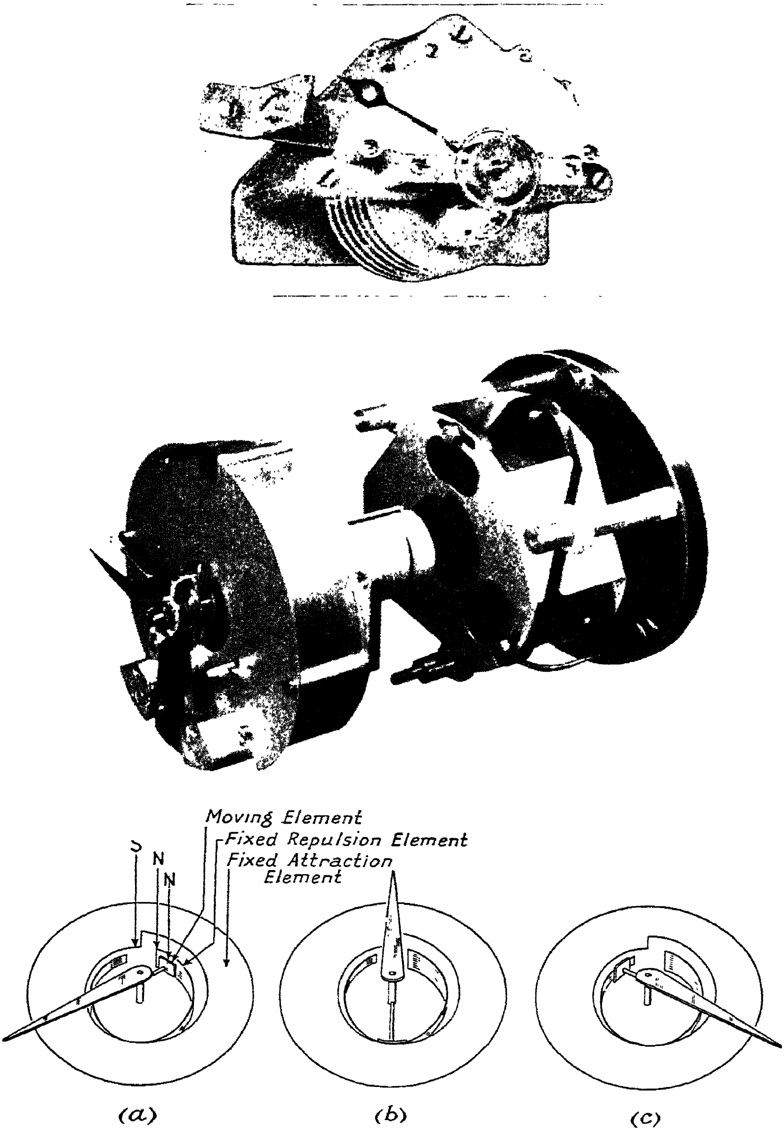
where k is constant.

If the instrument is spring controlled, the controlling torque T_C is proportional to θ, and we may write

$$T_C = k'\theta$$

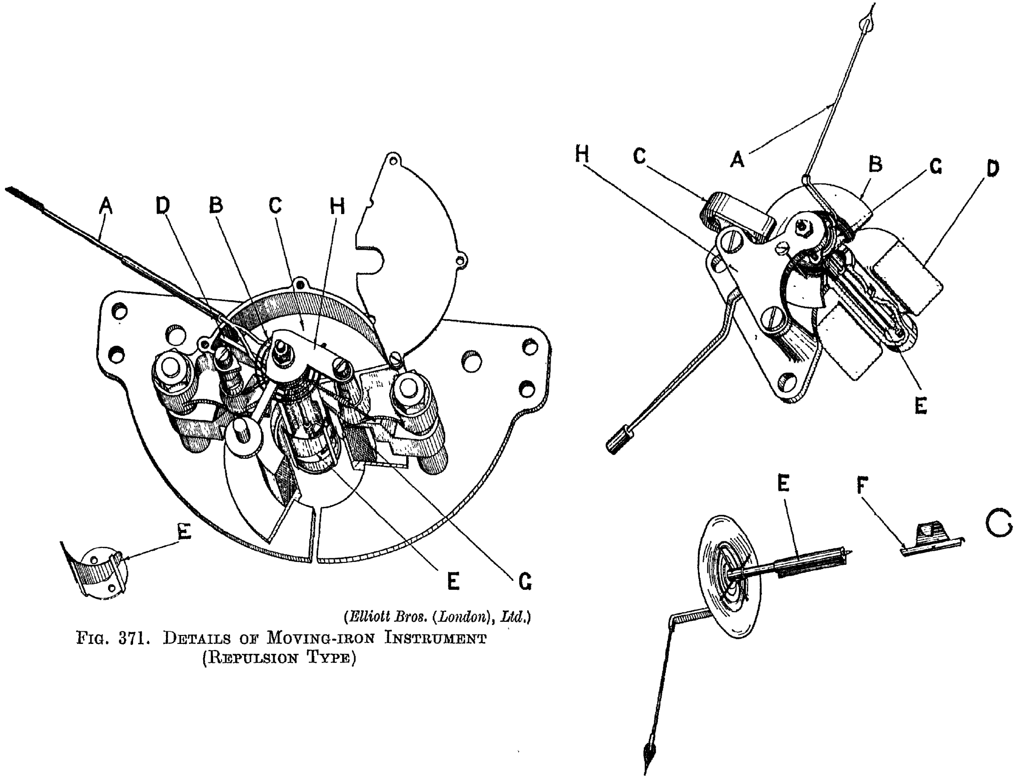
k' being a constant. A steady deflection is obtained when

$$T_C = T_D$$



(*Crompton Parkinson, Ltd.*)

FIG. 370. CIRCULAR SCALE MOVING-IRON INSTRUMENT SHOWING (CENTRE) ARRANGEMENT OF SHIELDING IRONS



(*Elliott Bros. (London), Ltd.*)

FIG. 371. DETAILS OF MOVING-IRON INSTRUMENT (REPULSION TYPE)

or when $$k'\theta = kI^2 \sin 2(\theta + \phi)$$

Hence $$I = \sqrt{\frac{k'\theta}{k \sin 2(\theta + \phi)}} = K\sqrt{\frac{\theta}{\sin 2(\theta + \phi)}} \qquad (350)$$

where K is another constant.

If gravity control is used

$$T_C = k'' \sin \theta$$

Thus, for a steady deflection θ,

$$k'' \sin \theta = kI^2 \sin 2(\theta + \phi)$$

or $$I = K' \sqrt{\frac{\sin \theta}{\sin 2(\theta + \phi)}} \quad . \quad . \qquad (351)$$

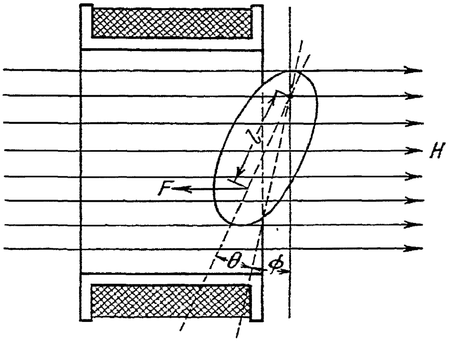


FIG. 372

The greatest range is obtained when the initial angle ϕ is zero. The deflecting torque per unit of current is obviously maximum when $\sin 2(\theta + \phi) = 1$, i.e. when $\theta + \phi = 45°$. The deflecting torque is zero when $(\theta + \phi)$ is 90°.

THEORY OF REPULSION TYPE MOVING-IRON INSTRUMENTS. Let the two irons—fixed and moving—of a repulsion instrument be straight round rods of equal length and diameter, and assume that the distance between is at all times small compared with their lengths. Suppose that the pole strengths of these rods, when the current in the coil is I, and the corresponding field strength inside the coil H, are m_1 and m_2 respectively. Referring to Fig. 373, l is the length and D the distance apart of these magnets, the angle of deflection being θ. Let the initial angular displacement of the rods be ϕ.

Then, if l is great compared with D, the forces of attraction between either of the pairs of diagonally opposite poles can be neglected in comparison with the forces of repulsion between the adjacent poles.

The total force of repulsion between the rods is thus $\frac{2m_1m_2}{D^2}$

Now $$D = 2R \,.\, \sin\frac{\theta + \phi}{2}$$

where R is the distance of the rods from the axis of rotation of the moving one. The deflecting torque is, therefore, given by

$$T_D = \frac{2m_1m_2}{D^2} \,.\, R \cos\frac{\theta + \phi}{2}$$

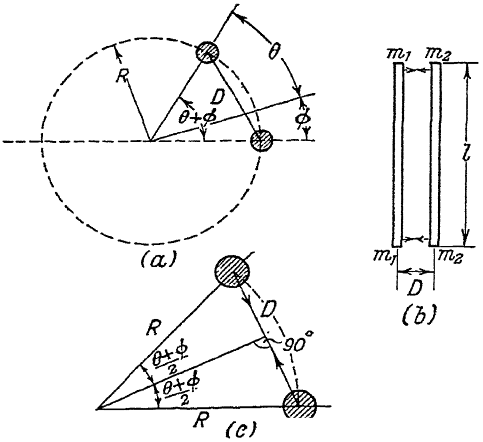


FIG. 373

or $$T_D = \frac{m_1m_2 \cos\frac{\theta + \phi}{2}}{2R \sin^2\frac{\theta + \phi}{2}} \quad . \quad . \quad . \quad . \quad . \quad (352)$$

Since both m_1 and m_2 are proportional to the current I,

$$T_D = \frac{KI^2 \cos\frac{\theta + \phi}{2}}{\sin^2\frac{\theta + \phi}{2}} \quad . \quad . \quad . \quad . \quad . \quad (353)$$

where K is a constant.

If gravity control is used, the controlling torque T_C is proportional to sin θ or $T_C = k \sin\theta$ where k is a constant. If spring control is used, $T_C = k'\theta$ where k' is another constant.

Therefore, if θ is the steady deflection for a current I, we have, for the gravity controlled instrument,

$$\frac{KI^2 \cos \frac{\theta + \phi}{2}}{\sin^2 \frac{\theta + \phi}{2}} = k \sin \theta$$

or

$$I = K' \sin \frac{\theta + \phi}{2} \sqrt{\frac{\sin \theta}{\cos \frac{\theta + \phi}{2}}} \qquad . \quad . \quad (354)$$

where K' is another constant.

For the spring-controlled instrument,

$$\frac{KI^2 \cos \frac{\theta + \phi}{2}}{\sin^2 \frac{\theta + \phi}{2}} = k'\theta$$

or

$$I = K'' \sin \frac{\theta + \phi}{2} \sqrt{\frac{\theta}{\cos \frac{\theta + \phi}{2}}}. \qquad . \quad . \quad (355)$$

The initial angular displacement ϕ is of the order of 20°.

Relationship between Torque and Inductance. If the *attraction* type of moving-iron instrument is considered, it can easily be seen that, since the inductance of the coil of the instrument is proportional to the flux linking with it, per ampere flowing in it, this inductance will change as the position of the moving iron changes. There is thus a relationship between torque and inductance and this is developed below.

Suppose the moving iron to be a rectangular plate of thickness t, breadth b, and length d, as shown in Fig. 374. Suppose also that the flux passes through the iron in straight lines (i.e. that the shape of the field of the coil is not altered by the presence of the iron) and that the permeability of the iron is constant.

Then, for any deflection $(\theta + \phi)$ from the vertical position, we have for the length of magnetic path in the iron $\frac{b}{\cos(\theta + \phi)}$ as can be seen by considering the elemental strip shaded in the figure. The cross-sectional area of iron, presented to the flux when the iron is in this position, is $td \cos (\theta + \phi)$. Hence, the reluctance of the path in the iron is $\dfrac{\frac{b}{\cos(\theta + \phi)}}{td \cos(\theta + \phi)\mu}$ where μ is the permeability.

The iron and air paths of the flux are in parallel and thus the total reluctance S of the path of the flux is given by

$$\frac{1}{S} = \frac{1}{S_a} + \frac{1}{\dfrac{b}{td\,\mu \cos^2(\theta + \phi)}}$$

where S_a is the reluctance of the air path. The total "permeance" $\left(\text{i.e. } \frac{1}{\text{Reluctance}}\right)$ is therefore

$$P = P_a + \frac{dt\,\mu \cos^2 (\theta + \phi)}{b}$$ where P_a is the permeance of the air path and is constant.

The total flux threading the coil, for a current I is given by

$$\phi = \frac{4\pi}{10} . NI . P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (356)$$

where N = No. of turns on the coil.

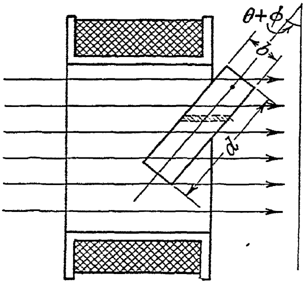


FIG. 374

The inductance L of the coil is given by $L = \frac{\phi . N}{10^8 I}$ where L is in henries and I is amperes. Then,

$$L = \frac{4\pi . NIP . N}{10^9\ I} = \frac{4\pi N^2}{10^9} . P$$

or
$$L = A\,[P_a + a \cos^2 (\theta + \phi)] \quad . \quad . \quad . \quad . \quad . \quad (357)$$

where A and a are constants. This may be written

$$L = A\left[P_a + a\left\{\frac{\cos 2(\theta + \phi) + 1}{2}\right\}\right]$$

or
$$L = A\left[P_a + \frac{a}{2} + \frac{a}{2}\cos 2(\theta + \phi)\right] \quad . \quad . \quad . \quad (358)$$

Now, the deflecting torque T_D is given by

$$T_D = kI^2 \sin 2(\theta + \phi)$$

Differentiating L with respect to $(\theta + \phi)$ we have

$$\frac{dL}{d(\theta + \phi)} = A' \sin 2\,(\theta + \phi)$$

where A' is constant and equals $-Aa$.

Substituting for $\sin 2(\theta + \phi)$ in the equation for T_D we have

$$T_D = \frac{kI^2}{A'} . \frac{dL}{d(\theta + \phi)}$$

or
$$T_D = GI^2 . \frac{dL}{d(\theta + \phi)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (359)$$

where $G = \frac{k}{A'}$.

Otherwise, the energy in the magnetic field of a coil of inductance L when a current I flows in the coil is $\frac{1}{2}LI^2$. If the moving iron rotates through a very small angle $d(\theta + \phi)$, thus changing the inductance of the coil by an amount dL, the change in the energy of the magnetic field is $\frac{1}{2}dL \,.\, I^2$. Also, if T_D is the deflecting torque, the work done during the movement of the iron is $T_D \,.\, d(\theta + \phi)$.

Hence $$T_D \,.\, d(\theta + \phi) = \tfrac{1}{2}dL \,.\, I^2$$

or $$T_D = \tfrac{1}{2}\, I^2 \,.\, \frac{dL}{d(\theta + \phi)} \quad . \quad . \quad . \quad (360)$$

If I and L are in electromagnetic C.G.S. units, T_D is in dyne-centimetres. Thus, when these units are employed, G, in the above expression for T_D, is equal to $\frac{1}{2}$.

Expressing T_D in gramme-centimetres, I in amperes, and L in henries, we have

$$T_D = \frac{1}{2 \times 981} \times \frac{I^2}{100} \times 10^9 \,.\, \frac{dL}{d(\theta + \phi)}$$

$$= 5{,}100\, I^2 \frac{dL}{d(\theta + \phi)}$$

Errors in Moving-iron Instruments. The causes of errors in these instruments may be divided into those which occur with either direct or alternating current and those which occur only with alternating current.

(*a*) WITH BOTH D.C. AND A.C. (i) *Hysteresis Error.* This is a serious source of error in moving-iron instruments. Owing to hysteresis in the iron of the operating system the readings are higher when descending values of current or voltage are measured than when ascending values are observed.

The error is reduced by making the iron parts short so that they demagnetize themselves or by choosing such (low) values of flux density in the iron that the hysteresis effect in the iron is small. A low flux density is, however, detrimental when the instrument is used with alternating current owing to the flattening of the B-H curve at the bottom end. A compromise is usually made.

Another effect of hysteresis is to cause an error due to the change in the position of the poles in the moving iron as its position changes. This "position" error is usually small.

(ii) *Stray Magnetic Fields.* Errors due to this cause may be serious, if not guarded against, owing to the weakness of the operating magnetic field. The error produced depends upon the direction of the stray field relative to that of the field of the instrument.

Such errors are minimized by magnetic screening of the working part of the instrument by an iron case or a thin iron shield.

(*b*) WITH A.C. ONLY. (i) *Frequency Errors.* Changes of frequency may produce errors due to changes of reactance of the working coil and also to changes of the magnitude of eddy currents set up in metal parts of the instrument near to the working portion. The magnitudes of these eddy currents and, consequently, their effect upon the magnetic field of the instrument, vary with frequency.

The change of impedance of the coil of the instrument, of resistance R (together with its series resistance r), due to change of frequency, is only of importance in the case of voltmeters. If L is the inductance of this coil circuit, the current for an applied voltage E will be given by

$$I = \frac{E}{\sqrt{(R+r)^2 + \omega^2 L^2}} \text{ where } \omega = 2\pi \times \text{frequency.}$$

Thus, if the frequency changes, the current for a given applied voltage changes, and hence an error in deflection is produced. In order that the error shall be small, L must be made small compared with $R + r$.

The time-constant for such voltmeters is usually of the order of 0·0005. There are several methods of compensating for frequency error, one of which consists in connecting a capacitor in parallel with r. It can be shown that the impedance of the whole circuit (including the working coil) is independent of frequency if $C = \frac{L}{r^2}$ where C is the the capacitance of the capacitor.

Total impedance of the instrument circuit when r is shunted by a capacitor

C is $$R + j\omega L + \frac{r}{1 + j\omega Cr} = R + j\omega L + \frac{r}{1 + \omega^2 C^2 r^2} - \frac{j\omega C r^2}{1 + \omega^2 C^2 r^2}$$

For independence of frequency this should equal $R + r$ and, equating real and imaginary terms, we have the conditions

$$r = \frac{r}{1 + \omega^2 C^2 r^2} \text{ and } C = \frac{L}{r^2}$$

The first condition is only fulfilled if $\omega^2 C^2 r^2$ is small compared with unity.

Although the above theory had long been generally accepted, it was pointed out* that while the circuit is thus made independent of frequency the resistance of the voltmeter is reduced from $R + r$ to $R + \frac{r}{1 + \omega^2 C^2 r^2}$ and that $\omega^2 C^2 r^2$ is not usually sufficiently small for this reduction to be negligible.

Both graphical and calculation methods of obtaining the correct value of C to avoid this are given in the article.

TABLE XIX

Quantity	Voltmeters	Ammeters
Weight of moving system	4 g.	3 g.
Torque at full deflection	0·2 g-cm.	0·2 g-cm.
$\frac{\text{Torque}}{\text{Weight}}$ ratio	0·05	0·067
Resistance	20 ohms per volt of range	0·01 ohm
Inductance	1 henry	—
Watts lost	8	4
Coil surface per watt	16 sq. cm.	
Ampere-turns at full deflection	300	300

DESIGN DATA FOR MOVING-IRON INSTRUMENTS. Table XIX, which is compiled from data given by various writers on measuring instruments and from figures supplied by instrument manufacturers, gives average values of the various quantities involved in the design of moving-iron instruments (see also Ref. (1)).

✓**Moving-coil Instruments.** There are two types of moving-coil instruments, namely, the permanent-magnet type, which can only be used for direct current measurements, and the dynamometer type, which can be used on either direct or alternating current circuits.

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* See *The Wireless Engineer*, October, 1940, p. 429.

1. PERMANENT-MAGNET TYPE. The principle of this type of instrument is the same as that of the moving coil, or D'Arsonval galvanometer. A light rectangular coil is pivoted so that its sides lie in the air gaps between the two poles of a permanent magnet and a soft-iron cylinder. When current passes through the coil a deflecting torque is produced owing to the reaction between the permanent-magnet field and the magnetic field of the coil. This is illustrated in Fig. 375. The air gap between the magnet poles and iron core is small (about 0·05 in.), and the flux density is uniform and is in a radial direction. If a current I flows in the moving coil in the direction shown, forces F, F, will act on the two sides of the coil which are in the field. The torque causing the coil to rotate is thus 2Fr, where r is the mean distance of the wires forming the sides of the coil, from the axis of rotation.

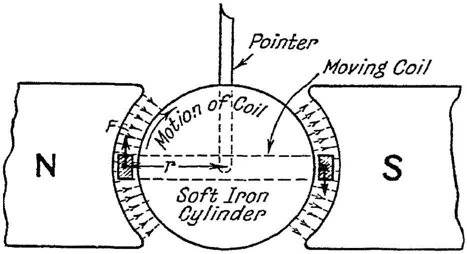


FIG. 375. ACTION OF PERMANENT-MAGNET MOVING-COIL INSTRUMENT

If there are N turns on the coil and the field strength in the air gap is H, the force F, in dynes, for a given current I (amperes) in the coil, is given by

$$\mathrm{F} = \frac{NHIl}{10}$$

where l is the active length in centimetres of the sides of the coil in the air gap.

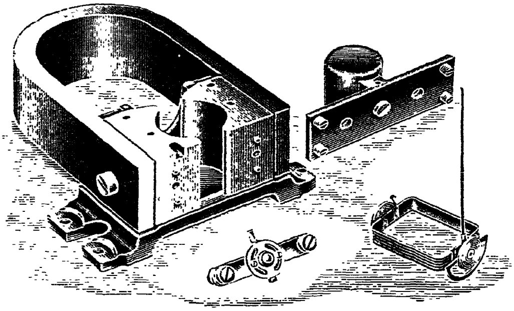
Since H is uniform, the deflecting torque is constant for all positions of the moving coil, provided its sides are within the pole arcs of the magnet.

The full expression for the torque in gramme-centimetres is

$$T = 2r \times \frac{NHIl}{10 \times 981} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (361)$$

$$= \frac{\text{Ampere-turns on coil}}{10 \times 981} \times \text{Area of coil} \times H$$

Constructional Details. Moving-coil instruments are spring-controlled. Two phosphor-bronze hair springs are used, these also serving as leads to the moving coil. Fig. 376 shows the construction of a moving-coil instrument. A bracket of non-magnetic material carries the iron cylinder which fits inside the coil. Since the deflecting torque is directly proportional to the current and the controlling torque of the springs is proportional to the deflection θ, we have



(*Nalder Bros. and Thompson, Ltd.*)

FIG. 376. PARTS OF MOVING-COIL INSTRUMENT

$$T_D \propto I$$

and $$T_C \propto \theta$$

But $$T_D = T_C$$

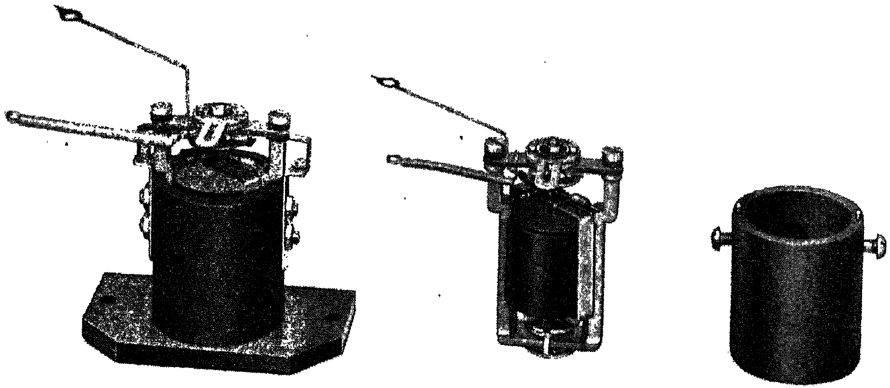
when the moving system is at rest in its deflected position,

$$\therefore \theta \propto I$$

Thus the scale is uniformly divided. Damping is by eddy currents induced in the aluminium former upon which the moving coil is wound.

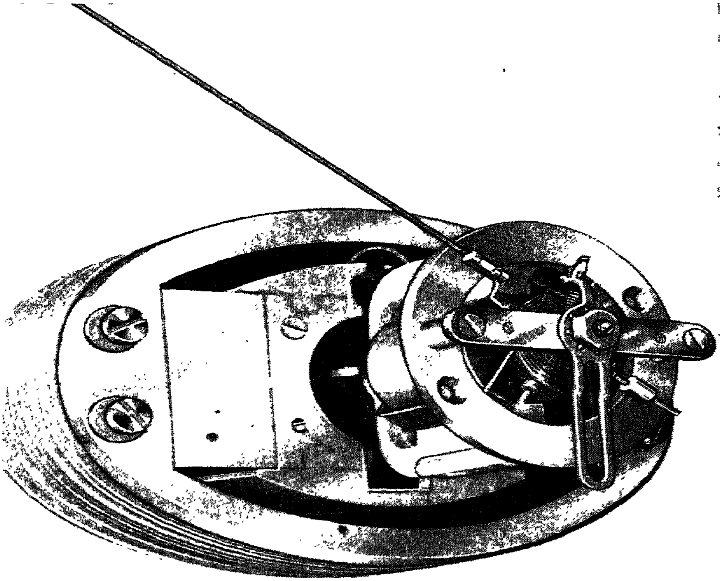
Types of Permanent-magnet Moving-coil Instruments. The form of instrument already described is perhaps the commonest. Its development for commercial use has been largely the work of Dr. Weston.

The other types are the same in principle, but different in the form of the magnet and in the relative positions of the magnet and moving system. The Nalder-Lipman patent moving-coil movement shown in Fig. 377A is unusual in that there is, inside the coil, a cylindrical magnet of nickel-aluminium-cobalt-steel alloy with very high coercive force. An outer ring of soft iron provides the return path for the magnetic flux and the coil moves in the uniform field between



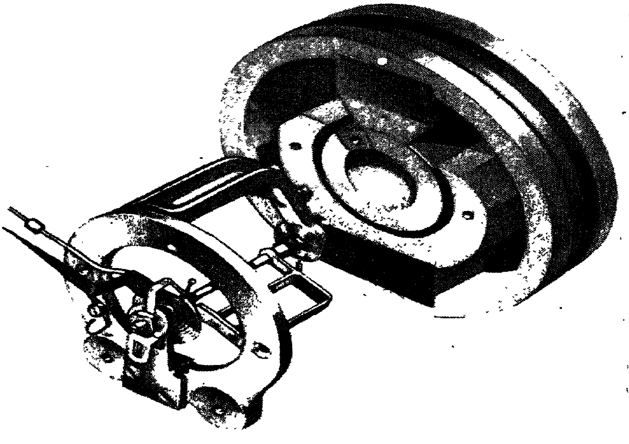
(*Nalder Bros. & Thompson, Ltd.*)

FIG. 377A. CONSTRUCTION OF NALDER-LIPMAN PATENT MOVING-COIL INSTRUMENT



(*Crompton Parkinson, Ltd.*)

FIG. 377B. MAGNET SYSTEM OF SELF-SHIELDING PATTERN MOVING-COIL INSTRUMENT WITH MOVEMENT EXTRACTED



(*Crompton Parkinson, Ltd.*)

FIG. 377C. MOVEMENT OF CIRCULAR-SCALE COIL INSTRUMENT

the two. The instrument is entirely free from interference by stray magnetic fields and has a very low power consumption. Figs. 377B and 377C show the internal construction of two Crompton-Parkinson instruments, the first having a circular scale and the second being of the self-shielding pattern. Fig. 378 shows the original arrangement of the Record "Cirscale" instrument. One pole of the permanent magnet lies inside the two parallel halves of the other pole. The coil, which is mounted to one side of the spindle, fits round the inner pole as shown, its two horizontal sides lying in the two inter-polar gaps. A very long scale is obtained with this type of instrument, the total deflection obtainable being about 300°. Fig. 379 shows the working system of an improved form of instrument of this type.

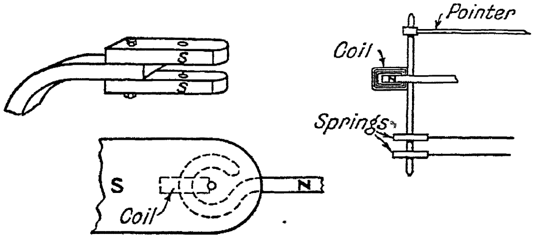
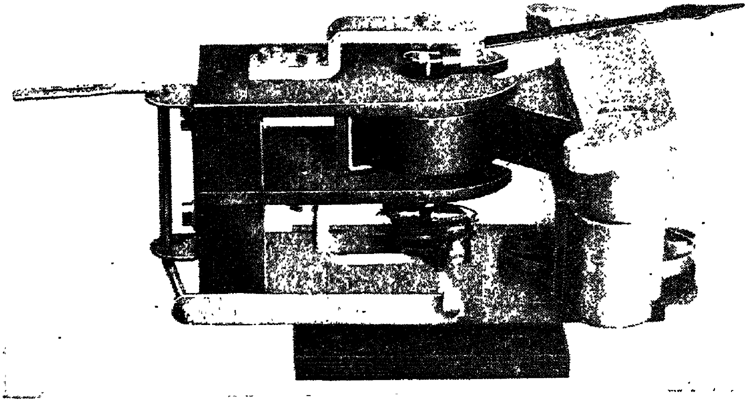


Fig. 378. Details of Record "Cirscale" Instrument

In the "Unipivot" instrument manufactured by the Cambridge Instrument Co. Ltd., the moving coil is circular instead of rectangular, and is carried by one pivot in the top of the iron core, which is spherical. The construction is shown in Fig. 380. The instrument is highly accurate, the friction torque being reduced by the elimination of one pivot. It is frequently used for current measurements when the current is small, but is not very robust, and needs rather careful handling. The current is led into the moving coil by the single control spring and passes out by a ligament at the bottom of the coil. Such instruments must, of course, be used only in a horizontal position.

Extension of Range of Moving-coil Instrument. Owing to the low current-carrying capacity of the moving coil, these instruments are very largely used in conjunction with shunts—when used as ammeters—and with a high series resistance when used as voltmeters.

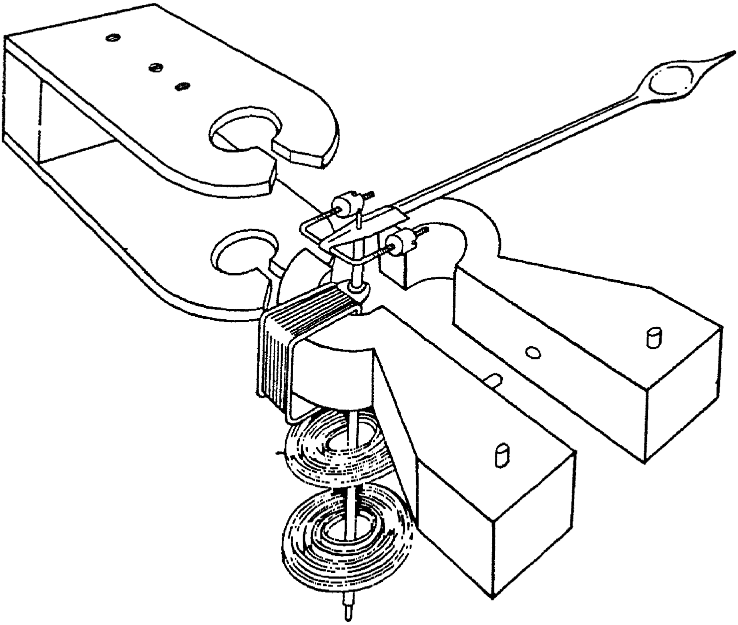
The extension of range of instruments will be discussed more fully in the next chapter. Ammeter shunts are simply low resistances which are connected in parallel with the instrument so that only a small fraction of the current to be measured actually passes through the latter. The shunt itself is usually of manganin strip or tube, and has, therefore, a very small temperature coefficient. The



(*Record Electrical Co., Ltd.*)

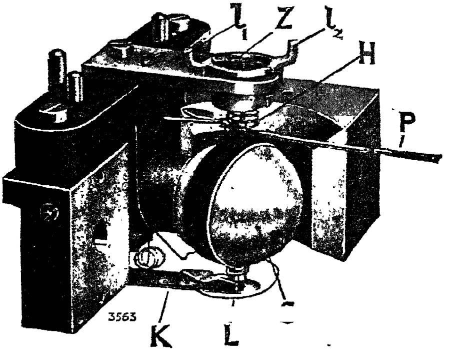
FIG. 379. RECORD "CIRSCALE" INSTRUMENT

(*above*) Movement complete
(*below*) Details of pole irons



instrument must also have a small temperature coefficient if the ratio $\frac{\text{Instrument current}}{\text{Total current}}$ is to remain constant.

B.S.I. Spec. 89 (1937) specifies a voltage drop, across the shunt and instrument in parallel, of 0·075 volt. An operating current of 0·015 amp. (15 milliamp.) has been largely adopted in these instruments. Thus, the total resistance of the instrument (including the leads which connect it to the potential terminals of the shunt) must be 5 ohms. Of this, only about 1 ohm should be in the coil and copper leads, the remaining 4 ohms being in the form of a "swamping" resistance of manganin. The advantage of this arrangement is shown in the following example.



(Cambridge Instrument Co., Ltd.)

FIG. 380. UNIPIVOT MOVING-COIL INSTRUMENT

Example. A moving-coil instrument whose resistance is 5 ohms and whose working current (for full-scale deflection) is 0·015 amp. is to be used, with a manganin shunt, to measure up to 100 amp. Calculate the error caused by a 10° C. rise in temperature—

(*a*) when the whole of the 5 ohms is in the copper of the instrument coil and leads,

(*b*) when a 4 ohm manganin swamping resistance is used with a coil and leads of 1 ohm resistance.

Instrument current = 0·015 amp.

∴ Shunt current = 99·985 amp.

Volt drop across the shunt = 0·075 volt

∴ Shunt resistance = $\frac{0\cdot075}{99\cdot985} = 0\cdot00075$

Take the temperature coefficient of copper as 0·0040 ohms/ohm/° C., and of manganin 0·00015 ohms/ohm/° C.

Then shunt resistance at 10° C. rise in temperature $= 0{\cdot}00075\,(1 + 10 \times 0{\cdot}00015)$

$= 0{\cdot}000751$

In case (*a*), instrument resistance after 10° C. rise in temperature $= 5\,(1 + 10 \times 0{\cdot}004)$

$= 5{\cdot}2$

Then, instrument current corresponding to 100 amp. in the main circuit $= \dfrac{0{\cdot}000751}{5{\cdot}200751} \times 100$

$= 0{\cdot}01444$ amp.

∴ Instrument reading $= 96{\cdot}3$ amp.

∴ Percentage error due to temperature rise $= 3{\cdot}7$ per cent

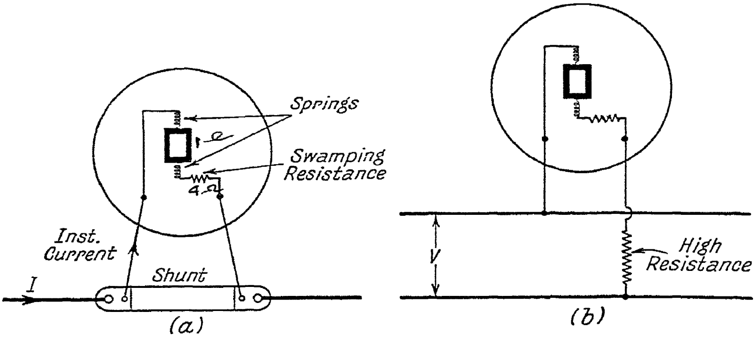


FIG. 381. CONNECTIONS OF MOVING-COIL INSTRUMENT

In case (*b*), after 10° C. rise in temperature, resistance of instrument circuit $= 1(1 + 10 \times 0{\cdot}004) + 4(1 + 10 \times 0{\cdot}00015)$

$= 5{\cdot}046$

∴ Instrument current corresponding to 100 amp. $= \dfrac{0{\cdot}000751}{5{\cdot}046751} \times 100$

$= 0{\cdot}001488$

∴ Instrument reading $= 99{\cdot}2$ amp.

∴ Percentage error due to temperature $= 0{\cdot}8$ per cent

Thus, the effect of using the manganin swamping resistance is to reduce the percentage error for 10° C. rise in temperature from 3·7 per cent to 0·8 per cent.

Fig. 381 shows the connections of a moving-coil instrument when used, (*a*) as an ammeter for large currents, (*b*) as a voltmeter. If the voltage drop across the shunt is 0·075 volt and the working current of the instrument 0·015 amp. the total power loss is $0{\cdot}075I$ when used as an ammeter (I being the current to be measured), and $0{\cdot}015V$ when used as a voltmeter, V being the voltage to be measured.

In general, the power loss in permanent-magnet moving-coil instruments is lower than that in other types of instruments.

Principal Advantages of Permanent-magnet Moving-coil Instruments. These are—

(*a*) Low power consumption.

(*c*) Uniformity of the scale and the possibility of a very long scale.

(*d*) The possibility of a single instrument being used, with shunt and resistances, to cover a large range of both current and voltage.

(*e*) Freedom from hysteresis errors and, very largely, from errors due to stray magnetic fields.

(*f*) Perfect damping, simply afforded by eddy currents induced in the metal former of the moving coil.

Errors. Friction and heating errors are, of course, present as in other types of instrument. The weakening of the permanent magnet with the passage of time may introduce a considerable error unless the magnet is carefully aged during manufacture. Thermo-electric e.m.f.'s may introduce errors when these instruments are used shunted, for current measurements, but with a well-designed shunt such error should be small.

Design Data. The following table gives approximate values of various quantities in connection with the design of permanent-magnet moving-coil instruments.

TABLE XX

Quantity	Voltmeters	Ammeters
Weight of moving system	3·5 g.	3·5 g.
Torque at full deflection	0·5 g-cm.	0·5 g-cm.
$\frac{\text{Torque}}{\text{Weight}}$ ratio	$\frac{1}{7}$th	$\frac{1}{7}$th
Resistance of coil and springs	50 ohms	1 ohm
Flux density in gap	1,200 lines/sq. cm.	1,200 lines/sq. cm.
Number of turns on coil	50	20
Ampere-turns at full deflection	1	1

2. Dynamometer Type Moving-coil Instruments. In dynamometer instruments the permanent magnet is replaced by either one or two fixed coils which carry the current to be measured (or a current proportional to the voltage to be measured), and which are connected either in series or in parallel with the moving-coil. The coils are usually air-cored, the use of iron being usually avoided in such instruments owing to its introduction of hysteresis, eddy

current, and other errors when the instrument is used for a.c. measurements. The general arrangement is shown in Fig. 382, in which the connections of a dynamometer ammeter (*b*) and of a dynamometer voltmeter (*c*) are also shown.

The torque of the instrument is dependent upon the strengths of the magnetic fields of both fixed and moving coils—i.e. in an ammeter the torque is roughly proportional to the current squared, and in a voltmeter to the voltage squared. Dynamometer instruments can thus be used on alternating current circuits, for which a square law is essential.

Two hair-springs are used for the control and as leads to the moving coil. Damping is often by air piston or enclosed vane, although in some cases eddy current damping, by an aluminium disc rotating in a permanent-magnet field, is used. In ammeters of this type the current which can be measured without the use of a shunt, is small owing to the difficulty of leading-in the heavy currents to the moving system, and also on account of the heavy moving coil which would be necessitated if this were to have a high current-carrying capacity. Thus the moving coil is usually connected, in series with its swamping resistance across a shunt together with the fixed coils, as shown in Fig. 382 (*b*). By this means the time constants of the two parallel branches can be more easily made equal—an essential requirement if the division of current between the two paths is to be independent of frequency when the instrument is used for alternating current measurements.

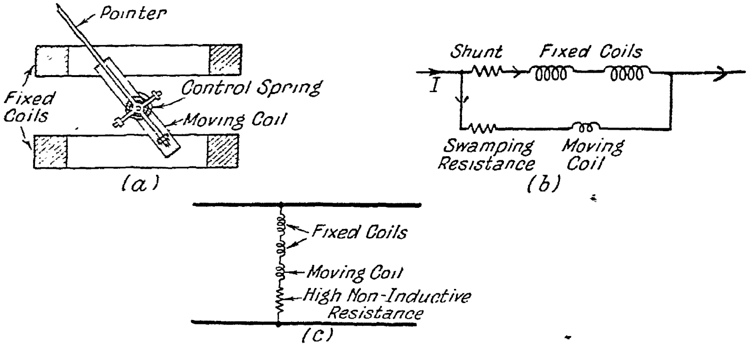


FIG. 382. DYNAMOMETER MOVING-COIL INSTRUMENT

Disadvantages for Direct Current Use. Although useful for precise measurements on alternating current circuits, these instruments compare unfavourably with the permanent-magnet type owing to the following disadvantages—

(*a*) The magnetic field strength obtainable is small, owing to the absence of iron. This means that a comparatively large number of ampere-turns must be used on the moving coil in order to obtain the necessary deflecting torque. A heavy moving system and high power loss are the results. The $\frac{\text{torque}}{\text{weight}}$ ratio is small, and hence friction errors tend to be serious as well as those due to internal heating.

(*b*) Owing to the deflecting torque varying with current according to an approximate square law, the scale is not uniform.

(*c*) Such instruments are more expensive than the types of ammeters and voltmeters already described.

For these reasons, ay and voltmeters are not in common use, especially

important application of the dynamometer principle is the dynamometer wattmeter.

For full descriptions of the various types of dynamometer instruments and for the theory of the instruments, the reader is referred to the works mentioned in Refs. (1) and (2).

A. H. M. Arnold (Ref. (31)) has discussed the performance limits of dynamometer instruments at audio frequencies. He gives curves showing the frequency errors of dynamometer ammeters both with air cores and with nickel-iron cores and shows the improvement which can be obtained through compensation by connecting a capacitance and resistance in series across the instrument terminals. Arnold shows that dynamometer voltmeters with nickel-iron cores are superior to those with air cores because of their greater inductance and that the effective frequency range can be extended by connecting a capacitor in parallel with the series resistor. K. A. Macfadyen and N. D. Hill (Ref. (32)) have developed a negative feed-back amplifier circuit for use with a dynamometer instrument in measurements at 50 c/s with distorted wave-forms. This dispenses with the need for a series resistor for the instrument.

✓**Hot-wire Instruments.** In these instruments the current to be measured, or a definite fraction of it, is passed through a fine wire, which expands due to the heating effect of the currents. If the resistance and coefficient of expansion of the wire are constant, the heating and expansion are both directly proportional to the square of the current. If the expansion is large enough, it can be used to produce deflections of a pointer which are also proportional to the square of the current.

Since such instruments have a square law, they can be used for alternating-current as well as for direct-current measurements. Moreover, since the deflections depend upon the heating effect of the alternating current, hot-wire instruments measure the r.m.s. values of the current, regardless of wave-form or frequency; this is their greatest advantage. The calibration is the same for alternating-current measurements as for direct-current measurements. Another advantage is that they are not affected by stray magnetic fields, since no magnetic effects are employed to produce their operation.

MAGNIFICATION OF THE EXPANSION. In commercial instruments, the hot wire is too short for the expansion to be used directly for the production of a deflection. A magnifying device is used, the theory of which is given below.

Referring to Fig. 383 (*a*), let AB be the hot wire, of length L, and let its expansion after reaching a steady temperature (when the heat produced by the current per second equals the heat radiated in the same time) be dL. Then, if tension is applied at the centre of the wire, the sag S is given by

$$S^2 = \left(\frac{L + dL}{2}\right)^2 - \left(\frac{L}{2}\right)^2 = \frac{2L \,.\, dL + (dL)^2}{4}$$

or

$$S = \sqrt{\frac{L \,.\, dL}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (362)$$

if $(dL)^2$ is neglected as being small compared with $L \,.\, dL$.

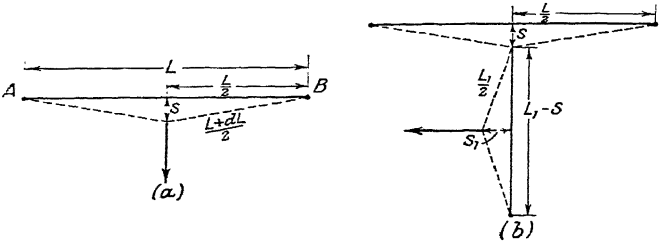


FIG. 383. SAG MAGNIFICATION IN HOT-WIRE INSTRUMENT

Since dL is proportional to the square of the current, the sag is thus proportional to the current, and the magnification of the expansion dL is

$$\frac{\sqrt{\frac{L \,.\, dL}{2}}}{dL} = \sqrt{\frac{L}{2dL}}$$

In Fig. 383 (*b*) further magnification is obtained by utilizing the sag of the downward wire. Let the length of this wire be L_1, and let it be pulled at its centre so as to take up the slack produced by the sag S of the hot wire. Let the "sag" in the downward wire be S_1. Then we have

$$S_1^2 = \left(\frac{L_1}{2}\right)^2 - \left(\frac{L_1 - S}{2}\right)^2 = \frac{2L_1S - S^2}{4}$$

or, neglecting S^2 compared with L_1S,

$$S_1 = \sqrt{\frac{L_1S}{2}}$$

Since

$$S = \sqrt{\frac{L \,.\, dL}{2}}$$

$$S_1 = \sqrt{\frac{L_1}{2}\sqrt{\frac{L \,.\, dL}{2}}} = \sqrt{\frac{L_1}{2\sqrt{2}}}\,\sqrt[4]{L \,.\, dL}$$

or

$$S_1 \propto \sqrt[4]{dL} \propto \sqrt{\text{current}} \quad . \quad . \quad . \quad . \quad . \quad (363)$$

The above theory assumes that there is no sag in either of the wires before current flows. As they are under tension from a spring, this is not so in practice, nor are the attachments to them exactly at their centres. The theory suffices, however, to show how the magnification is produced. The actual scale closely approaches to the square law scale, and is cramped at the bottom end.

CONSTRUCTION OF HOT-WIRE INSTRUMENTS (DOUBLE-SAG TYPE). The construction of the commonest form of double-sag hot-wire instrument is illustrated by Fig. 384. The hot wire is of platinum-iridium, so that it may withstand high temperatures without deterioration due to oxidation. The diameter of this wire is of the order of 0·1 mm. Attached to it is a phosphor-bronze wire W, and attached to this wire again is a fine silk thread T, which passes round a small pulley P (to which it is clamped) before being fastened to a spring which keeps the whole system taut. A light pointer and thin aluminium disc D are carried by the spindle upon which the pulley is mounted. The edge of the disc is situated in the air-gap of a permanent magnet, and provides damping. This damping is necessary, not so much to prevent oscillation when the instrument deflects originally (since the movement is somewhat sluggish), but to damp out rapid movements due to vibration or to sudden changes in the current, which would place excessive stresses upon the hot wire.

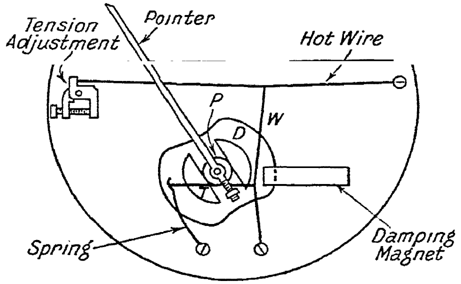


FIG. 384. CONSTRUCTION OF HOT-WIRE INSTRUMENT

When the hot wire expands, the slack in it, and in the wire W, is taken up by the spring, and the thread T causes the pulley to rotate and the pointer to deflect. The base of the instrument must be made up of materials which give a coefficient of expansion equal to that of the hot wire, so that changes in temperature external to the instrument shall not affect the deflections. Even when these coefficients of expansion are equal there is, however, a tendency for an error to be caused by the fact that the wire, being of small mass, changes in temperature quickly, while the temperature of the base takes some time to change.

The hot wire is made as thin as possible, so that it may attain a steady temperature quickly when current flows through it. It must at the same time have sufficient mechanical strength to withstand the stresses placed upon it by the tension of the spring and system.

RANGE. Instruments of this type are very limited as regards current-carrying capacity, owing to the fineness of the wire. They can be used as ammeters for current of the order of 1 amp. in the above form without a shunt. For currents up to about 5 amp. without a shunt, the device shown in Fig. 385 is used. This divides the hot wire electrically into four parallel paths, so that the current in any part of it is only one-quarter of the total instrument current. Above 5 amp. the instrument must be shunted for use as an ammeter.

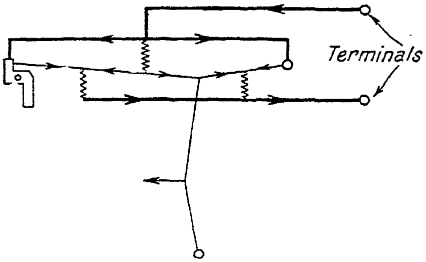


FIG. 385. ARRANGEMENT OF HOT-WIRE INSTRUMENT FOR 5 AMP. RANGE

When used as a voltmeter, a high non-inductive resistance is connected in series with the hot wire.

POWER CONSUMPTION. The power consumption in hot-wire instruments is high. The voltage drop across ammeters of this type is about 0·5 volt at full deflection, and the current in the wire for full deflection in the case of voltmeters is usually about 0·25 amp. Thus the power consumption depends directly on the range of the instrument. In the case of voltmeters, the wire cannot be reduced in diameter indefinitely on account of the mechanical strength required to withstand the stresses imposed upon it. A comparatively high voltmeter current must therefore be taken in order to heat the wire sufficiently to give the desired deflection.

DISADVANTAGES. Most of these have already been mentioned. Summarizing, they are—

(*a*) Sluggishness owing to the time taken for the wire to heat up.

(*b*) A cramped scale.

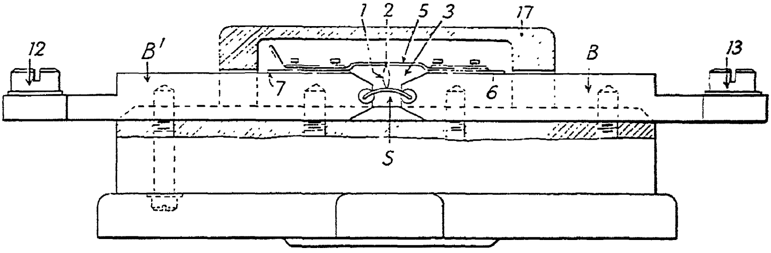
(*c*) Errors due to differences of temperature of the working wire and the base of the instrument, which cause shifting of the zero of the instrument. Such temperature variations and stretching of the wires require frequent zero adjustment.

(*d*) Inability to withstand overloads. The hot wire is so fine that it quickly fuses if the current exceeds the working value by any appreciable amount. Fuses do not give adequate protection, since the wire may melt before the fuse.

(*e*) Fragility.

(*f*) High power consumption.

The construction of the heating element of the Weston Thermo-ammeter for a range of 50 amp. is shown in Fig. 386. Between the terminals *BB′* a platinum alloy resistor strip *S* is hard soldered. The hot junction of a thermo-couple is hard soldered, or welded, to the centre of this strip. The cold ends of this thermo-couple are soldered to the centre points of two copper compensating strips, which are in thermal contact with the terminals *BB′*, but are insulated from them by thin mica strips. The compensating strips are proportioned so that they are thermally equivalent to the strip *S*. The thermo-couple has a very small heat capacity, so that the response to variations of current in *S* may be rapid. A surrounding case (17) shields the heater from external air currents.



(*Sangamo Weston, Ltd.*)

FIG. 386. HEATING ELEMENT FOR THERMO-AMMETER (ELEVATION)

This case, together with the compensating strips, ensures that the indications of the instrument are independent of external temperature variations and of air currents. The indicating instrument used in conjunction with the heating element is connected to two terminals brought out from the thermo-couple.

The full-load volt drop across the heating element is only about 150 millivolts, and the safe overload capacity is about 50 per cent. The instrument is free from zero shift and is especially useful for the measurement of very high-frequency currents and for currents whose wave-form is very far from sinusoidal.

Electrostatic Instruments. Such instruments are essentially voltmeters, and although they may be applied to the measurement of current and power, such applications involve the measurement of the voltage-drop across a known impedance.

Their advantages are that they give equally correct measurements on a.c. and d.c. circuits, and that since no iron is present in their working system, they are free from all errors in connection with magnetic fields in iron, such as those due to hysteresis and eddy currents. Wave-form, and frequency variations, also, are unimportant, and the power loss in such instruments is also extremely small.

They have the disadvantages, however, that the operating forces are very small, especially for low voltages (of the order of several hundred volts), their most useful range being from about 500 volts upwards to several hundred kilovolts.

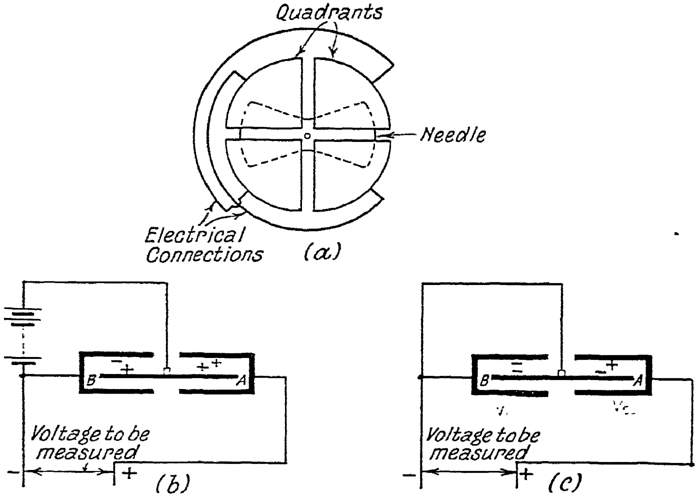


FIG. 387. QUADRANT ELECTROMETER

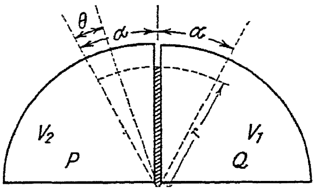
TYPES. There are two general types of electrostatic voltmeters, namely—

(a) The quadrant type.

(b) The attracted disc type.

Instruments of the former type are used for voltages up to 10 or 20 kilovolts, while the attracted disc type is general for voltages above this. Since these voltmeters are really modifications of Lord Kelvin's quadrant and attracted disc electrometers, the theory of these electrometers will now be discussed, the same theory holding for the voltmeters under consideration.

Quadrant Electrometer. The principle of this instrument is illustrated by the line diagrams in Fig. 387. There are four fixed metal double quadrants arranged so as to form a shallow circular box with short air gaps between the quadrants. Inside this (incompletely closed) box a thin metal "needle" is suspended by means of a thread of phosphor-bronze or silvered quartz. The needle is of a double-sector shape, and is suspended so as to be equidistant from the quadrant plates, which are above and below it. A plan view is shown in diagram (*a*) of Fig. 387. In diagrams (*b*) and (*c*), two methods of connection to the quadrants and needle are shown. In diagram (*b*) a high-tension battery is used to charge the needle to a potential considerably above that of the quadrants to which the negative of the voltage to be measured is connected. When so connected, the electrometer is said to be used "heterostatically."



Needle Potential = V

FIG. 388

When the needle is connected directly to one pair of quadrants, as in diagram (*c*), the electrometer is used "idiostatically."

The idiostatic connection is generally used in commercial instruments. With the polarities shown in diagram (*b*), end *A* of the needle is repelled by the fixed quadrant adjacent to it, while end *B* is attracted by its adjacent fixed quadrant, so that rotation of the needle is produced. In diagram (*c*), end *B* of the needle is repelled, and end *A* attracted, by the fixed quadrants near them.

The torque producing rotation will be shown (below) to be proportional to the square of the voltage to be measured in the case of the idiostatic connection. Hence, the instrument can be used to measure alternating voltages.

Actually, of course, the forces of attraction and repulsion are not purely rotational, but have components in the direction perpendicular to the needle—one upwards and one downwards at each end of the needle—but these components neutralize one another, leaving only the rotational components.

Theory. In considering the theory of the instrument it is simpler to consider one-half of the needle only, with the two (double) quadrants adjacent to it. Suppose the connection to be heterostatic.

Then, referring to Fig. 388, in which the half of the needle is taken simply as a sector of a circle whose radius is r, we have an arrangement which is essentially two capacitors side by side. Each of these capacitors is composed of portions of the upper and lower plates of one of the double quadrants, and a portion (both sides) of the needle. When the needle rotates, the capacitances will change, one becoming less and the other greater. Thus, in Fig.

388, suppose the needle rotates in a clockwise direction, then the capacitance of the right-hand capacitor will increase and that of the left-hand one will decrease.

Let the potential of the needle be V, that of quadrant P being V_2 and of quadrant Q V_1 where $V > V_2 > V_1$.

Suppose the capacitances of the right- and left-hand capacitors, when the needle has rotated through an angle θ from its zero position, to be C_1 and C_2 respectively.

Then, energy stored in right-hand capacitor $= \frac{1}{2}C_1(V - V_1)^2$

Energy stored in left-hand capacitor $= \frac{1}{2}C_2(V - V_2)^2$

Thus, total energy stored in any position θ $= \frac{1}{2}[C_1(V - V_1)^2 + C_2(V - V_2)^2$ $= W$

Let the torque, when the needle is in this position, be T_θ. Then, considering an infinitesimal advance $d\theta$ of the needle the work done on the moving system $= T_\theta \cdot d\theta$. Increase in stored energy $= dW$. But these two quantities must be equal. Therefore

$$T_\theta \cdot d\theta = dW$$

or

$$T_\theta = \frac{dW}{d\theta}$$

$$= \frac{d}{d\theta}[\tfrac{1}{2}C_1(V - V_1)^2 + \tfrac{1}{2}C_2(V - V_2)^2]$$

$$= \frac{(V - V_1)^2}{2}\frac{dC_1}{d\theta} + \frac{(V - V_2)^2}{2}\frac{dC_2}{d\theta} \quad . \quad . \quad . \quad (364)$$

Now, if d is the distance of the needle from either of the plates (upper or lower) of the quadrants, and if $2a$ is the angle of the needle sector, then, since the plates are in air,

$$C_1 = \frac{2[\frac{1}{2}r^2(a + \theta)]}{4\pi d} = \frac{r^2}{4\pi d}(a + \theta)$$

and

$$C_2 = \frac{2[\frac{1}{2}r^2(a - \theta)]}{4\pi d} = \frac{r^2}{4\pi d}(a - \theta)$$

$$\therefore \quad T_\theta = \frac{(V - V_1)^2}{2} \times \frac{r^2}{4\pi d} - \frac{(V - V_2)^2}{2} \times \frac{r^2}{4\pi d}$$

$$= \frac{r^2}{8\pi d}[(V - V_1)^2 - (V - V_2)^2] \quad ✓$$

or

$$T_\theta = \frac{r^2}{8\pi d}(V_2 - V_1)[2V - (V_1 + V_2)] \quad ✓ \quad . \quad (365)$$

This expression for the torque is positive only when $2V > V_1 + V_2$, and its magnitude, for given values of V_1 and V_2, obviously depends upon the value of V.

In the idiostatic connection the needle and quadrant Q are connected together, so that $V = V_1$ and the expression for the torque then becomes

$$T_\theta = -\frac{r^2}{8\pi d}(V_2 - V_1)^2$$

$$= -\frac{r^2}{8\pi d} \times v^2 \quad . \quad . \quad ✓ \quad . \quad . \quad (366)$$

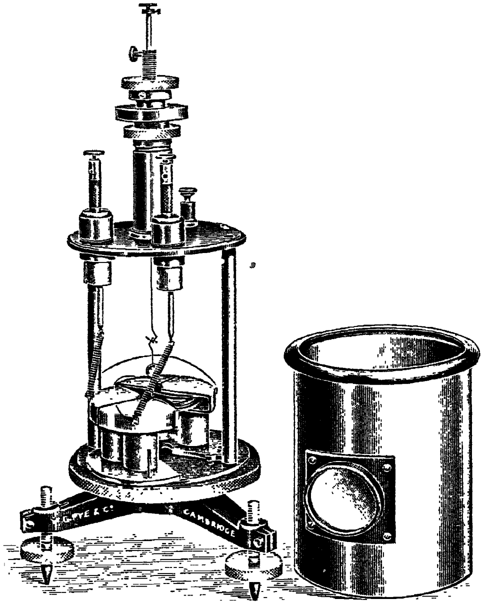
where v is the potential difference to be measured, and equals $V_2 - V_1$.

The torque is negative in this case, which means that the needle will rotate in an *anti-clockwise* direction as shown previously.

Considering now four quadrants and a needle of double-sector shape, the torque is given by

$$T_\theta = -\frac{r^2}{4\pi d} v^2 \qquad . \quad . \quad . \quad . \quad . \quad (367)$$

being double the above value. The torque is in dyne-centimetres if the potentials are expressed in electrostatic C.G.S. units, r and d being in centimetres. Converting to gramme-centimetres and volts, we have



(*W. G. Pye & Co., Ltd.*)

FIG. 389. DOLEZALEK ELECTROMETER

$$T_\theta = -\frac{r^2 E^2}{1{\cdot}11 \times 10^9 d} \text{ g-cm.}$$ ✓

where E is in volts (1 e.s.u. = 300 volts). Fig. 389 shows the construction of a quadrant electrometer of the Dolezalek pattern made by W. G. Pye & Co., Ltd. In the figure the fixed quadrants are shown opened out to disclose the needle.

Attracted Disc Electrometer. It was shown in Chapter I that if two parallel conducting plates are at a distance d apart and are charged so that a difference of potential V exists between them, then the force of attraction between them is given by

$$\mathrm{F} = \frac{A\kappa E^2}{8\pi}$$

where F is the force in dynes, A is the area of the plates in square centimetres, κ the permittivity of the medium, and E the field intensity (considered uniform) between them.

Now $$E = \frac{V}{d}$$

$$\therefore \qquad \mathrm{F} = \frac{A\kappa V^2}{8\pi d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (368)$$

Hence the potential difference between the plates in air is

$$V = d\sqrt{\frac{8\pi \mathrm{F}}{A}} \text{ e.s.u.} \quad . \quad . \quad . \quad . \quad . \quad (369)$$

Hence, $$V = 1{,}504d\sqrt{\frac{\mathrm{F}}{A}} \text{ volts}$$

This attraction between parallel plates is used as a measure of potential difference in the attracted-disc electrometer.

Kelvin Absolute Electrometer. This was one of the earliest instruments employing the attracted disc principle. The essential parts of the instrument are shown in Fig. 390.

The moving disc is carried by a spring, and is thus suspended from a micrometer head so that it is above the centre of a fixed disc. Surrounding the moving disc is a guard ring separated from the disc by a short air gap. The purpose of this guard ring is to render the field between the moving and fixed discs uniform by transforming the non-uniform fringing field from the edge of the moving disc to its own outer edge. It is electrically connected to the moving disc. The effective area of the moving disc is its actual area plus half the area of the air gap.

A fine cross-hair is carried by the moving disc, so that, by means of a sighting device consisting of lenses and two finely pointed rods, the zero setting of the disc may be accurately determined.

In use, the potential difference to be measured is applied between the two discs. The moving one is attracted downwards and is brought back to its zero position by turning the micrometer head, the movement required to return the disc to zero being observed.

The spring and micrometer head are calibrated by first short-circuiting the instrument, then setting the moving disc to its zero position, and adding known weights to the disc. The movements of the micrometer required to bring the disc back to zero are observed for different weights, a calibration being thus obtained. Thus, actually, the attractive force produced by a certain potential difference between the discs is measured by the instrument, and the potential difference itself is then determined in terms of the dimensions of the instrument and of this measured force.

The disadvantage of the electrometer, when used as above, is that when the potential difference to be measured is only a few hundred volts, the two discs must be very near together for any appreciable attractive force to be obtained. In such cases the measurement of their distance apart is difficult to carry out accurately. To overcome this difficulty, Lord Kelvin used an auxiliary high potential in conjunction with the instrument, the heterostatic connection, previously mentioned, being employed.

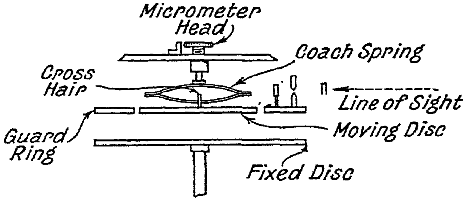


FIG. 390. KELVIN ABSOLUTE ELECTROMETER

In secondary instruments of the attracted-disc pattern, the guard ring is omitted and the voltage corresponding to a given deflection is obtained by calibration instead of by calculation from the dimensions of the instrument.

Commercial Forms of Electrostatic Voltmeters. Fig. 391 shows, diagrammatically, the construction of the Kelvin Multicellular Voltmeter. This instrument is essentially a quadrant electrometer with a large number of needles and of fixed quadrants, instead of the single needle and four fixed quadrants. It is suitable for a voltage range of about 100 to 1,000 volts, although instruments of this type having as low a range as 40 volts have been constructed.

The large number of quadrantal cells are necessary in order to obtain a sufficiently high working force with such low voltages.

By suspending the moving system, bearing friction is avoided. The coach-spring is fitted as a protection against fracture of the suspension due to vibration. A clamp is fitted to be used when transporting the instrument. The torsion head, which can be moved very slowly by a worm-wheel attachment, is for zero adjustment. The pointer and scale are of the "edgewise" pattern, and damping is by a vane dipping into an oil-dashpot.

There are several other features which have been omitted from the diagram for the sake of clearness. The suspended system has a safety collar just above the pointer to prevent such movements of the system as would cause the moving needles to touch the fixed quadrants and produce a short-circuit. Two vertical guard plates of tin-foil are fitted inside the case of the instrument. These plates are electrically connected to the moving system and to the case, which is metal.

Precision Forms of Kelvin Electrostatic Voltmeters. Work at the National Physical Laboratory during the past two or three decades has resulted in the development of this form of instrument as a transfer instrument, from direct to alternating current, of standard accuracy.

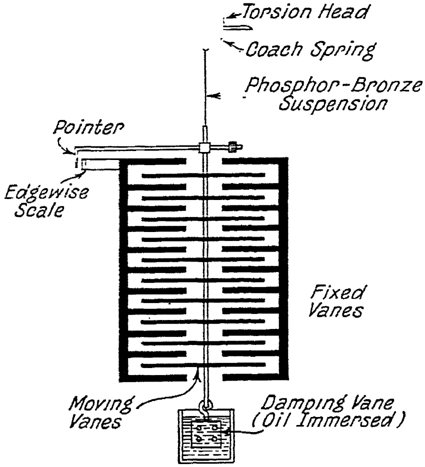


FIG. 391. KELVIN MULTICELLULAR VOLTMETER

There is no standard, for alternating voltages, comparable with the standard cell for direct voltages so that the approach to standardization in the former case must be through the use of a transfer instrument which reads the same on both a.c. and d.c. Although both thermal and dynamometer transfer instruments can be used, at N.P.L. the greatest reliance is placed on this electrostatic standard instrument developed by E. H. Rayner (Ref. (28)).

The instrument, which is a reflecting one with a scale about four metres long, is calibrated on d.c. by potentiometer and standard cell. Optimum sensitivity is around the 100-V point and the readability on the scale is there better than 1 part in 20,000. Applying the appropriate corrections, alternating voltages of about 100 V can be measured in terms of the absolute units to an accuracy of 1 part in 10,000 (Ref. (27)).

The use of electrostatic instruments for alternating power measurements is discussed in Refs. (29) and (30).

Kelvin Electrostatic Voltmeter for Voltages up to 10,000 *volts*. A simple form of voltmeter, which has the same principle as the one described above, but which has only one moving needle and is suitable for a voltage range of about 1,000 to 10,000 volts, is shown in Fig. 392.

The needle is of thin aluminium sheet, and is carried on a spindle which is supported by knife edges. This needle is connected to one terminal of the instrument, and is mounted in the space between two parallel, fixed, vanes, which are connected to the other terminal of the instrument. It carries a pointer as shown and is gravity-controlled. Mechanical damping, by a wire passing through the case of the instrument, is employed. The controlling torque may be increased by adding small weights to the bottom of the needle. This, of course, increases the range of the instrument.

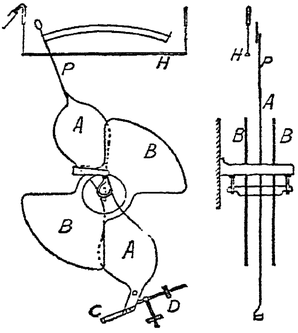


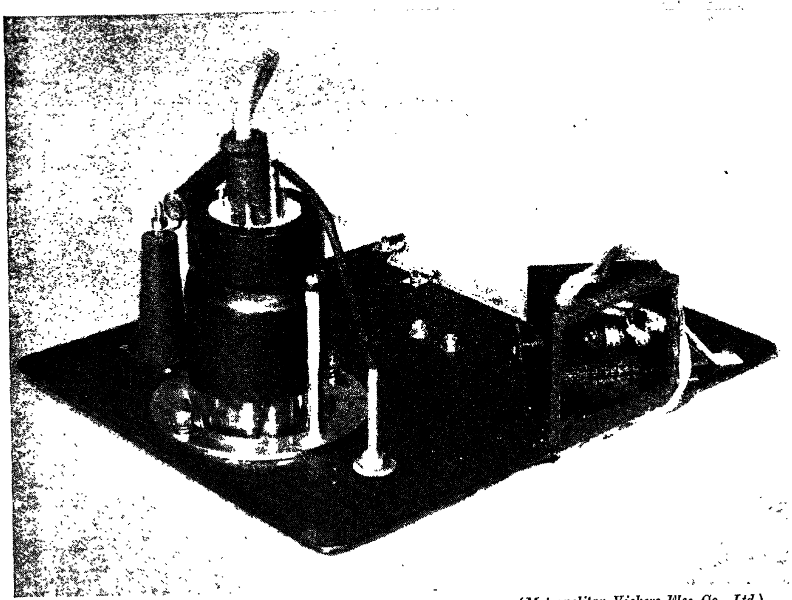
FIG. 392. PARTS OF ELECTROSTATIC VOLTMETER

(*From "Theory and Practice of Alternating Currents," Dover.*)

The instrument case has a glass front to which a strip of tin-foil is pasted, in order to lead away to earth any electrostatic charges which may accumulate on the glass, and so affect the reading of the instrument.

Vacuum-enclosed Electrostatic Voltmeters. Metropolitan-Vickers Electrical Co., Ltd. have recently developed instruments of this type for the measurement of high voltages. The interior construction of one for 12 kV is shown in Fig. 393. The active vanes are segments of cylinders, concentrically mounted on a framework. The fixed vane is at high voltage and the moving vane, mounted between jewel bearings, is earthed. The control springs are made from temper-hardening spring alloy and a concave mirror is carried by the moving system, a light pointer being used. For damping there are two permanent magnets operating on a radial portion of the moving vane.

The weight of the moving system is 1·25 grammes and the torque is 0·0125 gramme-centimetres per degree deflection. The full scale deflection of the movement is 30° (60° on the scale) and the torque/weight ratio is about 1/4. The movement is aperiodic with a response time from 1/5 to 1 second and the instrument is robust. Such instruments are calibrated to give an accuracy of 0·1 per cent or better.

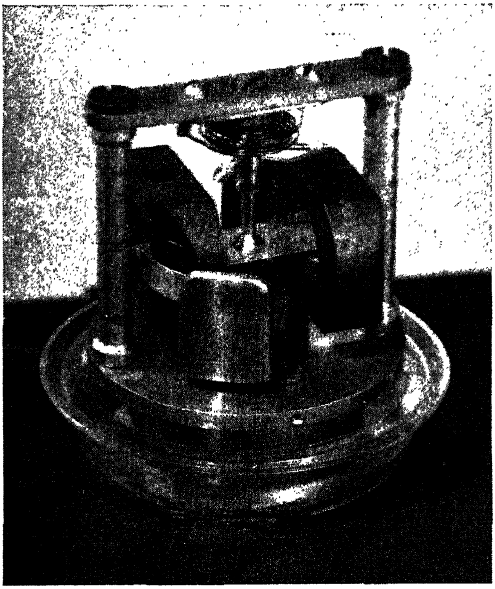


(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 393A. ELECTROSTATIC VOLTMETER

Other Forms. Fig. 394 shows the moving system of an Everett Edgcumbe single-vane voltmeter, three such movements being used in one case as a three-phase "leakage indicator" for 2,000 volts.

A leakage indicator is for the purpose of detecting a ground fault on a system. In the three-phase indicator one terminal of each movement is connected to earth and the remaining three terminals are connected to the three insulated lines of the system under observation. A difference between the readings of the three instruments contained in the indicator shows that a fault exists on one or more of the lines. The advantage of electrostatic instruments for this purpose is that they themselves do not take any appreciable current, and therefore do not constitute a "leak" when installed.



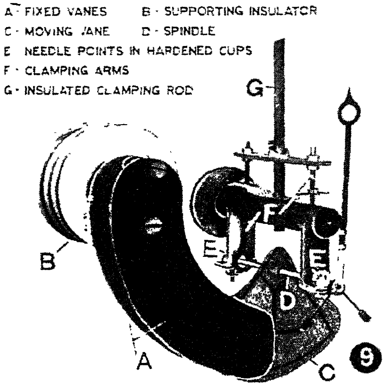
(Metropolitan-Vickers Elec. Co., Ltd.)

FIG. 393B. INTERNAL CONSTRUCTION OF THE ELECTROSTATIC VOLTMETER

The instrument shown consists of two fixed curved vanes about 1 in. wide and mounted parallel to one another about $\frac{1}{2}$ in. apart. A pivoted vane is pulled inside them when the potential difference is applied. The pivots are attached to the spindle at right angles to its axis, and rest in polished steel cups. Gravity control is employed.

Voltmeters for Extra-high Tension. Voltmeters for the measurement of voltages in excess of about 10,000 volts are almost always of the attracted disc type. Several forms of such instruments have already been described in Chapter XI. The Abraham voltmeter, which is manufactured by Messrs. Everett Edgcumbe & Co., for voltages up to 500,000 volts, is perhaps the most generally used form.

Induction Instruments. These instruments can only be used on alternating currents circuits. Their chief advantage is that a full-scale deflection of some 300° can be obtained, giving a long and open scale. The effect of stray magnetic fields upon their readings is small and the damping is good.



(*Everett Edgcumbe & Co., Ltd.*)

FIG. 394. MOVING SYSTEM OF SINGLE-VANE VOLTMETER

They have, however, several serious disadvantages which, for most purposes, outweigh their advantages. The large deflection means a greatly increased stress in the control spring, since this stress is proportional to the deflection. A serious error may be introduced by a variation of the supply frequency, unless a compensating device is employed. Temperature variations also, may produce considerable errors unless, again, compensation is employed. Other drawbacks are the fairly high power consumption and high cost of such instruments.

PRINCIPLE. All induction instruments depend, for their action, upon the torque produced by the reaction between a flux, whose magnitude depends upon the value of current or voltage to be measured, and eddy currents which are induced in a metal disc, or drum, by another flux, whose value again is dependent upon the current or voltage to be measured. Since the magnitude of the eddy current is proportional to that of the flux inducing it, the torque at any instant is proportional to the square of the current or voltage to be measured, and the mean torque is proportional to the mean-square value of this current or voltage.

Consider a flux $\phi = \phi_{max} \sin \theta$ producing a torque, by the force which it exerts upon an eddy current lagging in phase by an angle α behind this flux, i.e. whose law of variation is $i = {}_{max}I \sin (\theta - \alpha)$.

Then, since the instantaneous torque is proportional to the product of the instantaneous current and instantaneous flux, we have

$$\text{Instantaneous torque } T_{inst} \propto \phi i$$

The mean torque T_M is therefore proportional to $\frac{1}{\pi}\int_o^{\pi} \phi i d\theta$,

or

$$\begin{aligned} T_M &\propto \frac{1}{\pi}\int_o^{\pi} \phi_{max} I_{max} \sin \theta \sin (\theta - \alpha)\, d\theta \\ &\propto \frac{1}{\pi}\int_o^{\pi} \phi_{max} I_{max} \left[\frac{\cos \alpha - \cos (2\theta - \alpha)}{2}\right] d\theta \\ &\propto \frac{\phi_{max} I_{max}}{2\pi}\left[\theta \cos \alpha - \frac{\sin (2\theta - \alpha)}{2}\right]_o^{\pi} \\ &\propto \frac{\phi_{max} I_{max}}{2\pi}(\pi \cos \alpha) \end{aligned}$$

Thus $$T_M \propto \phi I \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (370)$$

where ϕ and I are r.m.s. values of flux and current.

Hence, in all induction instruments, some means must be provided for producing an eddy current which is either appreciably less than, or appreciably greater than, 90° out of phase with the flux with which it reacts, since, if α is 90°, $\cos \alpha$—and hence T_M—is zero, the torque being small, also, if α is not far from 90°.

There are two general methods of fulfilling this condition. One method is by splitting the winding of the electromagnet in which the flux exists into two portions, one of which is highly inductive, and the other non-inductive. The other method is by splitting the phase of the working flux by a copper band placed round a portion of the poles of the electromagnet. This leads us to the two general types of induction instruments, which are—

(*a*) The Ferraris type.

(*b*) The "shaded-pole" type.

(*a*) FERRARIS TYPE. This instrument operates on the same principle as the induction motor. A rotating field is produced by two pairs of coils wound upon a laminated magnet system, as shown in Fig. 395. These pairs of coils are both supplied from the same source, but a phase displacement of approximately 90° is produced in the currents flowing in them by connecting an inductance in series with one pair and a high resistance in series with the other pair. This rotating field induces currents in an aluminium drum, and causes this drum to follow its rotation. If the drum were free to rotate, it would do so at a speed slightly less than that of the rotating field, and in the same direction as the latter. If a control spring prevents such continuous rotation the drum will rotate only through some angle less than 360°—i.e. until the operating torque is balanced by the controlling torque of the spring.

The drum and moving system are carried by a spindle whose ends fit in jewelled cups or bearings. Inside the drum is a cylindrical laminated-iron core, to strengthen the magnetic field cutting the drum. The spindle also carries an aluminium damping disc, the edge of which moves in the air gaps of two permanent magnets.

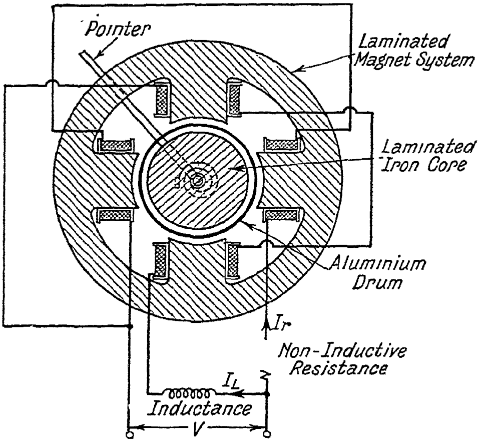


Fig. 395. Ferraris Type Induction Instrument

Theory. The theory of the instrument can be most easily worked out by considering the torque produced by the reactions between the two fluxes and the two eddy currents produced by them.

Saturation of the iron in the instrument, and iron losses, will be neglected so that the fluxes may be assumed to be in phase with, and proportional to, the currents producing them.

Referring to the vector diagram of Fig. 396, the vectors I_R and I_L represent the currents in the non-inductive and inductive windings of the instrument respectively, and the vectors ϕ_R and ϕ_L the fluxes produced by, and in phase with them. Let the phase angle between these currents (and fluxes) be β. This angle is large, but cannot be quite 90° owing to the inductance of the so-called "non-inductive" winding and to the resistance of the inductive winding. e_R and e_L—lagging 90° in phase behind ϕ_R and ϕ_L respectively—are the e.m.f's induced in the rotor drum, and i_R and i_L are the eddy currents set up by them. These currents lag behind the e.m.f.'s by a small angle α owing to the inductance of the eddy current paths.

As regards the torque of the instrument, it can be seen from Fig. 397 that there will be two components in opposite directions, one proportional to $\phi_L i_R$ in the direction of rotation of the rotating field, and one proportional to $\phi_R i_L$ in the opposite direction. In this figure both of the fluxes ϕ_R and ϕ_L are assumed to be decreasing, so that, from Lenz's Law, the currents i_R and i_L induced by them are in such directions that they tend to maintain the flux (as shown). The directions of the forces acting on the drum are obtained from the Left-hand Rule.

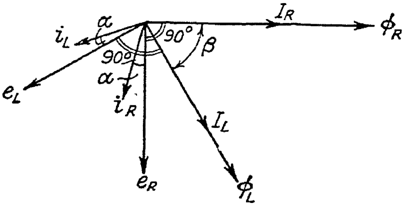
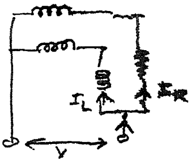


FIG. 396



The resultant torque is thus the *difference* between that due to $\phi_L i_R$ and that due to $\phi_R i_L$. Thus, if ϕ_R, ϕ_L, i_R, and i_L are r.m.s. values, the mean torque is given by

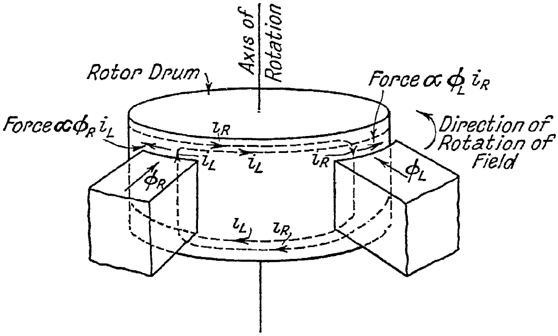


FIG. 397. ILLUSTRATING THE ACTION OF THE FERRARIS INDUCTION INSTRUMENT

$$T_M = k[\phi_L i_R \cos(90 + \alpha - \beta) - \phi_R i_L \cos(90 + \alpha + \beta)] \qquad . \quad . \quad (371)$$

where k is a constant.

$(90 + \alpha - \beta)$ is the phase angle between ϕ_L and i_R
and $(90 + \alpha + \beta)$,, ,, ,, ,, ϕ_R ,, i_L

Then, $T_M = k\{-\phi_L i_R \cos[90 - (\alpha - \beta)] + \phi_R i_L \cos[90 - (\alpha + \beta)]\}$
$= k[\phi_R i_L \sin(\alpha + \beta) - \phi_L i_R \sin(\alpha - \beta)]$

Now, $e_R \propto \underline{f \cdot \phi_R}$ and $e_L \propto \underline{f \cdot \phi_L}$ where f is the frequency.

Also, $i_R = \frac{e_R}{z}$ and $i_L = \frac{e_L}{z}$ where z is the impedance of the eddy-current paths in the drum.

$$\text{Hence, } T_M = \frac{k'\phi_R\phi_L f}{z}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$= \frac{2k'\phi_R\phi_L f}{z}\cos\alpha \, . \sin\beta$$

where k' is another constant.

Again, if ϕ_R and ϕ_L are directly proportional to the current I_R and I_L (as they are here assumed to be),

$$T_M \propto \frac{I_R I_L f}{z}\cos\alpha \, . \sin\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad (372)$$

or, since both I_R and I_L are proportional to the current I to be measured (in the case of an ammeter) or to the voltage to be measured (in the case of a voltmeter), we have

$$T_M \propto \frac{I^2 f}{z} \, . \cos\alpha \, . \sin\beta \quad . \quad . \quad . \quad (373)$$

Since I is a root-mean-square value, the torque is proportional to the mean-square value of the current to be measured.

It can be seen from the above that the torque is directly proportional to the sine of the angle between the two currents I_R and I_L, hence the necessity for making this angle large.

D. Connelly (Ref. (26)) has discussed, in some detail, the production of the torque in an induction instrument.

Compensation for Frequency and Temperature Errors. Since the torque is directly proportional to the frequency and also since z, $\cos\alpha$ and I_L are all dependent upon the frequency, it is necessary to consider compensation if serious errors, due to variation of frequency, are to be avoided. Some compensation can be provided by shunting the ammeter—in which instrument such errors are usually larger than in voltmeters—by a non-inductive shunt. If, then, the frequency increases, the increase in torque which would, as seen from the above expression, take place, is to some extent prevented by the fact that the impedance of the instrument windings increases and hence a greater proportion of the total current is taken by the non-inductive shunt, whose impedance remains constant for all frequencies.

In the case of a voltmeter the impedance of the inductive winding of the instrument increases with increasing frequency, and hence this winding takes a smaller current, which tends to compensate for the increase in torque due to the frequency increase.

Variations of temperature may produce serious errors, since the resistances of the eddy current paths in the rotor are dependent upon temperature. Compensation is obtained by shunting the instrument (in the case of an ammeter) with a shunt of material having a higher temperature coefficient than that of aluminium—of which the rotor is made. This shunt may be the same one as is used for frequency compensation. If, then, the temperature increases, the proportion of the total current which passes through the instrument is increased, and this compensates for the loss of torque due to the reduction of the eddy currents owing to the increased resistance of their path.

In voltmeters a combination of shunt and swamping resistance in series with the instrument is often used.

The frequency errors in induction instruments of all types are so serious, and are so difficult to compensate satisfactorily, that such instruments are most often used for switchboard purposes when frequency changes are small.

ADED-POLE TYPE. Fig. 398 illustrates the principle of the shaded-pole type of induction instrument. A thin aluminium disc is mounted on a spindle which is supported by jewelled bearings. The spindle carries a pointer and a control spring is attached to it. The edge of the disc moves in the air gap of a laminated electromagnet which is energized either by the current to be measured or by a current proportional to the voltage to be measured. A damping magnet is placed at the opposite side of the disc from the electromagnet, so that the disc serves for damping as well as for operating purposes. The poles of the electromagnet are split into two halves, with a thin copper band round one-half of each pole (on the same side), as shown. These two bands have currents induced in them by the alternating flux of the electromagnet, and the magnetic fields set up by these induced currents cause the flux, in the portions of the iron surrounded by the bands, to lag in phase by some 40 to 50 degrees behind the flux in the unshaded portions of the poles. This phase displacement is necessary for the production of torque, the shading bands serving the same purpose as the interposition of resistance and inductance between the windings and the terminals in the Ferraris type of instrument.

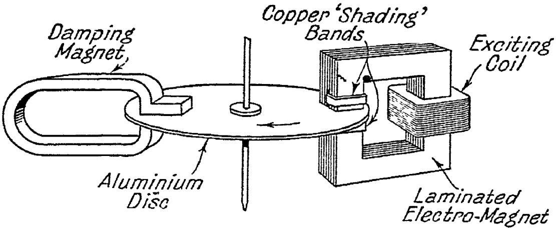


FIG. 398. ILLUSTRATING THE PRINCIPLE OF THE SHADED-POLE TYPE OF INDUCTION INSTRUMENT

The disc, instead of being circular, is sometimes made cam-shaped in order to improve the scale of the instrument. The scale is normally cramped at the lower end, since the deflecting torque T_{D} is proportional to the square of the current or voltage to be measured, and spring control is used. Then

$$T_{\mathrm{D}} \propto I^2 \text{ and } T_{\mathrm{C}} \propto \theta$$

where θ is the deflection.

For steady deflection $T_{\mathrm{C}} = T_{\mathrm{D}}$

so that $\theta \propto I^2$

Theory. Fig. 399 gives a vector diagram showing the relative phases of the fluxes, and the induced currents in the disc. The current induced in the shading band produces a number of "back" ampere-turns, and causes the flux ϕ_s in the shaded portion of the pole to lag in phase behind the flux ϕ_u in the unshaded portion. The resultant flux ϕ, in the main portion of the iron, is the vector sum of ϕ_u and ϕ_s, as shown.

These two fluxes induce voltages e_u and e_s in the disc, each of which is 90° in phase behind the flux inducing it. Eddy currents i_u and i_s flow in the disc as a result of these e.m.f's, each current lagging by a small angle α behind its voltage owing to the inductance of the path in the disc.

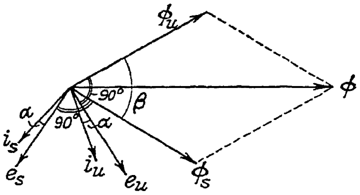


FIG. 399

From Fig. 400, which illustrates the operation of the instrument, it can be seen that the deflecting torque has two components, one due to ϕ_s reacting with i_u, and one due to ϕ_u with i_s. These torques are in opposite directions so that the resultant torque is the *difference* between these two component torques.

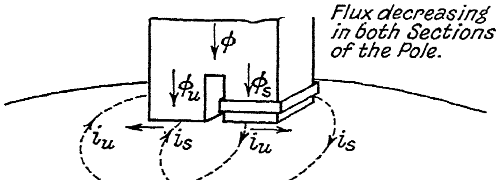


FIG. 400. ILLUSTRATING THE ACTION OF THE SHADED-POLE TYPE OF INDUCTION INSTRUMENT

If ϕ_s, ϕ_u, i_s and i_u are r.m.s. values, the mean torque T, deflecting the disc, is given by

$$T = k\{\phi_s i_u \cos[90 - (\beta - \alpha)] - \phi_u i_s \cos[90 + (\beta + \alpha)]\} \quad . \quad . \quad (374)$$

where k is a constant.

$90 - (\beta - \alpha)$ is the phase angle between ϕ_s and i_u

$90 + (\beta + \alpha)$ „ „ „ „ ϕ_u „ i_s

$$\therefore T = k[\phi_s i_u \sin(\beta - \alpha) + \phi_u i_s \sin(\beta + \alpha)]$$

As in the theory of the Ferraris instrument it follows that, if ϕ_s and ϕ_u are each directly proportional to the current I in the magnetizing coil, and z is the impedance of the eddy current paths, the torque is

$$\propto \frac{I^2 f}{z} \cos\alpha \sin\beta \quad . \quad . \quad (375)$$

For any given frequency,

$$T \propto I^2$$

Variation of frequency affects the torque for a given current directly, since the expression for T contains f. The values of sin β and of cos α and z also vary with frequency, so that compensation is applied both for frequency and temperature variations by shunting and by the use of non-inductive series resistance as described in dealing with the Ferraris instrument.

Thermionic Voltmeters. E. B. Moullin (Refs. (10), (11), (12)) has designed a thermionic voltmeter which is very useful for the measurement of low alternating voltages. The connections of the instrument, which is manufactured by the Cambridge Instrument Co., are shown in Fig. 401. It consists of a three-electrode valve used in conjunction with a unipivot galvanometer, the latter being operated by the rectified anode current obtained when the voltage to be measured is applied to the grid of the valve as shown. To allow for varying filament battery voltages it is necessary, before using the instrument, to adjust the pointer to zero on the scale by variation of the rheostat B_1.

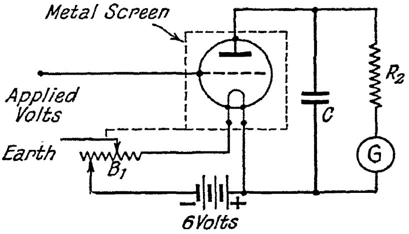


FIG. 401. MOULLIN THERMIONIC VOLTMETER

The instrument absorbs negligible power from the circuit under test, and has no frequency error. The error due to varying waveform is small and the accuracy of the instrument is usually to within 2 per cent.

Moullin's papers on the voltmeter should be consulted for descriptions of the various forms used. Methods of utilizing gas-filled triodes, or Thyratrons, for the measurement of peak voltages are described in Refs. (14) and (15).

Rectifier Instruments. During recent years copper-oxide rectifiers have been used fairly extensively in conjunction with moving-coil instruments for the measurement of small alternating currents and voltages. They are particularly suited to measurements on communication circuits and for light-current work when the voltage is low and the resistance high. It is essential in such cases that the current taken by voltmeters should not exceed (say) 1 mA at full scale and rectifier voltmeters are available having resistances of 1,000 or 2,000 ohms per volt.

These instruments consist of a permanent-magnet moving-coil instrument together with a full-wave copper-oxide rectifier incorporated within the case of the instrument. The connections are as shown in Fig. 402 (*a*). During both half-cycles of the a.c. wave current flows in the moving-coil instrument M in the direction shown, so that its deflection is proportional to the mean value of the current flowing through it.

On the assumption that the current is sinusoidal the scale of the instrument must be marked in terms of 1·11 times the current actually measured, to give r.m.s. values. It is clear at the outset that these instruments must be subject to considerable wave-form errors. A number of current wave-forms having the same mean value I_{av} (say), may have r.m.s. values which vary considerably, but in all cases the rectifier instrument will indicate the same r.m.s. value, namely, 1·11 I_{av}.

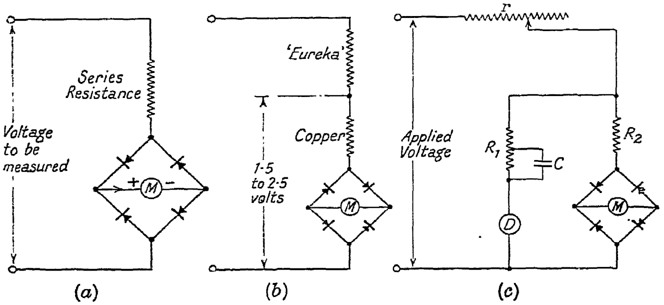


FIG. 402

The copper-oxide rectifiers used in such instruments are of three types, 1 mA, 5 mA, and 10 mA. As regards their characteristics it is important to realize that their resistance, even to current in the "forward" direction, is not constant, but depends upon the value of the current passing. The resistance in the "reverse" direction varies between 20,000 ohms and 80,000 ohms for a 10-mA rectifier, and between 0·3 and 0·6 megohm for a 1-mA unit. These figures are given by E. Hughes in a valuable paper on the subject of these rectifiers (Ref. (16)). The resistance in the "forward" direction is somewhat difficult to define, but in the case of a 10-mA rectifier this is of the order of 100 ohms when the rectified current is 6 mA, and between 300 and 500 ohms when the rectified current is 1 mA. The "reverse" current is thus negligible compared with the "forward" current. Dr. Hughes found that the resistance varied approximately as $(\text{current})^{-0\cdot 8}$ over a wide range, so that when a sinusoidal voltage $v = V_m \sin \theta$ is applied to a rectifier having a resistance R in series with it, the alternating current flowing is given by

. (376)

where k is a constant. Alternatively, the d.c. voltage v across the rectifier for a direct current i through it is given approximately by the equation

$$v = c + bi$$

where c and b are constants. On this basis the alternating current through the rectifier may be expressed as

$$i = \frac{V_m \sin \theta - c}{R + b}$$

The effect of the non-linear volt-amperes characteristic of the rectifier is to distort the wave-form of the alternating current flowing through it. Thus, if a sinusoidal voltage is applied the current flowing will be "peaked," its form-factor being greater than 1·11.* This means that the actual r.m.s. value of the current flowing will be greater than that indicated by the rectifier instrument, i.e. the instrument reads low. The effect of series resistance is to "swamp" the variable resistance of the rectifier itself, so that the greater the series resistance the less the error in reading on this account. When an alternating voltage is applied to a rectifier instrument having a large series resistance the reading of the instrument will thus be almost exactly proportional to the mean value of the voltage applied. As already pointed out, if the voltage wave-form is not sinusoidal an error is necessarily introduced. To correct for such non-sinusoidal wave-form, assuming that the instrument reads the true mean value of the wave, the instrument indication must be multiplied by $K_f/1{\cdot}11$ where K_f is the form-factor of the voltage wave-form. Dr. Hughes states that rectifier instruments actually read the r.m.s. value within 10 per cent for any shape of wave "except for a certain range and phase of the third harmonic."

The relationship between the applied voltage V and the rectified current I may be expressed in the form

$$I = KV^x \text{ where } K \text{ is a constant.}$$

With comparatively low values of series resistance the exponent x is approximately 2, so that low range voltmeters will have practically a square-law scale and will read roughly the r.m.s. value of the voltage whatever the wave-form.

Compensation for temperature errors in rectifier voltmeters may be carried out by using a combination of copper and Eureka for the series resistance as shown in Fig. 402 (*b*). The copper resistance is of such a value that the total volt drop across the rectifier and copper together is between 1·5 and 2·5 volts, the remaining series (Eureka) resistance being made up to the total value required, depending upon the applied voltage. The rectifier has a negative temperature coefficient and this is counterbalanced by the positive coefficient of the copper. The lowest voltage range for complete compensation is obviously of the order of 2 volts.

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* See the author's contribution to the discussion of Dr. Hughes's paper, *Journal I.E.E.*, Vol. 75, p. 477.

These instruments are subject, also, to frequency errors, the accuracy obtainable being some 2 per cent at full scale for frequencies between 20 and 4,000 cycles per second with ordinary wave-forms. The instrument indications decrease by about 0·5 per cent for every 1,000 cycles up to 35,000 cycles and, applying the necessary correction for this, the accuracy may be within 5 per cent when the wave-form approximates closely to a sine wave. By very careful construction to reduce the self-capacitance of the instrument it has been found possible to obtain an accuracy of within 1 per cent over a frequency range of 50 to 10,000 cycles per second and of 5 to 8 per cent between 50 and 100,000 cycles. While these errors may appear large it must be appreciated that the suitability of rectifier instruments for the measurement of small alternating currents having bad wave-forms, such as are met with in telephonic work, lies in their much greater sensitivity as compared with any other type of instrument available for the purpose.

As commonly manufactured the ranges covered by these instruments are—

Milliammeters 0–0·5 mA to 0–10 mA;
Voltmeters 0–1·0 to 0–300 volts.

The instrument scale is practically linear in the case of the higher voltmeter ranges, but is practically a square-law scale in the lower range instruments.

The use of rectifier instruments in conjunction with current transformers is also dealt with in the paper by E. Hughes already mentioned.

R. S. J. Spilsbury has described a direct-reading form-factor meter, devised by him (Ref. 17) which utilizes a rectifier instrument. The essential features of the device are shown in Fig. 402 (*c*). *D* is a low-range dynamometer voltmeter which is placed in parallel with the rectifier unit, *M* being, of course, a permanent-magnet moving-coil voltmeter. *C* is a capacitor which is connected across a portion of the series resistance of *D* to reduce the frequency error due to the inductance of the dynamometer instrument. R_2 is the series resistance of the rectifier (about 40,000 ohms) and r is a variable resistance for the purpose of adjusting the voltage across the combination of *D* and *M* to the same value whatever the voltage applied.

The dynamometer indicates the r.m.s. value of the voltage and the rectifier instrument its mean value. The resistance r is adjusted until D reads a standard value (actually 20 volts) so that M indicates the mean value of the standard voltage and its scale can thus be marked in terms of form-factor.

The range of form-factor measurable is from 1·0 to 1·32 and the indications are independent of wave-form, temperature and frequency (between 25 and 800 cycles per second) to an accuracy of 0·2 per cent. An important application of such an instrument is the measurement of form-factor in the magnetic testing of sheet steel with alternating currents (see page 401).

H. J. Lindenhovius, G. Arbelet and J. C. van der Breggen (Ref. (24)) describe a millivoltmeter, employing an amplifier and rectifying circuit with a moving-coil instrument, with a full deflection for 1 mV and covering the frequency range 1,000 to 30×10^6 c/s. Using an attenuator, voltages up to 1 kV can be measured.

Testing Sets. This chapter should not be concluded without mention of various "test-sets" which have been developed by a number of manufacturers as multi-range instruments for the measurement of current, voltage, power, resistance and other electrical quantities. Some of these consist of two moving-coil instruments in a common case together with appropriate shunts and multipliers. Others have moving-iron instruments for d.c. or a.c. measurements and there are also several sets, of unconventional design, intended for special measurements such as those of current in circuits which cannot conveniently be broken.

There is not space here to illustrate these sets nor, indeed, to give a full list of them but the following are some of the sets in common use—

(*a*) Avometers (The Automatic Coil Winder and Electrical Equipment Co., Ltd.). The "Universal" Avometer is an a.c.-d.c. instrument with fifty ranges of current, voltage, resistance, capacitance, power, etc.

A wide variety of instruments by the same makers include an "Electronic" test meter, valve-characteristic meter, high-sensitivity Avometers and a light meter.

(*b*) Weameter Test-set (Elliott Brothers (London), Ltd.). This is an a.c. test set for current, voltage and power with extensions, beyond its normal ranges, through current and voltage transformers.

The Elliott "Multiversal" test-set is particularly for insulation and fault location tests.

(*c*) Ferranti Test-sets (Ferranti, Ltd.) for current and voltage measurements (a.c. or d.c.) by two multi-range instruments contained in a single case.

(*d*) "Vampire" (Everett Edgcumbe & Co., Ltd.) a.c. test set for current, voltage, and power (up to 5 kW).

(*e*) Precision Multi-range meter (Electronic Instruments, Ltd.).

An instrument of sub-standard accuracy for current, voltage and resistance measurements.

(*f*) C.P. "Tong-test" (Crompton-Parkinson, Ltd.). An instrument, for both a.c. and d.c. measurements of current, which is used by placing the "tongs" round the current-carrying conductor.

(*g*) "Selectest" (General Electric Co., Ltd.). A multi-range instrument for d.c. and a.c. measurements of current and voltage at power and audio frequencies, and for resistance measurements up to 1 megohm.

(*h*) Component tester (Pye, Ltd.). This is an instrument for resistance, capacitance and insulation tests mainly on radio components in the workshop.

BIBLIOGRAPHY AND REFERENCES

(1) *Electrical Measuring Instruments,* Drysdale and Jolley, Vols. I and II.

(2) *Theory and Practice of Alternating Currents,* A. T. Dover.

(3) *Electrical Measurements,* F. A. Laws.

(4) *Electrical Measuring Instruments,* D. J. Bolton.

(5) *Commercial Electrical Measuring Instruments,* R. M. Archer.

(6) British Standard Specification for "Indicating Ammeters, Voltmeters, Wattmeters, Frequency and Power Factor Meters," No. 89 (1929).

(7) "Some Recent Advances in Alternating-current Measuring Instruments," K. Edgcumbe and F. E. J. Ockenden, *Journal I.E.E.,* Vol. LXV, p. 553 (1927).

(8) "Electrical Measuring Instruments," *Bureau of Standards Circular,* No. 20 (1915).

(9) "Electrical Measuring Instruments other than Integrating Meters," C. V. Drysdale, *Journal I.E.E.,* Vol. LXVI, p. 596.

(10) *Radio Frequency Measurements,* E. B. Moullin.

(11) E. B. Moullin, *Wireless World and Radio Review,* Vol. X, p. 1 (1922).

(12) *Journal I.E.E.,* Vol. LXI, p. 295.

(13) "A Thermionic Voltmeter for measuring the peak value and the mean value of an alternating voltage of any wave-form," E. B. Moullin, *Journal I.E.E.,* Vol. LXVI, p. 886.

(14) "Characteristics of the Thyratron, with special reference to the Application of Thyratrons to the Measurement of Peak Voltage," E. W. Golding, *Electrical Times,* 23rd Nov., 1933.

(15) "Measurement of Peak Values of Alternating Currents and Voltages by means of a Thyratron," E. Hughes, *Jour. Sc. Insts.,* Vol. X, June, 1933.

(16) "Copper-oxide Rectifiers in Ammeters and Voltmeters," E. Hughes, *Jour. I.E.E.,* Vol. LXXV, p. 453.

(17) "A Direct-reading Form Factor Meter," R. S. J. Spilsbury, *Jour. I.E.E.,* Vol. LXXV, p. 463.

(18) "Industrial Instruments," W. Phillips, *Jour. I.E.E.,* Vol. LXXIV, p. 161.

(19) "Use of Vacuum Tubes in Measurements," J. W. Horton, *Trans. A.I.E.E.,* Vol. 54, p. 93.

(20) "Considerations in the Use of the Electrostatic Peak Voltmeter," E. H. W. Banner, *Jour. Sc. Insts.,* July, 1934.

(21) "A Valve Ammeter for the Measurement of Small Alternating Currents of Radio Frequency," H. E. M. Barlow, *Jour. I.E.E.,* Vol. LXXVII, p. 612.

(22) "The Theory and Design of Hot-wire Ammeters for Frequencies of 25 to 100 Megacycles," C. L. Fortescue, *Jour. I.E.E.,* Vol. LXXIX, p. 179.

(23) "The Compensated Thermo Couple Ammeter," W. N. Goodwin, *Trans. A.I.E.E.,* Vol. LV, p. 23.

(24) "A Millivoltmeter for the Frequency Range from 1,000 to 30 × 10^6 c/s," H. J. Lindenhovius, G. Arbelet and J. C. van der Breggen, *Phillips Technical Review*, Vol. II, No. 7, p. 206.

(25) *Vacuum Tube Voltmeters*, J. F. Rider.

(26) "The Torque of an Induction-type Meter," D. Connelly, *BEAMA Journal*, December, 1950, p. 377, and February, 1951, p. 36.

(27) "The Accuracy of Measurement of Electrical Standards," A. Felton, *Proc. I.E.E.*, Vol. 98, Pt. II, p. 694.

(28) "A Precision Form of the Kelvin Electrostatic Voltmeter," E. H. Rayner, *Jour. I.E.E.*, Vol. LIX, p. 138.

(29) "The Electrostatic Voltmeter as a D.C./A.C. Transfer Instrument," R. S. J. Spilsbury and A. Felton, *Jour. I.E.E.*, Vol. LXXXIX, Pt. II, p. 129.

(30) "Precision A.C./D.C. Comparator for Power and Voltage Measurements," G. F. Shotter and H. D. Hawkes, *Jour. I.E.E.*, Vol. XCIII, Pt. II, p. 314.

(31) "Performance Limits in Electrical Instruments," A. H. M. Arnold, *Proc. I.E.E.*, Vol. 98, Pt. II, p. 70.

(32) "A Precision Thermionic A.C. Test Set," K. A. Macfadyen and N. D. Hill, *G.E.C. Journal*, 1041, Vol. II, p. 182.

CHAPTER XIX

EXTENSION OF INSTRUMENT RANGE

Ammeter Shunts. The use of low resistance "shunts" to be placed in parallel with an ammeter in order to measure a greater current than that which the instrument itself can carry, has been mentioned in dealing with moving-coil instruments, with which type of ammeter such shunts are most commonly used.

Let the ammeter resistance be R, and its current for maximum deflection be i. Suppose that R_s is the resistance of the shunt, which must be placed in parallel with the instrument in order to enable a maximum current of I to be measured, and let I_s be the current through the shunt during the measurement (see Fig. 403).

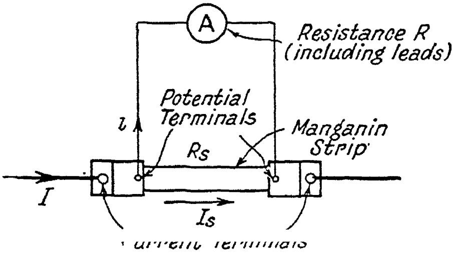


FIG. 403. SHUNTED AMMETER

Then $$i = I - I_s$$

Also, Volt-drop across shunt = Volt-drop across instrument

Hence, $$I_sR_s = iR$$

The resistance of shunt required is thus given by

$$R_s = \frac{i}{I_s} . R = \frac{i}{I-i} \times R = \frac{1}{\left(\frac{I}{i} - 1\right)} . R \quad . \quad (377)$$

The ratio of the total current to the instrument current $\left(\text{i.e. } \frac{I}{i}\right)$ is called the "multiplying power" of the shunt. Expressing this ratio as N, we have

$$R_s = \frac{1}{N-1} . R$$

or $$N = 1 + \frac{R}{R_s} \quad . \quad . \quad . \quad (378)$$

Example. Calculate the resistance of shunt required to make a milliammeter, which gives maximum deflection for a current of 15 milliamp., and which has a resistance of 5 ohms, read up to 10 amp.

$$\text{Multiplying power} = \frac{10}{0{\cdot}015} = 666{\cdot}7 = N$$

$$\therefore \qquad R_s = \frac{1}{666{\cdot}7 - 1} \times 5$$

$$= \frac{5}{665{\cdot}7} = 0{\cdot}007511 \text{ ohm}$$

or, otherwise,

Volt-drop across instrument for maximum deflection

$$\begin{aligned} &= 5 \times 0{\cdot}015 = 0{\cdot}075 \text{ volt} \\ &= \text{Volt-drop across shunt} \\ &= (10 - 0{\cdot}015)\, R_s \\ &= 9{\cdot}985 R_s \end{aligned}$$

$$\therefore \qquad R_s = \frac{0{\cdot}075}{9{\cdot}985} = 0{\cdot}007511 \text{ ohm}$$

ALTERNATING CURRENT OPERATION. If a shunt is to be used with an ammeter for alternating current work, the inductances of both the instrument and the shunt must be considered as well as their resistances.

In order that the division of current between the two parallel branches—instrument and shunt—shall remain the same for all frequencies, the ratio of the *impedances* of the two branches must remain constant.

Thus, let L be the inductance of the instrument and L_s that of the shunt, then, the ratio $\dfrac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{R_s^2 + \omega^2 L_s^2}}$ must be independent of frequency. To fulfil this condition, the time-constants of the shunt and instrument must be the same, i.e. $\dfrac{L_s}{R_s}$ must equal $\dfrac{L}{R}$.

Then, let $\dfrac{L_s}{R_s} = \dfrac{L}{R} = k.$

$$\text{We have now, } \frac{i}{I_s} = \frac{\sqrt{R_s^2 + \omega^2 L_s^2}}{\sqrt{R^2 + \omega^2 L^2}} = \frac{\sqrt{R_s^2 + \omega^2 k^2 R_s^2}}{\sqrt{R^2 + \omega^2 k^2 R^2}}$$

$$\therefore \qquad \frac{i}{I_s} = \frac{R_s\sqrt{1 + \omega^2 k^2}}{R\sqrt{1 + \omega^2 k^2}} = \frac{R_s}{R} \qquad . \quad . \quad . \quad (379)$$

The general expression for the multiplying power N, on alternating current circuits, is most easily obtained by the employment of the symbolic notation, owing to the phase difference between i and I_s when the time-constants of the two branch circuits are not equal.

Then, if v is the volt-drop across the instrument and shunt (in parallel),

$$i = \frac{v}{R + j\omega L} \text{ and } I_s = \frac{v}{R_s + j\omega L_s}$$

The total current I in the main circuit is thus given symbolically by

$$\frac{v}{R + j\omega L} + \frac{v}{R_s + j\omega L_s}$$

and the multiplying power $\frac{I}{i}$ by

$$N = \frac{v\left(\frac{1}{R + j\omega L} + \frac{1}{R_s + j\omega L_s}\right)}{v\left(\frac{1}{R + j\omega L}\right)} = \frac{R + R_s + j\omega(L + L_s)}{R_s + j\omega L_s}$$

Expressing this in real values, we have

$$N = \frac{\sqrt{(R + R_s)^2 + \omega^2(L + L_s)^2}}{\sqrt{R_s^2 + \omega^2 L_s^2}} \quad . \quad . \quad . \quad . \quad . \quad (380)$$

Now, if $\qquad \frac{L}{R} = \frac{L_s}{R_s} = k$

$$N = \frac{\sqrt{(R + R_s)^2 + \omega^2 k^2 (R + R_s)^2}}{\sqrt{R_s^2 + \omega^2 k^2 R_s^2}} = \frac{R + R_s}{R_s} = 1 + \frac{R}{R_s}$$

GENERAL REQUIREMENTS OF AMMETER SHUNTS. It is essential, whether on a.c. or d.c., that the temperature coefficients of the shunt and instrument shall be low, and as nearly as possible the same, in order that the multiplying power shall be independent of temperature. This question has already been discussed in the previous chapter.

Another requirement is that such shunts shall not vary in resistance with time, and this is ensured by careful annealing during manufacture. Again, the thermo-electric effect in the shunt must be small, and it must be able to carry the necessary current without appreciable heating.

These shunts are usually composed of one or more thin strips of manganin sheet, the ends of which are soldered to two heavy copper blocks which carry two terminals each—one current terminal and one potential terminal—the two current terminals being outermost. When such shunts are used with ammeters, the resistance of the copper leads from the potential terminals of the shunt to the ammeter terminals, forms part of the resistance of the ammeter circuit and must therefore be taken into account in calculating the required shunt-resistance. A special pair of leads is usually supplied with ammeters intended to be used shunted, and these leads should always be used for the purpose.

Voltmeter Multipliers. To increase the range of a voltmeter, a non-inductive resistance is connected in series with the instrument. Thus, if the voltmeter gives full-scale deflection when a current i passes through it, and if its resistance is r, V being the voltage to be measured, and R the resistance connected in series with the voltmeter, we have

$$\frac{V}{R+r}=i$$

or

$$R=\frac{V-ir}{i}=\frac{V}{i}-r \qquad . \quad . \quad . \quad . \quad . \quad (381)$$

The principal requirement of such multiplying resistances, or "multipliers," when used on d.c. circuits, is that their resistance shall be constant. Thus they must have a very small temperature coefficient, and, on account of the appreciable amount of power absorbed by them, ample provision for cooling must be made.

OPERATION ON A.C. CIRCUITS. When used with alternating currents, the total impedance of the voltmeter and its series resistance must remain as nearly constant as possible for different frequencies. The series resistance should, therefore, have as small an inductance as possible, in order to keep the total inductance of the circuit small. For this reason, the resistance coils are often wound upon flat mica strips to reduce the area enclosed by the turns of wire, and hence to reduce the enclosed flux for a given current. Sometimes non-inductively woven gauze resistances are used.

If r is the voltmeter resistance, l its inductance, and i the current for full-scale deflection, then the volt-drop across the instrument, for full deflection, is

$$v=i\sqrt{r^2+\omega^2l^2}$$

where

$$\omega=2\pi\times\text{frequency}$$

Let V be the voltage to be measured and R the resistance of the non-inductive series resistance.

Then, total resistance of the circuit $= r+R$

Total inductance of the circuit $= l$

$\therefore$ Total impedance of the circuit $=\sqrt{(r+R)^2+\omega^2l^2}$,

and the voltmeter current is given by

$$\frac{V}{\sqrt{(r+R)^2+\omega^2l^2}}=i$$

Hence,

$$v=\frac{V}{\sqrt{(r+R)^2+\omega^2l^2}}\sqrt{r^2+\omega^2l}$$

and the multiplying ratio

$$\frac{V}{v}=\frac{V}{\dfrac{V\sqrt{r^2+\omega^2l^2}}{\sqrt{(r+R)^2+\omega^2l^2}}}=\frac{\sqrt{(r+R)^2+\omega^2l^2}}{\sqrt{r^2+\omega^2l^2}} \quad . \quad (382)$$

An appreciable error may thus be introduced by neglecting the inductance of the instrument.

Multipliers for Electrostatic Voltmeters. Such multipliers are either in the form of a resistance potential-divider or a capacitor-multiplier.

Resistance potential-dividers are used for voltages of a few thousand volts only. For higher voltages they become expensive to manufacture, and are wasteful of power and capacitor multipliers are, therefore, used.

RESISTANCE DIVIDERS for such purposes are high non-inductive resistances (often oil-immersed), across a comparatively small portion of which the electrostatic voltmeter is connected (see Fig. 404).

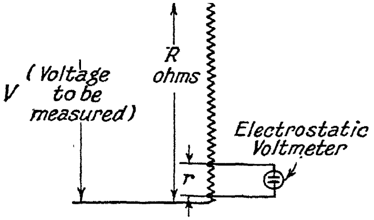


FIG. 404. RESISTANCE POTENTIAL DIVIDER

If the capacitance of the voltmeter is neglected, the multiplying power is

$$\frac{V}{v} = \frac{R}{r}$$

where V is the voltage to be measured and v the voltage at the voltmeter terminals. R and r are the total resistance (connected across V) and the resistance across which the voltmeter is connected, respectively.

Taking the capacitance of the voltmeter into account, let this be C, and suppose the voltmeter to be used with alternating current of ordinary commercial frequency. Then,

Impedance of voltmeter and r in parallel $= \dfrac{r}{1 + j\omega Cr} = z$ using the symbolic method.

The total impedance connected across the voltage V is thus

$$z_T = R - r + \frac{r}{1 + j\omega Cr}$$

$$z_T = \frac{R + j\omega Cr\,(R - r)}{1 + j\omega Cr}$$

The multiplying power is thus,

$$\frac{z_T}{z} = \frac{\dfrac{R + j\omega Cr(R - r)}{1 + j\omega Cr}}{\dfrac{r}{1 + j\omega Cr}} = \frac{R + j\omega Cr(R - r)}{r}$$

$$= \frac{R}{r} + j[\omega C\,(R - r)]$$

Numerically,

$$\frac{z_T}{z} = \sqrt{\left(\frac{R}{r}\right)^2 + \omega^2 C^2 (R - r)^2}$$

or,

$$\frac{z_T}{z} = \frac{R}{r}\sqrt{1 + \frac{\omega^2 C^2 r^2 (R - r)^2}{R^2}} \quad . \quad . \quad . \quad . \quad (383)$$

This will have the value $\frac{R}{r}$ if ωCr is small.

Capacitor Multipliers. When used with a capacitor multiplier, the voltmeter may be simply connected in series with a suitable capacitor (as in Fig. 405 (*a*)), and the voltage to be measured applied across the whole circuit. Again, a number of capacitors may be connected, in series, across the voltage to be measured, and the voltmeter connected across one of them (as in Fig. 405 (*b*)).

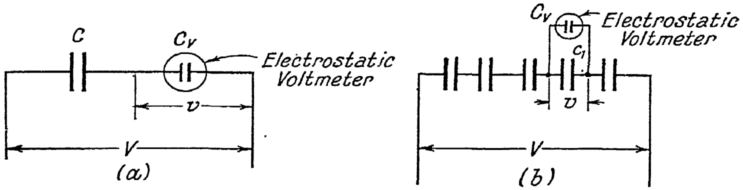


Fig. 405. Capacitor Multipliers

Referring to the first method of use, let C_v be the capacitance of the voltmeter and C that of the capacitor in series with it. The total capacitance of the circuit is $\frac{C_v C}{C + C_v}$ and its impedance on a.c. is $\frac{C + C_v}{\omega C_v C}$. The impedance of the voltmeter itself is $\frac{1}{\omega C_v}$. The multiplying ratio is

$$\frac{V}{v} = \frac{\dfrac{C + C_v}{\omega C_v C}}{\dfrac{1}{\omega C_v}} = \frac{C + C_v}{C} = 1 + \frac{C_v}{C} \quad . \quad . \quad . \quad (384)$$

The capacitance C_v varies with the deflection (i.e. the position of the moving vane) of the voltmeter. Thus the voltmeter must be calibrated together with the series capacitor. It should be noted that, in order that the multiplying ratio shall be large, the capacitance of the series capacitor must be small compared with that of the voltmeter.

Referring to the second method of connection, let C_1 be the capacitance of the capacitor across which the voltmeter—of capacitance C_v—is connected, and let C_s be the capacitance in series with

C_1 and C_v in parallel. Then, total impedance of the circuit connected across the voltage V

$$= \frac{C_s + C_1 + C_v}{\omega C_s(C_1 + C_v)}$$

Again, the impedance of C_v and C_1 in parallel is $\dfrac{1}{\omega(C_1 + C_v)}$. Thus the multiplying ratio

$$\frac{V}{v} = \frac{\dfrac{C_s + C_1 + C_v}{\omega C_s(C_1 + C_v)}}{\dfrac{1}{\omega(C_1 + C_v)}} = \frac{C_s + C_1 + C_v}{C_s}$$

$$= 1 + \frac{C_1 + C_v}{C_s} \quad . \quad . \quad . \quad (385)$$

In this case, again, C_v is variable, so that the voltmeter should be calibrated together with its multiplier, although if the capacitance C_1 is large compared with that of the voltmeter, the multiplying ratio may be approximately constant.

✓**Instrument Transformers.** A very general method of increasing the range of a.c. instruments is to use instrument transformers in conjunction with them. Such transformers are either (*a*) current (or series) transformers, or (*b*) voltage (or potential) transformers. The former are used for increasing the current range of instruments and the latter for increasing the voltage range.

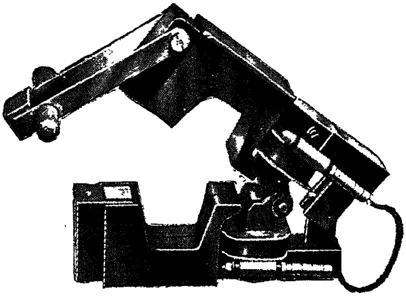
ADVANTAGES OF INSTRUMENT TRANSFORMERS AS COMPARED WITH SHUNTS AND MULTIPLIERS. The most important of these advantages are—

(i) Single-range instruments can be used to cover a large current, or voltage, range, when used with a suitable multi-range current or voltage transformer, or with several single-range transformers. The usual ranges of ammeters and voltmeters for use with such transformers are 5 amp. and 110 volts.

(ii) The indicating instruments can be located at some distance from the circuit, the current or voltage of which is to be measured. This is a great advantage, especially when the circuit under test is a high-voltage one, as far greater safety can be obtained for the observer than when the instruments are connected directly to the high-voltage circuit. Also, the instruments need not be insulated for high voltages.

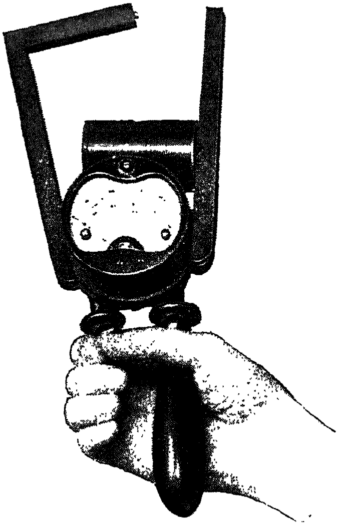
(iii) By the use of a current transformer with a suitably split and hinged core, upon which the secondary winding is wound, it is possible to measure the current in a heavy-current busbar without breaking the current circuit. The split core of the transformer is simply fitted round the busbar, which acts as the primary winding of the transformer.

Fig. 406 shows a current transformer of this type, while Fig. 407 shows a combined ammeter and current transformer which can be



(*Elliott Bros.* (*London*), *Ltd.*)

FIG. 406. SPLIT-CORE TYPE CURRENT TRANSFORMER



(*Ferranti, Ltd.*)

FIG. 407. COMBINED AMMETER AND SPLIT-CORE CURRENT TRANSFORMER

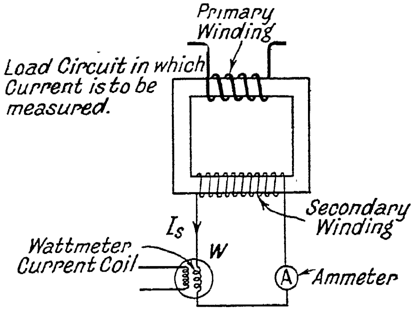


FIG. 408. CONNECTIONS OF A CURRENT TRANSFORMER

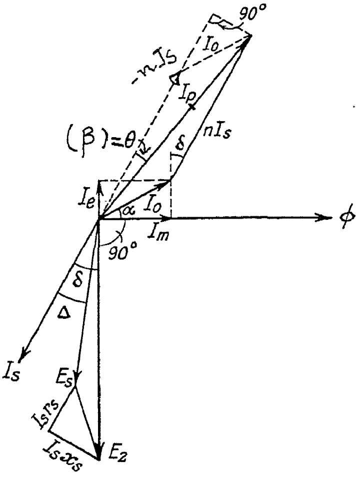


FIG. 409. VECTOR DIAGRAM OF A CURRENT TRANSFORMER

used in the same way. In this case the ammeter is, of course, calibrated together with the transformer. Another interesting type of split-core current transformer is made by Price and Belsham, Ltd. This is designed to be fixed round a fuse when no other part of the circuit is available.

Current Transformers. The primary winding of a current transformer is connected in series with the load circuit, the current in which is to be measured. There is no appreciable voltage between the two terminals of this winding, and also—and this is very important—the primary current is obviously not determined by the secondary load on the transformer. The ammeter, or wattmeter current-coil (in the case of power measurements), is connected directly across the secondary winding of the current transformer, as shown diagrammatically in Fig. 408.

THEORY OF CURRENT TRANSFORMERS. Fig. 409 represents the general vector diagram for a current transformer. The magnitudes of the magnetizing component I_m and the iron-loss component I_e, of the exciting current I_o, are exaggerated for convenience in drawing.

Then n = turns ratio $= \dfrac{\text{number of secondary turns}}{\text{number of primary turns}}$

r_s = resistance of the secondary winding
x_s = reactance of the secondary winding
E_2 = induced secondary voltage
T_p = number of primary turns
T_s = number of secondary turns
E_s = voltage at secondary terminals
I_s = secondary current
I_p = primary current
θ = the "phase angle" of the transformer
ϕ = working flux of the transformer
δ = angle between secondary current and secondary induced volts
= phase angle of the total burden (including impedance of the secondary winding)
Δ = phase angle of secondary *load* circuit
α = angle between I_0 and the working flux

Current Transformation Ratio. From the vector diagram,

$$\begin{aligned} I_p^2 &= (I_e + nI_s \cos\delta)^2 + (I_m + nI_s \sin\delta)^2 \\ &= (I_0 \sin\alpha + nI_s \cos\delta)^2 + (I_0 \cos\alpha + nI_s \sin\delta)^2 \end{aligned}$$

$$\therefore I_p = [(I_0 \sin\alpha + nI_s \cos\delta)^2 + (I_0 \cos\alpha + nI_s \sin\delta)^2]^{\frac{1}{2}}$$

∴ Transformation ratio

$$R = \frac{I_p}{I_s} = \frac{[(I_0 \sin\alpha + nI_s \cos\delta)^2 + (I_0 \cos\alpha + nI_s \sin\delta)^2]^{\frac{1}{2}}}{I_s}$$

$$= \frac{[I_0^2 \sin^2\alpha + 2nI_s \cos\delta \,.\, I_0 \sin\alpha + n^2 I_s^2 \cos^2\delta + I_0^2 \cos^2\alpha}{I_s}$$

$$+ \frac{2nI_s \sin\delta \,.\, I_0 \cos\alpha + n^2 I_s^2 \sin^2\delta]^{\frac{1}{2}}}{I_s}$$

Neglecting terms containing I_0^2, this becomes

$$\frac{[n^2I_s^2(\cos^2\delta + \sin^2\delta) + 2nI_sI_0(\cos\delta\sin\alpha + \sin\delta\cos\alpha)]^{\frac{1}{2}}}{I_s}$$

$$= \frac{[n^2I_s^2 + 2nI_sI_0\sin(\alpha + \delta)]^{1/2}}{I_s}$$

which, to a very close approximation,

$$= \frac{nI_s + I_0\sin(\alpha + \delta)}{I_s}$$

since I_0 is small compared with nI_s.

$$\therefore \qquad R = n + \frac{I_0}{I_s}\sin(\alpha + \delta) \text{ (very nearly)} \quad . \quad . \quad (386)$$

Although only approximate, this expression is sufficiently accurate for almost all purposes. The above theory refers, of course, to the case when the power factor of the secondary load (or burden) is lagging. This is the most usual condition in practice.

The expression can be expanded still further, as

$$R = n + \frac{I_0}{I_s}(\sin\alpha\cos\delta + \cos\alpha\sin\delta)$$

or

$$R = n + \frac{I_e\cos\delta + I_m\sin\delta}{I_s} \quad . \quad . \quad . \quad . \quad (387)$$

since $I_0\sin\alpha = I_e$ and $I_0\cos\alpha = I_m$

PHASE ANGLE. It can be seen from the vector diagram that the secondary current of a current transformer is displaced in phase by almost 180° from the primary current. If this angle were exactly 180°, no phase error would be introduced when a current transformer is used with a wattmeter for power measurements. The existence of the magnetizing and iron-loss components of the primary current causes the angle to be (usually) less than 180°, so that the transformer usually does, in practice, introduce a phase error.

The angle by which the secondary current vector, when reversed, differs in phase from the primary current, is the phase angle of the transformer; this angle is reckoned as positive if the reversed secondary current *leads* the primary current. On very low power factors the phase angle may be negative.

From the vector diagram (Fig. 409)—

$$\tan\theta = \frac{I_0\sin[90 - (\alpha + \delta)]}{nI_s + I_0\cos[90 - (\alpha + \delta)]}$$

$$= \frac{I_0\cos(\alpha + \delta)}{nI_s + I_0\sin(\alpha + \delta)}$$

from which $\theta = \dfrac{I_0\cos(\alpha + \delta)}{{}_s + I_0\sin(\alpha +}$ radians (approx.) . (388)

since θ is a small angle.

The error in the phase angle made by using this approximation is less than 5 per cent up to 3° phase angle, and less than 1 per cent up to 9° phase angle. Converting this expression to a more convenient form, we have

$$\theta = \frac{I_0(\cos \alpha \cos \delta - \sin \alpha \sin \delta)}{nI_s + I_0 \sin (\alpha + \delta)} \text{ radians}$$

$$\doteqdot \frac{I_m \cos \delta - I_e \sin \delta}{nI_s}$$

since $I_o \sin (\alpha + \delta)$ is small compared with nI_s.

$$\therefore \qquad \theta = \frac{180}{\pi}\left(\frac{I_m \cos \delta - I_e \sin \delta}{nI_s}\right) \text{ degrees} \quad . \quad . \quad (389)$$

ERRORS INTRODUCED BY CURRENT TRANSFORMERS. When used for current measurement only, the only essential requirement of a current transformer, in order that it shall not introduce an error into the measurement, is that the secondary current shall be a definite and known fraction of the primary current. It can be seen from the above theory that the current ratio R of the transformer differs from the turns ratio n by an amount which depends upon the magnitude of the exciting current of the transformer, and upon the current, and power factor, of the secondary circuit. The current ratio is, therefore, not constant under all conditions of load and of frequency, and the error so introduced may be of considerable importance.

Again, in power measurements, it is necessary that the phase of the secondary current shall be displaced by exactly 180° from that of the primary current. It can be seen that, in general, this condition is not fulfilled, but that the transformer has a "phase angle" error θ, which may introduce an appreciable error in power measurements.

In general, it may be said that the ratio error—expressed by $\dfrac{\text{Nominal ratio} - \text{Actual ratio}}{\text{Actual ratio}}$—is largely dependent upon the value of the iron loss component I_e of the exciting current and phase angle error upon the value of the magnetizing component I_m.

Thus, if the angle δ, which is usually fairly small, is assumed zero, then, the ratio

$$R = n + \frac{I_e}{I_s} \quad . \quad . \quad . \quad . \quad (390)$$

and the phase angle

$$\theta = \frac{I_m}{nI_s} \quad . \quad . \quad . \quad . \quad . \quad (391)$$

The ratio error is considered to be positive when the actual ratio of the transformer is less than the nominal ratio—i.e. when the secondary current, for a given primary current, is high.

DESIGN CONSIDERATIONS. (*a*) *Number of Primary Ampere-turns.* In order that the number of exciting ampere-turns I_oT_p shall be a small proportion of the total primary ampere-turns (this being necessary to keep the ratio and phase-angle errors small), the number of primary ampere-turns should be of the order of 500–1,000. In the case of current transformers having a single bar as their primary winding, the number of primary ampere-turns is, of course, limited by the primary current. When "Stalloy" is used as the material for the transformer core, satisfactory performance for measurement purposes cannot be expected with this type of transformer when the primary current is less than several hundred amperes, depending upon the magnitude of the burden (see British Standard 81, 1936, p. 30), although if a high-permeability, low-loss alloy, such as "Mumetal" is used for the core, the primary current of a bar-primary transformer may be as low as 100 amp. and still give a reasonably good performance.

(*b*) *Core.* To minimize the exciting ampere-turns required, the core must have a low reluctance and small iron loss. The flux density used in the core should also be small (not greater than 1,000 lines per square centimetre). "Mumetal" cores are now commonly used when it is essential that the transformer errors shall be very small. Mumetal, a nickel-iron alloy containing copper, has the properties of very high permeability, low loss, and small retentivity—all of which are advantageous for current transformer work —but has the disadvantage that its maximum permeability (about 90,000) occurs with a flux density of only about 3,500 lines per square centimetre, as compared with a maximum permeability of about 4,500 in the case of silicon steel, occurring with a flux density of about 5,000 lines per square centimetre (Ref. (40)). It is used in the construction of the movements of indicating instruments, for magnetic shields and in other apparatus used in electrical measurements. G. A. V. Sowter (Ref. (43)) has given very full information on Mumetal and other high permeability alloys including a series of curves giving magnetization and loss characteristics for these materials together with the variation of their magnetic properties with temperature.

The figures in Table XXI, for Mumetal, are abstracted from Sowter's paper.

TABLE XXI

MAGNETIC PROPERTIES OF MUMETAL

Peak flux density (gauss) . . .	25	50	100	250	500	1000	2500	5000
Magnetizing force at 15° C. (oersteds)	0·00101	0·00190	0·00355	0·00750	0·0128	0·0220	0·0440	0·125
Loss (mW/lb.) at 15° C. . . .	0·00062	0·00280	0·0122	0·0870	0·385	1·62	9·20	37·5

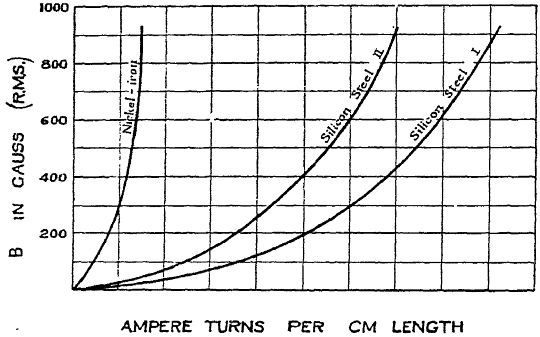
Incidentally the alloy Permendur (49 per cent iron, 49 per cent cobalt and 2 per cent vanadium) has a very high saturation point of some 24,000 gauss as compared with 7,000–8,000 gauss for other high-permeability alloys.

Its price is a disadvantage of Mumetal and, in this connection, Sowter's figures of comparative costs are interesting. These are given in Table XXII.

TABLE XXII

Alloy	Approximate Relative Costs per lb.
Hot-rolled silicon iron . .	0·4
Cold-rolled silicon iron . .	1·0
36/64 alloy	3·1
Radiometal . . .	3·4
Special radiometal . .	4·2
Mumetal	5·3

Fig. 410* gives curves of magnetizing ampere-turns per centimetre, at different flux densities, for silicon sheet steel of ordinary grade (Curve I); for special silicon steel (Curve II); and for a nickel-iron alloy.



(*Everett Edgcumbe & Co., Ltd.*)

FIG. 410. CURVES OF MAGNETIZING AMPERE-TURNS FOR VARIOUS FLUX DENSITIES IN THE CASE OF THREE DIFFERENT MATERIALS

When Stalloy punchings are used, their width is usually about 1 in. to 1½ in., and their individual thickness 0·014 in. The punchings are often annealed, to remove "punching strains." The total width of the punching being small, the strained area may be an appreciable part of the total cross-section (Ref. (10)).

The length of magnetic path in the core should be as small as is consistent with good mechanical construction and with insulation requirements, in order to reduce the core reluctance. For the same reason, joints in the core should be avoided as far as possible, and such joints as do exist should be rendered as efficient as possible by careful assembly.

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* See K. Edgcumbe and F. E. Ockenden, *I.E.E. Journal*, Vol. LXV, p. 553.

Fig. 411 shows the three forms of core most commonly used. (*a*) is the rectangular form, built up of L-shaped punchings. The windings are placed on one of the shorter limbs, with the primary usually wound over the secondary, an advantage being that there is ample space for insulation, so that this form is suitable for high-voltage work.

The "shell" form in Fig. 411 (*b*) gives considerable protection to the windings, but is more difficult to build up than the other forms. The windings are placed on the central limb.

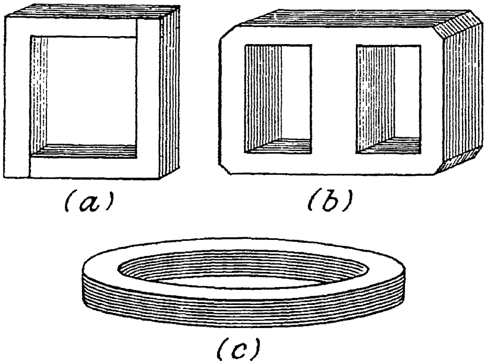


Fig. 411. Forms of Cores

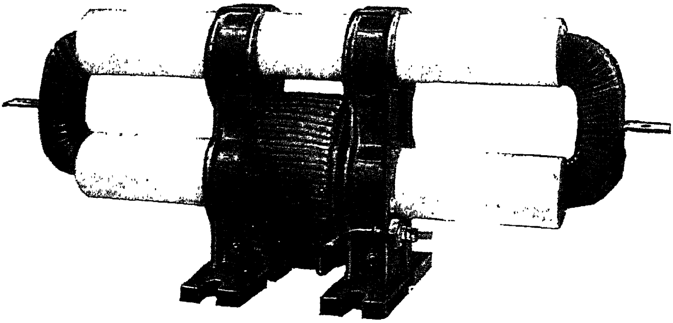
The ring form (*c*) is very commonly used when the primary current is large. The secondary winding is uniformly distributed round the ring, and the primary winding is a single bar. This is a very robust construction, and has the further advantages of a jointless core (giving low reluctance) and of very small leakage reactance.

(*c*) *Windings.* The windings should be close together in order to reduce the secondary leakage reactance, as this increases the ratio error. No. 14 S.W.G. copper wire is frequently used for the secondary winding, copper strip being used for the primary winding, the dimensions of which depend, of course, upon the primary current.

The windings must be designed with a view to their withstanding, without damage, the very large forces which are brought into play when a short-circuit takes place on the system in which the transformer is connected. The bar-primary, ring-core construction is generally recognized as the most satisfactory from this point of view.

(*d*) *Insulation.* The windings are separately wound, and are insulated by tape and varnish for the lower line voltages. For voltages of 7,000 volts and upwards, the transformers are oil-immersed or compound-filled. In the former case the windings are wound over one another on cylinders of Bakelite or similar material, and the transformer is mounted in a sheet steel tank filled with transformer oil. In the latter case the transformer is enclosed in a tank which is filled up solid with an insulating compound. The compound introduces a difficulty in that the cooling is poor. Some high-voltage transformers have porcelain tubes as the insulation for their primary winding and are not oil-immersed. Such a transformer is shown in Fig. 412, and other types of current transformers are shown in Fig. 413(*a*)—(*f*).

DESIGN DATA. The following figures in connection with the design of current transformers do not refer to any particular type, but are intended to give a general idea of the usual magnitudes of the quantities involved.



(*Chamberlain & Hookham, Ltd.*)

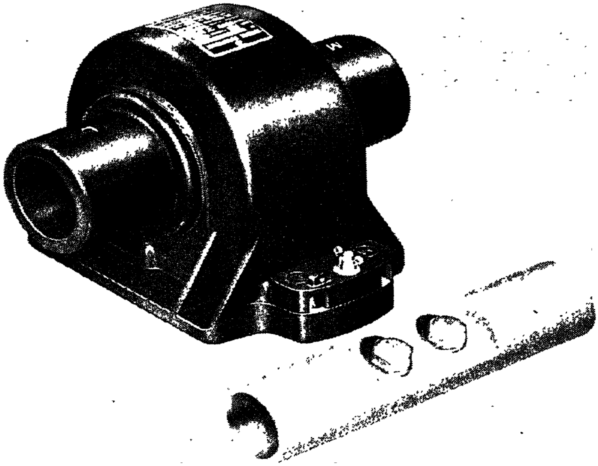
FIG. 412. HIGH VOLTAGE TYPE AIR-COOLED CURRENT TRANSFORMER

Total primary ampere-turns	1,000
Exciting ampere-turns	Not more than 1 per cent of the total primary ampere-turns at full rated current
Full rated secondary current	5 amp.
Core flux density	Not more than 1,000 lines/sq. cm.
Length of mean magnetic path in the core	About 40 cm. (less than this if the line voltage is low)
Current density in primary winding	1 to 2 amp. per sq. mm.
Secondary winding	14 S.W.G. (d.c.c.) copper wire
Resistance of secondary winding	About 0·5 ohm
Inductance of secondary winding	Varies according to type from a negligible quantity (in bar-primary, ring-core type) to about 15 millihenries

(Note that the resistance and inductance of the primary winding are not considered, since they do not affect the performance of the transformer.)

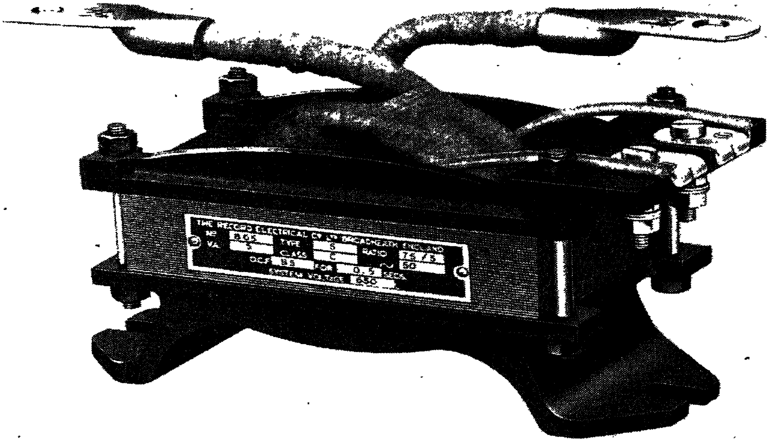
Turns Compensation is used in most current transformers in order to obtain a transformation ratio more nearly equal to the nominal one than would be the case if the "turns ratio" were made equal to the nominal ratio of the transformer. It has been shown that if the phase angle of the secondary circuit is zero, the actual ratio of transformation is given by

$$R = n + \frac{I_e}{I}$$



(*Sangamo Weston, Ltd.*)

(*a*)



(*Record Electrical Co., Ltd.*)

(*b*)

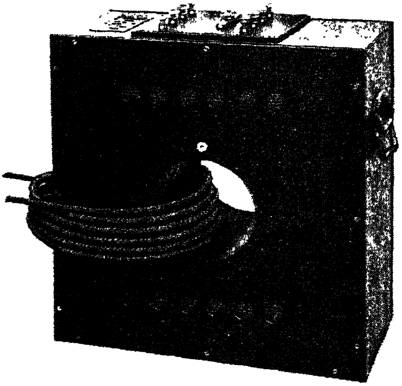
FIG. 413

(*a*) Air-cooled Current Transformer without Primary Winding.

(*b*) Shell-type Current Transformer.

where n is the "turns ratio." The effect of reducing the number of secondary turns by (say) 1 per cent, will obviously be to reduce the transformation ratio by an approximately equal percentage. By this means, partial compensation for the effect of I_e, in increasing the ratio, can be obtained. Usually, the best number of secondary turns is one or two less than that number which would make n equal to the nominal ratio of the transformer. For example, in a 1,000/5 current transformer of the bar-primary type the number of secondary turns would be either 199 or 198 instead of 200.

The phase angle error is very little affected by a change of one or two in the number of secondary turns.



(*Everett Edgcumbe & Co., Ltd.*)

(*c*)

FIG. 413. (*contd.*)

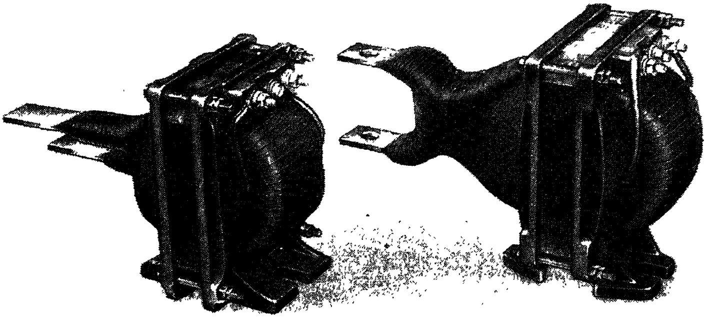
(*c*) Portable Omni-range Current Transformer.

Standard Ratios, Rating and Permissible Errors. British Standards Institution, Specification No. 81 (1936) for "Instrument Transformers," gives a list of twenty-five rated primary currents, ranging from 5 to 6,000 amps., the rated secondary current normally being 5 amps., although it may be 1 amp. or even 0·5 amp. under certain conditions.

The burden of the current transformer is expressed as the output in volt-amperes at the rated secondary current. Thus, for a burden of 15 VA, the rated secondary current being 5 amps., the secondary terminal voltage will be 3 volts and the impedance of the connected load 0·6 ohm. Rated burdens for various classes of current transformer are 1½, 2½, 5, 7½, 15, and 30 VA. The standard frequency is 50 cycles per second. There are nine classes of current transformer mentioned in the specification: Classes A, B, C, and D for use with substandard and commercial instruments of various grades; classes AL and BL for precision testing and laboratory work; and classes AM, BM, and CM for industrial metering.

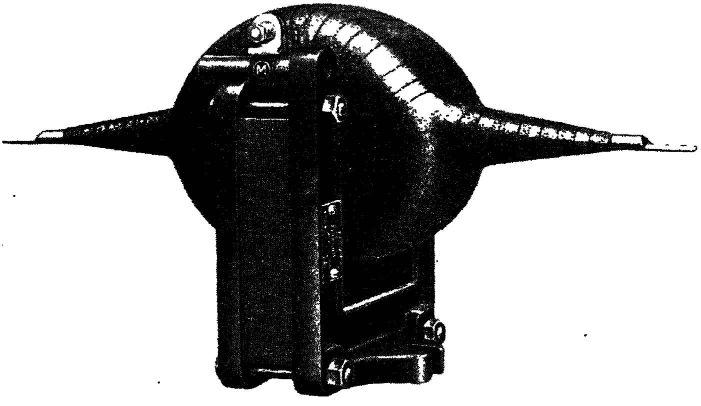
Various standard rated burdens and permissible errors are stated for these classes, the latter varying from 0·15 per cent ratio error and 3 min. phase angle (in the AL class) down to 1 per cent and 120 min. (in class C). No phase-angle error is specified in class D, for which the permissible ratio error is 5 per cent. It should be pointed out that the rating of these transformers is upon an accuracy basis, and is not a thermal rating.

CHARACTERISTICS OF CURRENT TRANSFORMERS. Fig. 414 shows the ratio-error and phase-angle characteristics of a 1,000/5 bar-primary ring-core type of current transformer, with a burden of 15 VA, unity power factor, 50 cycles frequency. These errors are plotted against various values of secondary current. Two curves of ratio error and of phase angle are shown, one pair relating to 199 secondary turns and the other to 198 turns. The errors were



(*Metropolitan Vickers Elec. Co., Ltd.*)

(*d*)



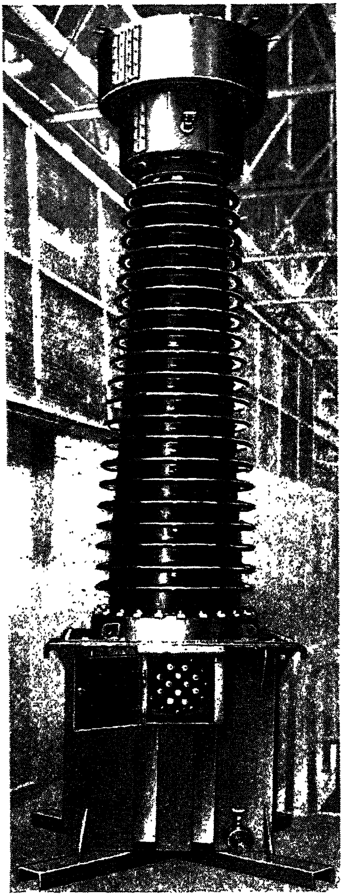
(*Ferranti. Ltd.*)

(*e*)

FIG. 413. (*contd.*)

(*d*) Wound Primary Current Transformer for 6,600 and 11,000 Volts.
(*e*) Heavy-current Type Current Transformer for 6,600 Volts.

calculated for a transformer designed to comply with the B.S.I. specification for a class B transformer. Obviously the 199 turn secondary winding gives the better performance.



(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 413. (*contd.*)
(*f*) Out-door Multiple-secondary Current Transformer for 275 kV.

The shapes of the error curves are characteristic of all current transformers. The "droop" of the ratio error curve and rise of the phase angle curve for low values of secondary current are due to the fact that, when this current is reduced—with consequent proportional reduction of the flux density in the core—the iron-loss and magnetizing ampere-turns are not reduced proportionately, and so form a greater proportion of the total primary ampere-turns than they do for higher values of I_s (and of I_p, since the reduction of the secondary current I_s is produced by a reduction of the primary current I_p).

The small effect of the alteration of the number of secondary turns upon the phase angle error can be seen from these curves. The order of magnitude of the "droop" of the ratio error curve depends upon the value of $\frac{I_e}{I_s}$ compared with the value of the turns ratio n, and the rise of the phase angle curve upon the value of I_m compared with nI_s.

Effect of Variation of Power Factor of the Secondary Burden upon the Performance.

Referring to Fig. 409, reduction of the power factor of the load circuit increases the angle Δ—and hence δ—and thus brings the vectors nI_s and I_o more nearly into phase with one another. This increases the value of I_p for any given value of I_s and thus the transformation ratio $\frac{I_p}{I_s}$ is increased. The ratio error is made less positive as the power factor is reduced, up to the point when I_s is in phase with I_o reversed, when the ratio $\frac{I_p}{I_s}$ will be maximum and the ratio error least positive (or most negative).

The phase angle error is obviously reduced with reduction of power factor, since nI_s moves more into phase with I_0 (Fig. 409) as δ is increased. This reduces the phase angle θ, which becomes zero when $\delta = 90 - \alpha$, nI_s and I_0 then being in phase. Fig. 415 shows the ratio and phase angle curves for the 1,000/5 transformer, to which the curves of Fig. 414 refer, when the secondary load power factor is 0·8 lagging, the burden being 15 VA and the frequency 50 ∼.

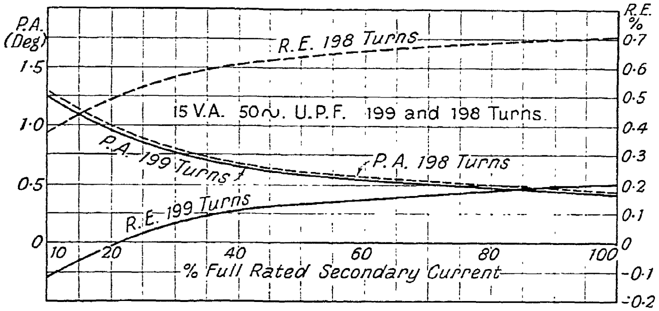


FIG. 414. RATIO ERROR AND PHASE ANGLE CURVES FOR 1,000/5 BAR PRIMARY RING CORE CURRENT TRANSFORMER

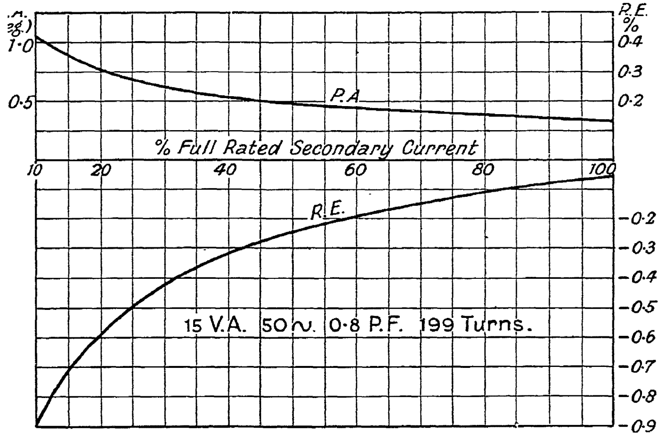


FIG. 415

Fig. 416 shows the effect of variation of secondary load power factor upon the ratio error and phase angle of the same transformer for 10 per cent and 100 per cent full rated secondary current, the burden being 15 VA and frequency 50 ∼.

The ranges of values (between 10 per cent and 100 per cent full rated current) covered by the ratio error and phase angle curves for any given power factor can be obtained by taking the intercepts between the two ratio error or phase angle curves at the given power factor. For example, at 0·6 power factor the ratio error curve has a range of from − 0·15 per cent at 100 per cent full rated current to − 1·20 per cent at 10 per cent full current (from intercept *CD*, Fig. 416). The phase angle ranges from 0·27° (at 100 per cent full rated current) to 0·85° (at 10 per cent full current) as obtained from intercept *AB*.

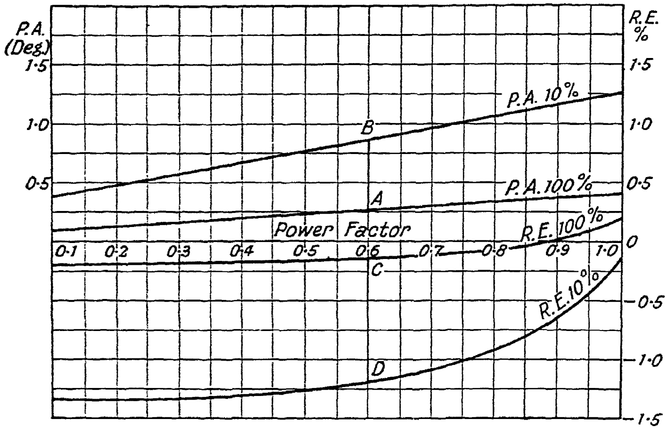


FIG. 416. EFFECT OF VARIATION OF THE SECONDARY LOAD POWER FACTOR

Effect of Variation of the Secondary Burden upon the Performance of a Current Transformer. Increase of the secondary burden (in volt-amperes) obviously necessitates an increase of secondary terminal voltage for a given value of secondary current. This again means an increase in the induced secondary voltage and consequently in the core flux and flux density. The exciting current I_0 (Fig. 409) is thus increased, and this increases the ratio of the transformer, causing the ratio error to become less positive for any given values of power factor and frequency. The phase angle, also, is considerably increased by increase of burden.

The ratio error and phase angle curves for the 1,000/5 current transformer already mentioned, with burdens of 5, 15, and 40 VA, are shown together in Fig. 417 for comparison. Fig. 418 shows the general effect of variation of secondary burden upon the ratio and phase angle errors, the curves referring to the same transformer.

Effect of Frequency upon Performance. A current transformer is seldom called upon to operate at a frequency very different from that for which it was designed (usually 50 ∼), so that the consideration of the effect of frequency variation is less important than that of variation of load and power factor. The variation of the magnetizing and iron-loss ampere-turns per centimetre, with variation of

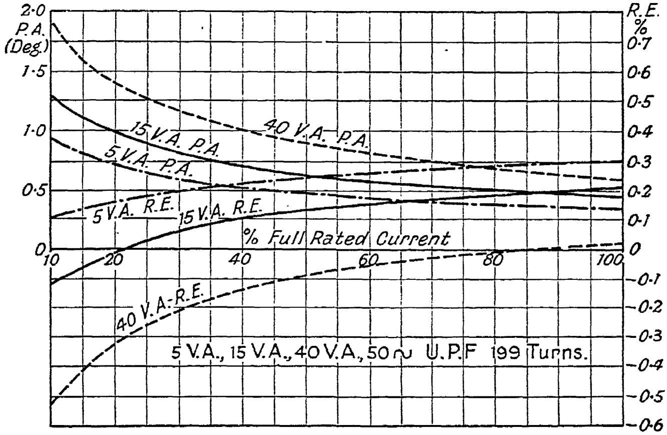


FIG. 417. RATIO ERROR AND PHASE ANGLE CURVES AT DIFFERENT BURDENS

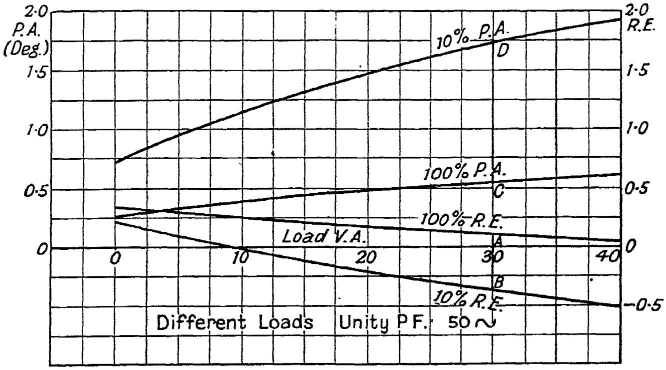


FIG. 418. EFFECT OF VARIATION OF SECONDARY BURDEN

frequency, is comparatively small if the frequency variation does not exceed 10 or 20 cycles per second, so that, *as an approximation*, they may be considered independent of frequency.

For a given volt-amperes load and power factor, the secondary induced voltage is constant. Since this is proportional to the product of frequency and flux density, it follows that an increase of frequency will result in a proportionate decrease in flux density. Thus, in general, the effect of increase of frequency is similar to that produced by a decrease in the impedance of the secondary burden. It is roughly true that doubling the frequency has the same effect upon the performance of the transformer as halving the secondary burden for a fixed frequency.

In addition to the use of a low-loss, high-permeability, nickel-iron core for the reduction of current transformer errors, several special devices have been suggested by various investigators. H. B. Brooks and F. C. Holtz (Ref. (11)), H. W. Price and C. Kent-Duff (Ref. (12)), and J. G. Wellings and C. G. Mayo (Ref. (13)) have all described methods of minimizing these errors.

USE OF CURRENT TRANSFORMERS. It can be seen from the preceding paragraphs that the errors introduced by a current transformer vary with the magnitude and power factor of the secondary burden and with the frequency. It cannot, therefore, be assumed that the errors so introduced are constant, when the transformer is used with ammeters, wattmeters, etc., whose impedances and power factors differ from the impedance and power factor of the load with which the transformer was calibrated. This must be taken into consideration when selecting instruments for use with the transformer. If, however, the transformer and instrument with which it is to be used are calibrated together, these variations in the characteristics of the former are unimportant. Otherwise, if r is the ratio error—expressed as a percentage—the actual ratio of the transformer R is given by

$$R = \frac{N}{\frac{r}{100} + 1} \quad . \quad . \quad . \quad . \quad (392)$$

where N = nominal ratio.

If I_s is the measured value of the secondary current, the current in the circuits to be measured (i.e. the primary current) is

$$\left(\frac{N}{\frac{r}{100} + 1}\right) I_s$$

Correction for the phase angle error is unnecessary when current [illegible] to be measured.

[illegible] OF THE SECONDARY CIRCUIT OF A CURRENT [illegible] In a current transformer the number of primary ampere-turns is a fixed quantity (assuming a constant primary current) and *is thus not reduced, when the secondary circuit is opened,* as it is in the case of a power transformer. If, therefore, a current transformer has its secondary circuit opened when current is flowing in its primary circuit, a very high flux density is produced in the core owing to the absence of "back" ampere-turns due to current in the secondary winding. This high flux density results in a very high induced voltage in the secondary winding, with consequent severe strain on the insulation and danger to the operator (see Ref. (8)).

Moreover, the high magnetizing forces acting upon the core may, if suddenly removed, leave behind considerable residual magnetism in the core, so that the ratio and phase angle errors, obtained after such an open circuit, may differ appreciably from those before it occurred. For these reasons, care must be taken to ensure that, even when not in use for measuring purposes, the secondary circuit is *closed* if the primary current is flowing. The secondary winding may, of course, be short-circuited quite safely, since when used for measurement it is practically short-circuited, the impedance of the burden being merely that of the ammeter or wattmeter current coil used.

It should be noted also that a direct current passed through either of the current transformer windings will produce magnetization of the core which may alter the transformer errors. This is especially the case when a direct current is passed through the secondary winding—as in the measurement of its resistance, for example—since the secondary usually has many more turns than the primary winding. An appreciable number of magnetizing ampere-turns may thus result from the passage of even a small current through the secondary winding.

Demagnetization after an Open Circuit. If the secondary circuit of a current transformer is accidentally opened while the primary current is flowing, the transformer should be demagnetized before being used again. There are several simple ways of carrying out such demagnetization. The following two methods appear to be equally satisfactory in reducing the transformer to its original state of magnetization.

The first method is to pass through the primary winding a current at least equal to that which was passing through it when the open circuit took place, the secondary circuit being left open. (This primary current should be supplied from a motor-alternator set which can be shut down as required.) The motor driving the supply alternator is then shut down with the alternator field still on. As the set slows down, the alternator voltage falls gradually to zero, and the iron of the transformer core is passed through a large number of cycles of magnetization, of gradually decreasing amplitude, finishing at zero magnetization.

Another method is to connect, across the secondary winding of the current transformer, a resistance which is sufficiently high to amount almost to an open circuit as regards the secondary "back ampere-turns" obtained. The full current is passed through the primary winding and the secondary resistance (several hundred ohms) is then gradually reduced to zero as uniformly as possible.

By this means the magnetization of the transformer core is reduced from a very high value to its normal value very gradually.

Upon testing a current transformer, after demagnetization, by either of these methods, the author has found that the alterations of ratio error and phase angle, due to an open circuit, have been entirely eliminated.

Voltage (or Potential) Transformers. The errors introduced by the use of voltage transformers are, in general, less serious than those introduced by current transformers, and the theory of such transformers is essentially the same as that of the ordinary power transformer.

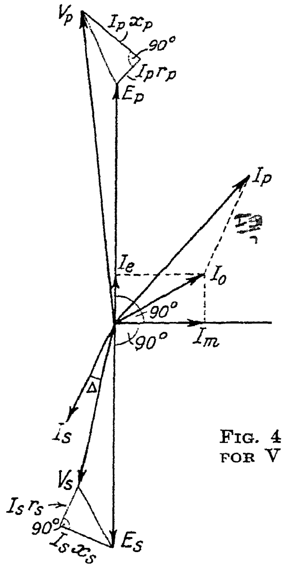


FIG. 419. VECTOR DIAGRAM FOR VOLTAGE TRANSFORMER

The main difference between the voltage transformer and the power transformer is that, in the former, the secondary current is of the same order as that of the magnetizing current of the transformer.

THEORY. Fig. 419 shows the vector diagram of a potential transformer.

ϕ = working flux in the transformer core
I_m = magnetizing component of the no-load current
I_e = iron-loss component of the no-load current
I_0 = no-load current
E_s = voltage *induced* in the secondary winding
V_s = secondary terminal voltage
I_s = secondary current

$I_s r_s$ = voltage drop in secondary winding resistance r_s (in phase with I_s)
$I_s x_s$ = voltage drop in secondary winding reactance x_s (90° in advance of I_s in phase)
I_p = primary current
= vector sum of I_0 and I_s reversed
E_p = that portion of the primary applied voltage which is available for transformation
= primary terminal (or applied) voltage V_p less the voltage drop in impedance of the primary winding
$I_p r_p$ = voltage drop in the resistance of the primary winding
$I_p x_p$ = voltage drop in the reactance of the primary winding
Δ = the phase angle of the secondary load circuit (usually a voltmeter)

Since the resistance of most voltmeters is very much higher than their reactance, Δ will usually be small.

In the vector diagram the magnitudes of the voltage-drop vectors $I_s r_s$, $I_s x_s$, $I_p r_p$, and $I_p x_p$ have been exaggerated for clearness in the drawing.

The secondary terminal voltage V_s is obtained by subtracting (vectorially) the voltage drops in the secondary winding from the voltage E_s induced in the secondary. Again, the voltage available for transformation in the primary—i.e. the component of the primary applied voltage which is in phase opposition to E_s—is obtained by subtracting (vectorially) the voltage drops in the primary winding from the applied primary voltage V_p.

The rest of the diagram is self-explanatory.

Ratio Expression. The expression for the voltage ratio $\frac{V_p}{V_s}$ of the transformer is best derived by considering the diagram shown in Fig. 420, in which all the vectors concerned are referred to the primary side. Secondary voltages, when referred to the primary side, become n times their actual (secondary side) value, n being the "turns ratio" of the transformer.

$$n = \frac{\text{Number of primary turns}}{\text{Number of secondary turns}} = \frac{E_p}{E_s}$$

When secondary currents are referred to the primary side their actual value must be *divided* by n.

Referring to Fig. 420, θ is the phase angle of the transformer—i.e. the phase angle between V_p and V_s reversed. Δ is the phase angle of the secondary load circuit and β the phase angle between V_s reversed and I_p.

Then, projecting V_p on to V_s (reversed) produced, we have

$$\begin{aligned} V_p \cos\theta &= V_s n + I_s r_s n \cos\Delta + I_s x_s n \sin\Delta + I_p r_p \cos\beta + I_p x_p \sin\beta \\ &= V_s n + nI_s(r_s \cos\Delta + x_s \sin\Delta) + I_p(r_p \cos\beta + x_p \sin\beta) \end{aligned}$$

Since θ is usually very small, both V_s (reversed) and V_p may be considered as at right angles to ϕ, so that

$$I_p \cos\beta = I_e + \frac{I_s}{n}\cos\Delta$$

and

$$I_p \sin\beta = I_m + \frac{I_s}{n}\sin\Delta$$

$\cos\theta = 1$ (very nearly) since θ is very small (less than 1°)

$$\therefore \quad V_p = V_s n + nI_s(r_s \cos\Delta + x_s \sin\Delta) + r_p\left(I_e + \frac{I_s}{n}\cos\Delta\right) + x_p\left(I_m + \frac{I_s}{n}\sin\Delta\right)$$

$$V_p = nV_s + I_s \cos \Delta \left(nr_s + \frac{r_p}{n}\right) + I_s \sin \Delta \left(nx_s + \frac{x_p}{n}\right) + I_e r_p + I_m x_p$$

$$V_p = nV_s + \frac{I_s}{n} \cos \Delta (n^2 r_s + r_p) + \frac{I_s \sin \Delta}{n} (n^2 x_s + x_p) + I_e r_p + I_m x_p$$

∴ The voltage ratio

$$\frac{V_p}{V_s} = n + \frac{\frac{I_s}{n}[R_p \cos \Delta + X_p \sin \Delta] + I_e r_p + I_m x_p}{V_s} \quad . \quad . \quad (393)$$

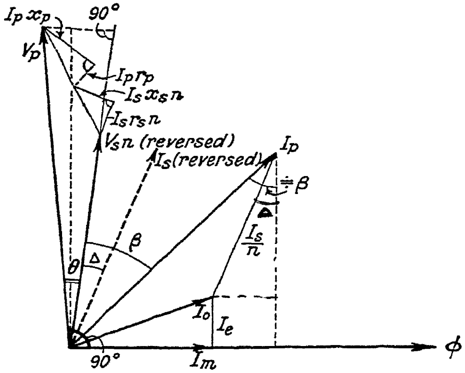


FIG. 420

where R_p = equivalent resistance of the transformer *referred to the primary*
and X_p = equivalent reactance of the transformer *referred to the primary*

Otherwise,

$$V_p = nV_s + nI_s \cos \Delta \left(r_s + \frac{r_p}{n^2}\right) + nI_s \sin \Delta \left(x_s + \frac{x_p}{n^2}\right) + I_e r_p + I_m x_p$$

$$\text{or} \quad \frac{V_p}{V_s} = n + \frac{nI_s[R_s \cos \Delta + X_s \sin \Delta] + I_e r_p + I_m x_p}{V_s} \quad . \quad (394)$$

where R_s and X_s are the equivalent resistance and reactance of the transformer *referred to the secondary.*

The difference between the actual voltage ratio and the turns ratio n is thus

$$\frac{\frac{I_s}{n}[R_p \cos \Delta + X_p \sin \Delta] + I_e r_p + I_m x_p}{V_s}$$

$$\text{or} \quad \frac{nI_s[R_s \cos \Delta + X_s \sin \Delta] + I_e r_p + I_m x_p}{V_s} \quad . \quad . \quad (395)$$

Phase Angle. From Fig. 420,

$$\sin\theta = \frac{I_p x_p \cos\beta - I_p r_p \sin\beta + I_s x_s n \cos\Delta - I_s r_s n \sin\Delta}{V_p}$$

$$= \theta \text{ (radians)} \quad \text{(very nearly, since } \theta \text{ is small)}$$

Otherwise,

$$\tan\theta = \frac{I_p x_p \cos\beta - I_p r_p \sin\beta + I_s x_s n \cos\Delta - I_s r_s n \sin\Delta}{V_s n + n I_s r_s \cos\Delta + I_s x_s n \sin\Delta + I_p r_p \cos\beta + I_p x_p \sin\beta}$$

$$= \frac{I_p x_p \cos\beta - I_p r_p \sin\beta + I_s x_s n \cos\Delta - I_s r_s n \sin\Delta}{V_s n}$$

since the terms in the denominator involving I_s and I_p are small compared with $V_s n$, or

$$\tan\theta = \frac{x_p\left(I_e + \frac{I_s}{n}\cos\Delta\right) - r_p\left(I_m + \frac{I_s}{n}\sin\Delta\right) + I_s x_s n \cos\Delta - I_s r_s n \sin\Delta}{V_s n}$$

$$= \frac{I_s \cos\Delta\left(\frac{x_p}{n} + x_s n\right) - I_s \sin\Delta\left(\frac{r_p}{n} + r_s n\right) + I_e x_p - I_m r_p}{V_s n}$$

Thus

$$\tan\theta = \frac{\frac{I_s \cos\Delta}{n}(x_p + x_s n^2) - \frac{I_s \sin\Delta}{n}(r_p + r_s n^2) + I_e x_p - I_m r_p}{V_s n} \quad . \quad (396)$$

Thus, since $\tan\theta = \theta$ (radians) very nearly, since θ is small,

$$\theta_{(radians)} = \frac{\frac{I_s \cos\Delta}{n} X_p - \frac{I_s \sin\Delta}{n} R_p + I_e x_p - I_m r_p}{V_s n}$$

$$\text{or,} \quad \theta_{(radians)} = \frac{n I_s \cos\Delta\, X_s - n I_s \sin\Delta\, R_s + I_e x_p - I_m r_p}{V_s n}$$

$$= \frac{I_s}{V_s}(X_s \cos\Delta - R_s \sin\Delta) + \frac{I_e x_p - I_m r_p}{V_s n} \quad . \quad (397)$$

where R_p and X_p are the equivalent resistance and reactance referred to the primary, and R_s and X_s the same referred to the secondary.

ERRORS INTRODUCED BY VOLTAGE TRANSFORMERS. It can be seen from the above that, like current transformers, voltage transformers introduce an error, both of magnitude and of phase, in the measured value of the voltage. The "ratio" error only is important when measurements of voltage are to be made, the "phase angle" error being of importance only when power is being measured.

Obviously, the voltage applied to the primary circuit of the transformer cannot be obtained correctly simply by multiplying the voltage measured by the voltmeter in the secondary circuit by the "turns ratio" n of the transformer.

The divergence of the actual ratio $\frac{V_p}{V_s}$ from n depends upon the reactance and resistance of the transformer windings as well as upon the value of the exciting current of the transformer.

The phase angle error depends largely upon the same factors.

DESIGN CONSIDERATIONS. In order to reduce the errors introduced by these transformers, the reluctance of the transformer core should be as small as possible, and the flux density in the core should be fairly low (under 10,000). The exciting current can then be made small.

Since the reactance of the transformer windings depends upon the magnitudes of the primary and secondary leakage magnetic fluxes, these should be kept small by placing the two windings as close together as is consistent with insulation requirements. The resistance of the windings should also be as small as possible.

Turns compensation can be applied in the voltage transformer even more successfully than in the current transformer owing to the large number of primary turns, the difference in ratio caused by the addition or removal of one primary turn being thus comparatively small. Since the actual ratio $\frac{V_p}{V_s}$ of the transformer is greater than the turns ratio n, this actual ratio can be made more nearly equal to the nominal ratio by adjusting the numbers of primary and secondary turns so that n is *less* than the nominal ratio.

As regards insulation, voltage transformers for voltages of 7,000 volts and over are oil-immersed. Adequate insulation must be provided, in the form of porcelain or other bushings, for the primary terminals, since, unlike the case of the current transformer, the full line voltage is applied to the primary winding.

The load on such transformers is usually very small, consisting as it does merely of a voltmeter and/or the pressure coil of a wattmeter, so that there is little heat produced in the transformer.

Fig. 421 shows the construction of some types of voltage transformers.

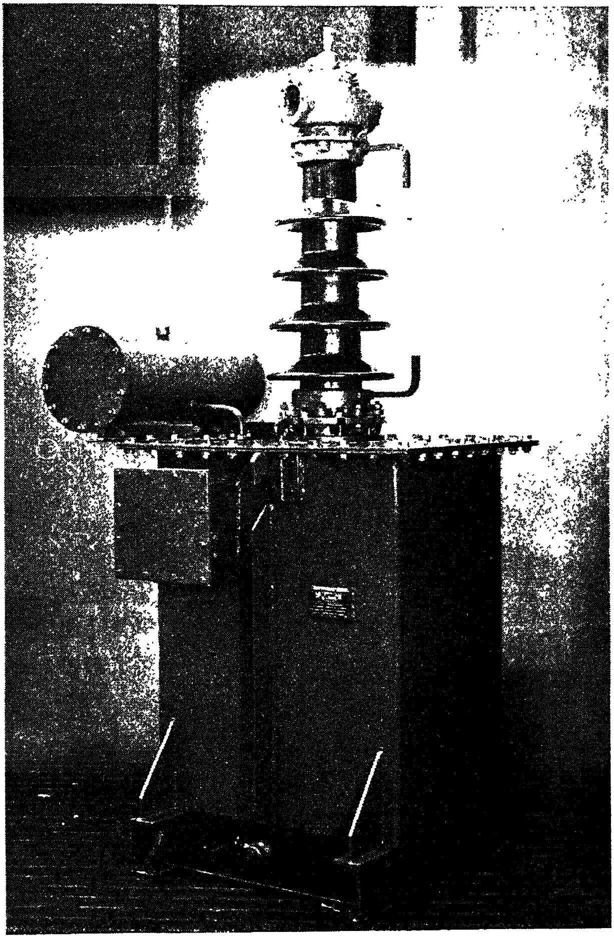
Standard Ratio and Permissible Errors. B.S.I. Specification No. 81 (1936), already referred to, specifies a secondary voltage of 110 volts when the rated primary voltage is applied to the transformer.

The standard ratios of transformation are as follows—

TABLE XXIII

Primary Volts	Secondary Volts	Primary Volts	Secondary Volts
110	110	33,000	110
440	,,	44,000	,,
550	,,	55,000	,,
660	,,	66,000	,,
2,200	,,	88,000	,,
3,300	,,	110,000	,,
5,500	,,	132,000	,,
6,600	,,	165,000	,,
11,000	,,	220,000	,,
22,000	,,		

These voltages apply when the transformer is connected between lines. If connected between line and neutral, the voltages should be divided by $\sqrt{3}$.



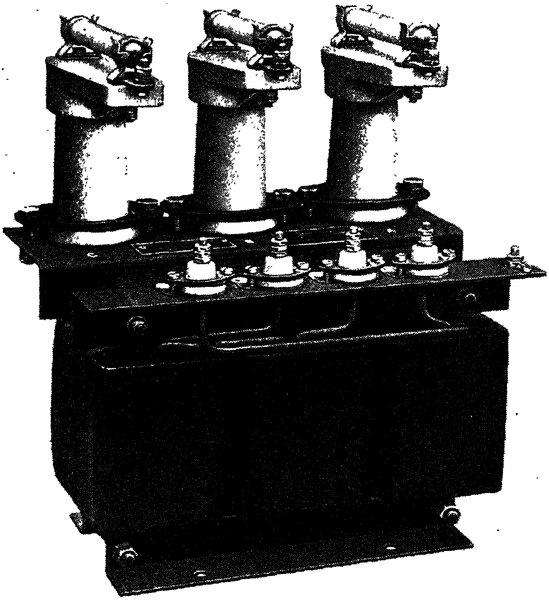
(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 421A. OUTDOOR TYPE VOLTAGE TRANSFORMER, RATIO $\dfrac{66{,}000}{\sqrt{3}} \Big/ \dfrac{110}{\sqrt{3}}$

Six classes of potential or voltage transformers are mentioned in the specification: A, B, C, D, and AL and BL. The rated burdens range from 15 to 200 VA for classes A, B, and C (no standard burden for class D); while for classes AL and BL 10 VA burden is specified. Limits of error range from

0·5 per cent ratio error and 20 min. phase angle to 2 per cent (5 per cent, class D), and 30 min. for classes A, B, and C. For class AL, 0·25 per cent ratio error and 10 min. phase angle; and for class BL, 0·5 per cent and 20 min. are permissible.

CHARACTERISTICS OF VOLTAGE TRANSFORMERS. Fig. 422 shows typical ratio and phase angle curves for a voltage transformer, the errors being plotted against values of secondary voltage. It can be seen that there is comparatively little change in these errors with change of voltage. The secondary terminal voltage on no load will, of course, be the same as the secondary induced voltage. As the applied primary voltage is reduced, the flux density in the core will be correspondingly reduced, involving a reduction (although not quite proportional) in I_0 (see Fig. 419). This, again, means a reduction in the voltage drops in the primary winding. If the two components, I_m and I_e, of I_0 were reduced in exact proportion as ϕ and E_s are reduced, the effect would be merely that of reducing the whole of the primary voltage triangle to scale and the ratio would remain constant. Actually, owing to the disproportionality of I_m and I_e, the ratio changes very slightly as the voltage is reduced.



(*Ferranti, Ltd.*)

FIG. 421B. OPEN TYPE, AIR-COOLED VOLTAGE TRANSFORMER

The same considerations apply when the transformer is on load, except that some modification is necessary on account of the existence of the secondary current vector I_s.

The reason for the phase angle being small under no-load conditions is that the vector $I_p x_p$ representing the primary reactance volt-drop—which vector is chiefly responsible for the phase displacement—is then perpendicular to the direction of I_0, and is thus more nearly in phase with E_p than when a secondary current exists (see Fig. 419).

Effect of Variation of Secondary Burden. Considering the case of full-rated voltage applied to the primary winding, then the secondary voltage will be approximately the same for all values of secondary volt-amperes burden. Increase of this burden will thus mean a proportional increase in the secondary current. Thus, if the burden is 15 VA, with full-rated voltage applied to the transformer, the secondary current will be $\frac{15}{110} = 0{\cdot}136$ amp. (assuming that the rated secondary voltage is 110 volts). If the burden is increased to 50 VA with the same applied voltage, the secondary current will be $\frac{50}{110} = 0{\cdot}45$ amp.

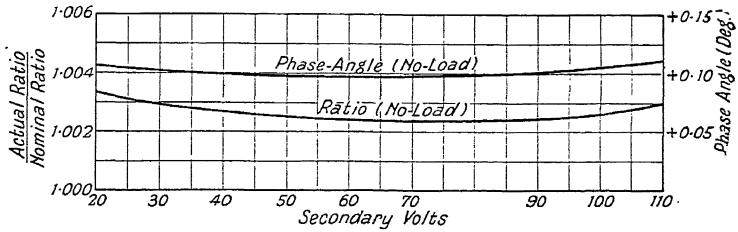


FIG. 422. POTENTIAL TRANSFORMER RATIO AND PHASE ANGLE CURVES

Thus, with increased burden I_s is increased, and hence the primary current I_p is increased. I_0 is reduced slightly, but its variation is unimportant. Both the primary and secondary voltage-drops are increased, and thus, for a given value of V_p (Fig. 419), E_p, E_s, and V_s are reduced by increase of burden. The effect is therefore to increase the actual ratio $\frac{V_p}{V_s}$ of the transformer as the burden increases.

With regard to the phase angle, the voltage V_p is advanced in phase relative to the flux ϕ (or, more strictly, ϕ is retarded in phase relative to V_p) as the secondary burden increases, owing to the increased voltage-drops and to the phase advance of I_p as I_s increases. Again, V_s is retarded in phase, relative to ϕ, on account of the increased secondary voltage drops. Hence, the phase angle is increased (lagging) as the burden increases.

Effect of Power Factor of Secondary Burden. As the power factor of the secondary burden is reduced—i.e. angle Δ (Fig. 419) is increased—I_p becomes more nearly in phase with I_0. The voltages V_p and V_s move more nearly into phase with E_p and E_s respectively, and since the voltage drops in the windings, for a given volt-amperes load, are very little influenced by the power factor of the load, the result is an increase in V_p relative to E_p (or, strictly, a reduction of E_p relative to V_p), and a reduction of V_s relative to E_s. Hence, the ratio of the transformer increases as the power factor of the burden decreases.

Since V_s is advanced in phase and V_p retarded in phase, thereby the phase angle of the transformer is reduced (lagging) or increased (leading).

Effect of Frequency. For a given applied voltage, reduction of frequency results in an increase in the core flux, with a corresponding increase in the exciting current I_0. Since the exciting current does not influence the transformer errors very seriously, the effects of variation of frequency are not so great as in the case of current transformers. The reactance volt-drops in the windings are also proportional to frequency.

Thus, the results of a reduction of frequency are, in the case of ratio error, dependent upon the relative values of I_0 and the reactances of the windings, since increase in I_0 tends to increase the ratio, whilst the reduction of the reactance volt-drops tends to reduce it.

As regards phase angle error, both effects retard V_p in phase relative to ϕ, and the reduction of secondary reactance advances V_s in phase. Thus the phase angle is reduced (lagging) as the frequency is reduced.

W. Gray and A. Wright (Ref. (42)), in a very informative paper, discuss the use of voltage and current transformers with switchgear on h.v. circuits. They conclude that capacitor voltage transformers —consisting of a tuned capacitor-reactor divider network together with an electromagnetic voltage transformer—are uneconomical for systems below 110 kV. These authors deal also with the transient performance of current transformers and with details of construction required for high-voltage operation.

A. H. M. Arnold (Ref. (44)) stresses the very low errors introduced by current transformers having nickel-iron cores. He states that voltage transformers for 14 kV have been built with a ratio error of only 0·01 per cent and a phase-angle error of less than 1 minute.

Tes Transformers. (*a*) CURRENT TRANS STING. Since it is often necessary, in the precision testing of current transformers, to determine the ratio to within $\frac{1}{10}$ of 1 per cent and the phase angle to within $\frac{1}{20}$ of 1 degree, special apparatus is necessary for such purposes.

These testing methods may be divided into two classes: (*a*) and (*b*) secondary, or . In the absolute methods the transformer errors are determined in terms of the constants—resistance, inductance, and capacitance—of the testing circuits; whereas, in the secondary methods, the errors of the transformer under test are compared with those of a standard current transformer. Two absolute methods, and three secondary methods are given below.

References to other methods are given at the end of the chapter.

Method. This method has been used, with modifications, by Sharp and Crawford, Agnew and Silsbee, and others. It has been developed largely by the American Bureau of Standards. The essential connections are given in Fig. 423.

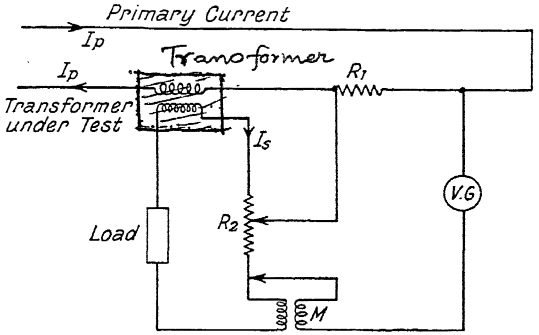


FIG. 423. MUTUAL INDUCTANCE METHOD OF CURRENT TRANSFORMER TESTING

R_1 and R_2 are low-resistance, non-inductive shunts, the latter having a slide wire in series with it for fine adjustment. R_1 is of fixed value. Neglecting, for the moment, the question of phase, the vibration galvanometer will indicate zero deflection when the voltage-drop I_pR_1 is equal (and opposite) to the voltage-drop I_sR_2. Thus, R_1 and R_2 must be chosen so that $\frac{R_2}{R_1}$ is approximately equal to the nominal ratio of the current transformer. Actually, as stated above, R_1 is suitably chosen to carry the full primary current, and R_2 is adjusted as required to render the two voltage-drops equal. In addition, it is necessary, in order to obtain zero galvanometer deflection, to adjust the mutual inductance M, on account of the phase displacement between the currents I_p and I_s.

Theory. Referring to the vector diagram of Fig. 424, I_p and I_s are vectors representing the primary and secondary currents of the transformer respectively. I_pR_1 and I_sR_2 are the voltage-drops across R_1 and R_2, and are in phase with these currents. The resultant voltage e (vector sum of I_pR_1 and I_sR_2) would exist in the galvanometer circuit if the value of the mutual inductance M were zero. The vector $I_s\omega M$ represents the voltage induced in the secondary of M (which is air-cored) by the current I_s. Obviously this vector lags in phase 90° behind I_s. When M and R_2 are such that the vectors e and $I_s\omega M$ are equal, and opposite in phase, no current flows through the galvanometer.

Then, from the vector diagram, the phase angle θ of the transformer is given by

$$\tan\theta = \frac{e}{I_pR} = \frac{I_s\omega M}{} = \frac{\omega M}{} \qquad (398)$$

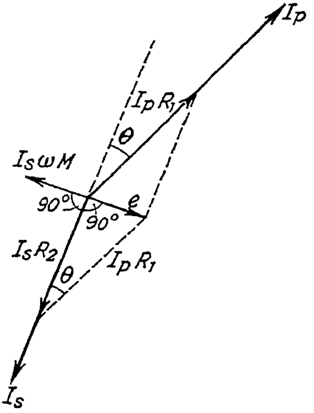


FIG. 424

Also,
$$\frac{I_sR_2}{I_pR_1} = \cos\theta$$

or Transformer Ratio
$$= \frac{I_p}{I_s} = \frac{R_2}{R_1\cos\theta} = \frac{R_2}{R_1} \text{ (very nearly)} \quad . \quad (399)$$

since θ is usually very small.

Obviously a measurement of the supply frequency is needed for the determination of the phase angle. It should be noted, also, that the impedance of the secondary load circuit includes the resistance R_2, and the impedance of the primary of M, as well as the impedance marked "load." This must be taken into account in stating the burden to which the transformer errors measured apply. R_1

and R_2 are assumed, in the above theory, to be entirely non-inductive.

The Leeds & Northrup Co. manufacture a set of apparatus for this method of testing. The mutual inductance is of the Brooks and Weaver type, and a range of low-resistance, non-inductive shunts is supplied for use, as required, in the primary circuit. The current-range covered is from 30 to 1,000 amp.

The makers claim for this apparatus that it will give ratio values which are accurate to within $\frac{1}{10}$ of 1 per cent, and phase angle values to within 5 min.

Biffi Method. This method, described initially by E. Biffi (Ref. (22)), is a very convenient one, since the apparatus required is of a simple character. W. I. Place (Ref. (23)) has investigated the method, and has given a full account of its operation and of the requirements of the apparatus for general current-transformer testing.

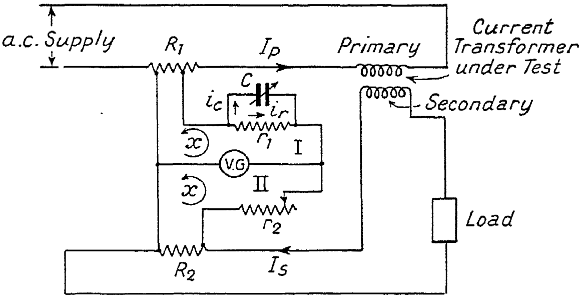


FIG. 425. BIFFI METHOD OF TESTING CURRENT TRANSFORMERS

Fig. 425 shows the connections of the apparatus for this method. r_1 and r_2 are non-inductive resistances, both being of the order of 100 to 400 ohms, and the latter being variable. C is a variable capacitor of about 1 microfarad, shunting r_1. R_1 and R_2 are non-inductive shunts, in the primary and secondary circuits of the current transformer respectively, of such magnitudes that the voltage-drops across them are approximately equal and are about 0·5 or 0·75 volt. V.G. is a vibration galvanometer. The arrangement forms what is essentially a bridge network.

The method of procedure in testing a current transformer, after having tuned the vibration galvanometer, is to adjust r_2 and C until no deflection of the galvanometer is observed. The values of the transformer ratio and phase angle, under the conditions of load and frequency existing during the test, are found by substituting, in the expressions given below, the values of r_2 and C for balance. The supply frequency also should be measured for use in the expression for phase angle.

Ratio Expression

$$\frac{I_p}{I_s} = \frac{R_2(R_1 + r_1)}{R_1(R_2 + r_2)\sqrt{1 + \omega^2 r_1^2 C^2}} \quad . \quad . \quad . \quad (400)$$

$\omega = 2\pi \times$ frequency.

Phase Angle

$$\tan\theta = \omega C r_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (401)$$

$$= \theta \text{ radians (very nearly)}$$

where θ is the phase angle of the transformer.

Theory. Let x and x be the two mesh currents in the two circuits of the network under balanced conditions (galvanometer current then equals $x - x = 0$).

Then, in mesh I,

$$R_1(x - I_p) + x\left(\frac{r_1}{A} - \frac{j\omega C r_1^2}{A}\right) = 0 \quad . \quad . \quad . \quad . \quad (402)$$

where $A = 1 + \omega^2 r_1^2 C^2$.

NOTE. For r_1 and C in parallel we have,

$$\text{total impedance} = \frac{r_1}{1 + j\omega C r_1}$$

Rationalizing, we have,

$$\text{impedance} = \frac{r_1(1 - j\omega C r_1)}{1 + \omega^2 C^2 r_1^2} = \frac{r_1}{A} - \frac{j\omega C r_1^2}{A}$$

In mesh II,

$$R_2(x - I_s) + x r_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (403)$$

I_p and I_s are the primary and secondary currents of the transformer under test.

From the above equations we have

$$\frac{I_p}{I_s} = \frac{R_2}{R_1}\left[\frac{R_1 + \frac{r_1}{A} - \frac{j\omega C r_1^2}{A}}{R_2 + r_2}\right]$$

or, converting to scalar quantities from the symbolic, we have

$$\frac{I_p}{I_s} = \frac{R_2[(R_1 A + r_1)^2 + \omega^2 C^2 r_1^4]^{\frac{1}{2}}}{R_1(R_2 + r_2)A}$$

$$= \frac{R_2[R_1^2(1 + \omega^2 C^2 r_1^2)^2 + 2R_1 r_1(1 + \omega^2 C^2 r_1^2) + \omega^2 C^2 r_1^4 + r_1^2]}{R_1(R_2 + r_2)(1 + \omega^2 C^2 r_1^2)}$$

$$= \frac{R_2[(1 + \omega^2 C^2 r_1^2)\{R_1^2(1 + \omega^2 C^2 r_1^2) + 2R_1 r_1 + r_1^2\}]^{\frac{1}{2}}}{R_1(R_2 + r_2)(1 + \omega^2 C^2 r_1^2)}$$

or

$$\frac{I_p}{I_s} = \frac{R_2(R_1 + r_1)}{R_1(R_2 + r_2)\sqrt{1 + \omega^2 C^2 r_1^2}} \text{ (very nearly)} \quad (404)$$

since $\omega^2 C^2 r_1^2$ is very small compared with unity.

The *phase angle expression* is most easily derived from the vector diagram of Fig. 426, which refers to balance conditions. In this diagram, the magnitudes of x and of the phase angle θ of the transformer have been exaggerated for clearness. Since, for balance $(I_s - x)R_2$ must be equal, both in magnitude and phase, to xr_2, it follows that the current x in mesh II is in phase with I_s. At balance also we have that $i_r r_1 = (I_p - x)R_1$, the subtraction of x from I_p being now vectorial, since I_p and x are not quite in phase (actually they are displaced from one another in phase by the phase angle θ). i_r is the component of the current x which passes through the resistance r_{1r}, i_c being the component of x passing through the capacitor C. Since x is usually quite negligible compared with I_p, its vectorial subtraction from I_p has been omitted in the vector diagram, the value I_p being used instead of the vector difference $(I_p - x)$.

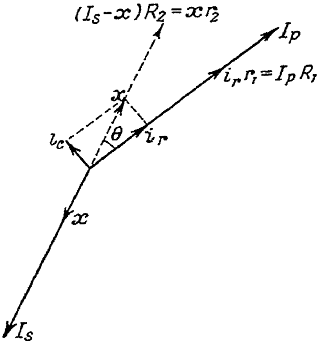


FIG. 426

We have, then,

$$\tan \theta = \frac{i_c}{\cdot}$$

Now, $$\frac{I_p R_1}{r_1}$$

and $$\frac{\frac{I_p R_1}{1}}{\omega C} = I_p R_1 \omega C$$

$$\tan \theta = \frac{I_p R_1 \omega C}{\underline{I_p R_1}} = \omega C r_1 \qquad . \quad (405)$$

Example. In order to obtain an idea of the quantities involved, suppose a current transformer whose nominal ratio is 300/5 is under test, with a supply frequency of 50 cycles per second. Let r_2 be 400 ohms, and R_2 be 0·15 ohm.

Then, at full rated current, from equation (403),

$$0{\cdot}15(x - 5) + 400x = 0$$

or $$x = 0{\cdot}002 \text{ amp.}$$

Obviously this is quite negligible compared with 300 amp. ($= I_p$), so that no appreciable error is introduced by taking the current through R_1 as equal to I_p as above. The value of R_1 to give a voltage drop equal to that across R_2 (namely, 0·75 volts) at full rated current would be 0·0025 ohm.

Again, assuming a phase angle of 1° in the current transformer, and supposing the value of C for balance to be exactly $\frac{1}{2}$ microfarad, we have

$$\tan 1^\circ = 2\pi \times 50 \times \frac{0{\cdot}5}{10^6} \times r_1$$

from which $$r_1 = 111 \text{ ohms}$$

Silsbee Method. Unlike the two methods described above, this is a secondary, or comparison, method. The ratio error and phase angle of a current transformer under test are obtained by comparison with the constants of a standard transformer of the same nominal ratio as the one to be tested. Such comparisons may be carried out very accurately, and without a large amount of expensive and complicated apparatus. The method is exceedingly useful, especially for such purposes as workshop testing of the constants of current transformers during manufacture.

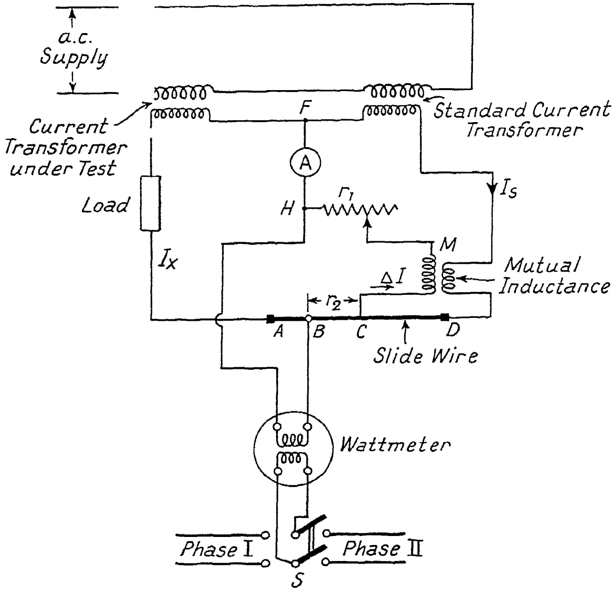


FIG. 427. SILSBEE METHOD OF TESTING

Fig. 427 gives a diagram of connections as given originally by F. B. Silsbee (Ref. (24)).

The transformer under test, and a standard transformer of the same nominal ratio, are connected as shown, with their primary windings in series and their secondaries also in series. The polarities of the secondary windings should be such that the ammeter *A* (10 amp.) indicates only a very small current when the full primary current is supplied to the two transformers. An impedance marked "Load" is inserted to bring the impedance of the secondary circuit of the test transformer up to the burden with which the performance of the transformer is required. An ammeter, the primary winding of the mutual inductance *M*, and a slide-wire resistance, are connected in the secondary circuit of the standard transformer. A dynamometer wattmeter is used as the detector, its pressure coil being supplied from one or other of two a.c. supplies—which are in quadrature—through the double-throw switch S. The current coil of the wattmeter is connected in parallel with a variable non-inductive resistance r_1, the secondary winding of the mutual inductance M, and a portion of the slide-wire resistance r_2.

In carrying out a test r_1, r_2, and M are adjusted until the wattmeter gives no deflection for either of the two positions of the switch S.

Then, when the sliding contact C is to the right of B,

$$\begin{aligned}\frac{R_x}{R_s} &= 1 + \frac{r_2}{r_1} - \frac{\omega^2M^2}{2r_1^2} - \frac{\omega^2ML_1}{r_1^2}\\ &= 1 + \frac{r_2}{r_1} - \frac{\omega^2M}{r_1^2}\left(\frac{M}{2} + L_1\right) \quad . \quad . \quad . \quad . \quad (406)\end{aligned}$$

where R_x = ratio of the transformer under test

R_s = ratio of the standard transformer

$\omega = 2\pi \times$ frequency

M = the setting of the mutual inductance for zero wattmeter deflection

r_1 = total resistance C to H, through M

r_2 = resistance of slide-wire B to C

L_1 = inductance of the secondary coil of M

Also,

$$\tan(\alpha_x - \alpha_s) = \frac{\omega M}{r_1} + \frac{\omega L_1 r_2}{r_1^2} \quad . \quad . \quad . \quad (407)$$

where α_x = phase angle of transformer under test

α_s = phase angle of standard transformer

If the setting of C for zero wattmeter deflection is to the left of B, then

$$\frac{R_x}{R_s} = 1 - \frac{r_2}{r_1} + \frac{r_2^2}{r_1^2} - \frac{\omega^2M^2}{2r_1^2} - \frac{\omega^2ML_1}{r_1^2} \quad . \quad . \quad (408)$$

and
$$\tan(\alpha_x - \alpha_s) = \frac{\omega M}{r_1} - \frac{\omega L_1 r_2}{r_1^2} - \frac{\omega M r_2}{r_1^2} \quad . \quad . \quad . \quad . \quad (409)$$

Silsbee gives the resistance r_1 as about 30 ohms, the total slide-wire resistance about 0·2 ohm, and the mutual inductance M about 600 microhenries. These need only be calibrated with the accuracy to which it is required to measure the *difference* between R_x and R_s (about 1 per cent).

Theory. Let the vector difference between the secondary current I_x of the transformer under test and the secondary current I_s of the standard be ΔI. This current flows through the branch containing r_1 and M when zero wattmeter deflection is obtained.

Substituting for ΔI, we have,

$$I_s(r_1 + j\omega L_1 + j\omega M) = I_x(r_1 + r_2 + j\omega L_1)$$

If α_s and α_x are the phase angles of the two transformers,

$$R_s = \frac{I_p}{I_s}(\cos\alpha_s + j\sin\alpha_s)$$

and $$R_x = \frac{I_p}{I_x}(\cos\alpha_x + j\sin\alpha_x)$$

where I_p is the common primary current.

Then $$\frac{R_x}{R_s} = \frac{I_s(\cos\alpha_x + j\sin\alpha_x)}{I_x(\cos\alpha_s + j\sin\alpha_s)}$$

or $$\frac{I_s}{I_x} = \frac{R_x(\cos\alpha_s + j\sin\alpha_s)}{R_s(\cos\alpha_x + j\sin\alpha_x)}$$

Multiplying numerator and denominator of the right-hand side by $(\cos\alpha_x - j\sin\alpha_x)$ we have,

$$\frac{I_s}{I_x} = \frac{R_x(\cos\alpha_s + j\sin\alpha_s)(\cos\alpha_x - j\sin\alpha_x)}{R_s(\cos^2\alpha_x + \sin^2\alpha_x)}$$

$$= \frac{R_x}{R_s}(\cos\alpha_s\cos\alpha_x + \sin\alpha_s\sin\alpha_x + j\sin\alpha_s\cos\alpha_x - j\cos\alpha_s\sin\alpha_x)$$

$$= \frac{R_x}{R_s}[\cos(\alpha_s - \alpha_x) + j\sin(\alpha_s - \alpha_x)]$$

But $$\frac{I_s}{I_x} = \frac{r_1 + r_2 + j\omega L_1}{r_1 + j\omega L_1 + j\omega M} = \frac{1 + \frac{r_2}{r_1} + \frac{j\omega L_1}{r_1}}{1 + \frac{j\omega L_1}{r_1} + \frac{j\omega M}{r_1}}$$

$$= \frac{1 + a + jc}{1 + j(b + c)} \text{ where } a = \frac{r_2}{r_1},\ b = \frac{\omega M}{r_1}, \text{ and } c = \frac{\omega L_1}{r_1}$$

or $$\frac{I_s}{I_x} = 1 + a - bc - b^2 \ldots + j(-b - ac - ab \ldots)$$

The succeeding terms are negligibly small

$$\therefore \frac{R_x}{R_s}[\cos(\alpha_s - \alpha_x) + j\sin(\alpha_s - \alpha_x)]$$
$$= 1 + a - bc - b^2 + j(-b - ac - ab)$$

Equating real and imaginary quantities, we have,

$$\frac{R_x}{R_s}\cos(\alpha_s - \alpha_x) = 1 + a - bc - b^2$$

and $$\frac{R_x}{R_s}\sin(\alpha_s - \alpha_x) = -b - ac - ab$$

From which we have

$$\frac{R_x}{R_s} = \sqrt{(1 + a - bc - b^2)^2 + (-b - ac - ab)^2}$$

$$= 1 + a - \frac{b^2}{2} - bc$$

By division, also,

$$\tan(\alpha_s - \alpha_x) = \frac{-b - ac - ab}{1 + a - bc - b^2} = b + ac \ldots$$

Since the second-order terms are usually small, it is sufficiently accurate for most testing purposes (to $\frac{1}{10}$ per cent) to write

$$\frac{R_x}{R_s} = 1 \pm a = 1 \pm \frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad . \quad (410)$$

and $$a_x - a_s = b = \frac{\omega M}{r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (411)$$

Arnold Method. The Silsbee method is not sufficiently sensitive for tests upon the more accurate modern current transformers such as those of the AL type. A. H. M. Arnold has described (Ref. (38)) an appropriate method for this purpose, developed and used at the National Physical Laboratory, which is essentially a modification of the Silsbee method. The circuit is shown in Fig. 428, which also gives the vector diagram for the network under balance conditions. Details of the circuit are as follows.

S and *X* are the standard and test transformers respectively, these being of the same nominal ratio. In the N.P.L. apparatus, three standard transformers are available, these covering the range of ratios from 5/5 to 12,000/5. Full details of the construction and calibration of the standard transformers are given in Arnold's paper.

T is a 5/5 current transformer having negligibly small ratio and phase-angle errors, and is for the purpose of insulating the measuring circuit from the main secondary circuit.

M is a variable astatic mutual inductance with a range of ± 2·4 microhenries.

R consists of three resistances, of 0·01, 0·10, and 1 ohm respectively, connected in series and being fitted with short-circuiting plugs so that any two may be short-circuited, depending upon the sensitivity and range desired.

r is a variable resistance of ± 500 microhms. This consists of a constantan tube having a return conductor passing down the centre of the tube. One potential connection is made at the centre of the tube and the other is a sliding contact.

An impedance giving a range of burdens for the test transformer and an ammeter, having a Mumetal movement enabling the secondary current to be set to 0·1 per cent of full-scale value, are included.

Testing Procedure. Inductive interference in the measuring circuit is first eliminated by disconnecting the potential leads from *R* and joining them together. Then, with *r* and *M* set at zero, and full primary current flowing, the apparatus and leads in the measuring circuit are arranged so that no deflection (indicating no inductive interference) is obtained on the vibration galvanometer.

A check upon the polarities is then made, replacing *R* temporarily by a resistance of 0·0004 ohm, to which the potential leads are connected as in Fig. 428.

Balance is finally obtained, using R as shown, by adjustment of M and r.

At balance,

$$K_x - K_s \doteqdot r/R$$

and

$$\theta_x - \theta_s \doteqdot \omega M/R$$

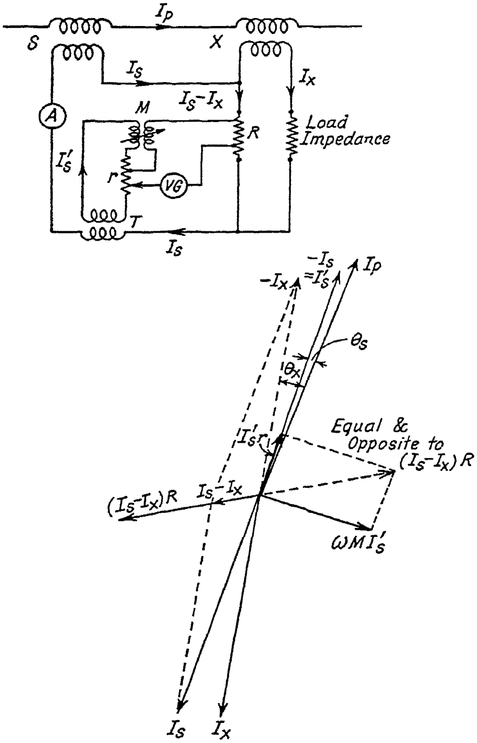


FIG. 428

K_x and K_s are the ratios $\dfrac{\text{actual ratio}}{\text{nominal ratio}}$ for the test and standard transformers respectively, and θ_x and θ_s are their phase angles.

Theory

$$I_s = I_s(\cos\theta_s + j\sin\theta_s) = -\frac{1}{R_s}(\cos\theta_s + j\sin\theta_s)\,.\,I_p$$

$$\doteqdot -\frac{1}{R_s}[1 + j\theta_s]I_p$$

where R_s = the actual ratio of the standard.

The approximation is justifiable, since θ_s is very small.

Similarly, $$I_x = -\frac{1}{R_x}[1 + j\theta_x]I_p$$

where R_x = actual ratio of test transformer.

Thus, $$I_x = \frac{R_s}{R_x}[1 + j(\theta_x - \theta_s)]I_s$$

Now, at balance,

$$R(I_s - I_x) - rI_s + j\omega MI_s = 0 \text{ (since } I'_s = -I_s\text{)}$$

Substituting for I_x in terms of I_s gives the equation

$$R\left[1 - \frac{R_s}{R_x}\left\{1 + j(\theta_x - \theta_s)\right\}\right] - r + j\omega M = 0$$

Equating real and imaginary terms

$$R[1 - R_s/R_x] = r$$

or $$1 - R_s/R_x = r/R$$

Hence $$1 - K_sN/K_xN = r/R$$

where N = the nominal ratio.

Thus, to a close approximation, since K_s and K_x are nearly unity,

$$K_x - K_s = r/R$$

Again, $$\frac{R_s}{R_x}(\theta_x - \theta_s) = \omega M/R$$

or $$\theta_x - \theta_s \doteqdot \omega M/R$$

Messrs. H. Tinsley & Co. manufacture a precision current transformer testing equipment based on the above principles. This equipment is designed for a range of ratios of 5/5 to 10,000/5, the accuracy obtainable being 2 in 100,000 in ratio measurements and 0·05 min. phase angle. In routine tests the accuracies are 1 in 20,000 and 0·2 min. respectively.

Petch-Elliott Testing Set. Messrs. Elliott Bros. have introduced the Petch-Elliott current transformer testing set for the testing of precision current transformers at a given frequency. As in the Arnold method, the transformer under test is compared with a standard transformer of the same nominal ratio.

The "difference" or "spill" current (called $I_s - I_x$ in the description of the Arnold method above) is passed through a separate winding on a toroidal transformer which carries three other windings. One of these supplies the vibration galvanometer, and the other two are supplied, through slide-wire potential dividers, from the standard transformer, through an auto-transformer.

This set has the advantages that it is capable of testing transformers designed for either 5 amp., 1 amp., or 0·5 amp. secondary currents. It is unaffected by stray magnetic fields, is portable, and has a range of + 0·5 per cent to − 0·5 per cent ratio and + 20 min. to − 20 min. phase angle on its normal range.

(*b*) VOLTAGE TRANSFORMER TESTING. Tests for the determination of the ratio and phase angle errors of voltage transformers may be divided into two classes, just as current transformer tests were divided; namely, absolute tests, and comparison tests.

Absolute Methods. In these methods, the voltage of the transformer secondary winding is compared with a suitable fraction of the voltage applied to the primary winding, this fraction being obtained by the use of a non-inductive and non-capacitative resistance potential divider with a variable tapping. The magnitude and phase of the difference between these two voltages are measured and the transformer errors are obtained therefrom. If the primary voltage of the transformer under test is very high it is very difficult to eliminate the effect of distributed capacitance in the potential divider. A complicated system of screening becomes necessary, and the divider is likely to be a very expensive item in the testing apparatus. B. G. Churcher has described a capacitor potential divider for voltages up to 132 kV (Ref. (25)), and his original paper should be referred to for the description of an absolute method of voltage transformer testing, using such a divider. The question of distributed capacitance in resistance potential dividers for such tests is discussed in a paper by C. Dannatt (Ref. (26)), which also gives details of a testing apparatus designed for routine tests upon voltage transformers.

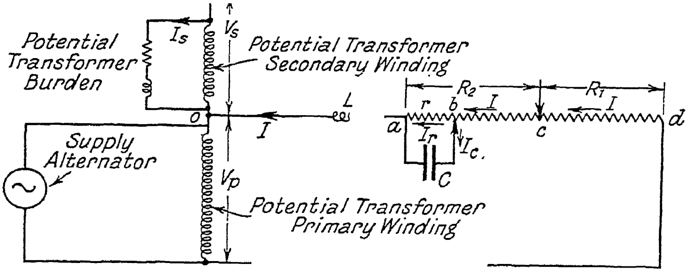

Potential Transformer Burden
I_s
V_s
Potential Transformer Secondary Winding
R_2
R_1
L
r
b
I
I
o
I
a
c
d
I_r
I_c.
C
Supply Alternator
V_p
Potential Transformer Primary Winding

FIG. 429. ABSOLUTE METHOD OF TESTING POTENTIAL TRANSFORMERS

There are several absolute methods of testing in which the transformer errors are determined in terms of the impedances of the various parts of the testing circuit. These methods are all essentially the same in principle, the transformer secondary voltage being, in all such methods, balanced by a fraction of the primary applied voltage, a vibration galvanometer being used as a detector to indicate exact balance, and various adjustable impedances being used to vary the magnitude and phase of the fraction of the primary voltage until such a balance is obtained.

One such method is described below, references to similar methods being given in Refs. (3), (25), (26), at the end of the chapter.

Fig. 429 gives the connections of this method of testing. The burden with which the transformer is to be tested is connected across the secondary winding, and the normal primary voltage, at normal frequency, is applied to the primary winding. One end of the secondary is connected to one end of the primary winding. A non-inductive and non-capacitative potential divider is connected across the primary winding in series with an inductance L. A capacitor C shunts a small part r of the resistance of the potential divider, which has two adjustable contacts b and c as shown. V.G. is a vibration galvanometer.

In carrying out the test the positions of these contacts b and c are adjusted until the vibration galvanometer gives no deflection.

Then the transformer ratio is given by

$$\frac{V_p}{V_s} = \frac{R_1 + R_2}{R_2} \text{ (approximately)} \quad . \quad . \quad . \quad (412)$$

where V_p = primary terminal voltage
V_s = secondary terminal voltage

and R_1 and R_2 are the resistances shown in Fig. 429.

The phase angle θ of the transformer is given by

$$\sin\theta = \omega[L - Cr^2]\left[\frac{1}{R_2} - \frac{1}{R_1 + R_2}\right]$$

or

$$\theta = \sin^{-1}\left[\omega(L - Cr^2)\left(\frac{1}{R_2} - \frac{1}{R_1 + R_2}\right)\right] \quad . \quad (413)$$

r being the resistance a to b, as shown.

Theory. Let I = current passing from the supply through the potential divider as shown.

When the vibration galvanometer indicates zero deflection, the current through the section co is the same as that through section dc. Then the secondary terminal voltage

$$V_s = \text{volt drop } co = IZ_{co}$$

where Z_{co} is the impedance of the section co.

Again, the primary terminal voltage

$$V_p = IZ_{do}$$

where Z_{do} is the impedance of the section d to o.

Using the symbolic notation, we have

$$Z_{co} = (R_2 - r) + j\omega L + Z_{ba}$$

where Z_{ba} is the impedance of r and C in parallel.

In evaluating Z_{ba} we have

$$\frac{1}{Z_{ba}} = \frac{1}{r} + \frac{1}{\frac{-j}{\omega C}} = \frac{1}{r} + j\omega C$$

$$\therefore \quad Z_{ba} = \frac{r}{1 + j\omega Cr} = \frac{r(1 - j\omega Cr)}{1 + \omega^2C^2r^2}$$

$$\therefore \quad Z_{co} = (R_2 - r) + j\omega L + \frac{r}{1 + \omega^2C^2r^2} - \frac{j\omega Cr^2}{1 + \omega^2C^2r^2}$$

$$= R_2 - r + \frac{r}{1 + \omega^2C^2r^2} + j\omega\left[L - \frac{Cr^2}{1 + \omega^2C^2r^2}\right]$$

Now,
$$Z_{do} = Z_{co} + Z_{dc} = Z_{co} + R_1$$
$$= R_1 + R_2 - r + \frac{r}{1 + \omega^2 C^2 r^2} + j\omega\left[L - \frac{Cr^2}{1 + \omega^2 C^2 r^2}\right]$$
$$\therefore \frac{V_p}{V_s} = \frac{IZ_{do}}{IZ_{co}} = \frac{R_1 + R_2 - r + \dfrac{r}{1 + \omega^2 C^2 r^2} + j\omega\left[L - \dfrac{Cr^2}{1 + \omega^2 C^2 r^2}\right]}{R_2 - r + \dfrac{r}{1 + \omega^2 C^2 r^2} + j\omega\left[L - \dfrac{Cr^2}{1 + \omega^2 C^2 r^2}\right]}$$

Since $\omega^2 C^2 r^2$ is small compared with unity,
$$\frac{r}{1 + \omega^2 C^2 r^2} = r \text{ (approx.)}$$

and also $\omega\left[L - \dfrac{Cr^2}{1 + \omega^2 C^2 r^2}\right]$ is small compared with R_1 and R_2.

Thus,
$$\frac{V_p}{V_s} = \frac{R_1 + R_2}{R_2} \text{ (very nearly)}$$

Phase Angle. The vector diagram corresponding to balanced conditions is shown in Fig. 430, in which I_c and I_r are the two components of I flowing through the capacitor C and resistance r respectively. θ is the phase angle of the transformer. The vectors representing the voltage drops across the parts of the circuit *oa*, *ab*, *cb*, and *cd* are marked thus— v_{oa}— in addition to their markings in terms of current and impedance.

Now, in the triangle whose sides are V_p, V_s, and v_{cd}
$$\frac{v_{cd}}{\sin\theta} = \frac{V_s}{\sin\alpha}$$
or
$$\frac{IR_1}{\sin\theta} = \frac{V_s}{\sin\alpha}$$
Hence,
$$\sin\theta = \frac{IR_1 \sin\alpha}{V_s}$$

The angle α is the phase angle of the whole circuit d to o; i.e. the angle between V and I.

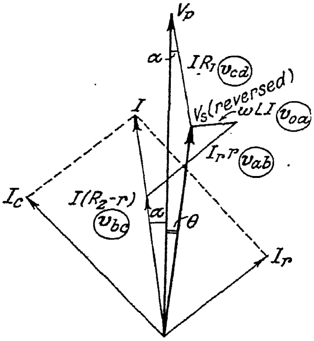


FIG. 430

$$\therefore \sin\alpha = \frac{\omega\left[L - \dfrac{Cr^2}{1 + \omega^2 C^2 r^2}\right]}{Z_{do}}$$
$$\therefore \sin\theta = \frac{IR_1}{V_s} \cdot \frac{\omega\left[L - \dfrac{Cr^2}{1 + \omega^2 C^2 r^2}\right]}{Z_{do}}$$

or, substituting $\dfrac{V_p}{Z_{do}}$ for I, we have
$$\sin\theta = \frac{V_p}{Z_{do}} \cdot \frac{R_1}{V_s} \cdot \frac{\omega\left[L - \dfrac{Cr^2}{1 + \omega^2 C^2 r^2}\right]}{Z_{do}}$$
$$= \frac{V_p}{V_s} \cdot \frac{R_1}{Z_{do}^2} \cdot \omega\left[L - \frac{Cr^2}{1 + \omega^2 C^2 r^2}\right]$$

Now, $\frac{V_p}{V_s} = \frac{R_1 + R_2}{R_2}$, and, taking $R_1 + R_2 = Z_{do}$, since the other terms involved in Z_{do} are small compared with $R_1 + R_2$, we have

$$\sin\theta = \frac{R_1 + R_2}{R_2} \cdot \frac{R_1}{(R_1 + R_2)^2} \cdot \omega \left[L - \frac{Cr^2}{1 + \omega^2 C^2 r^2}\right]$$

$$= \frac{R_1}{R_2(R_1 + R_2)}\, \omega \left[L - \frac{Cr^2}{1 + \omega^2 C^2 r^2}\right]$$

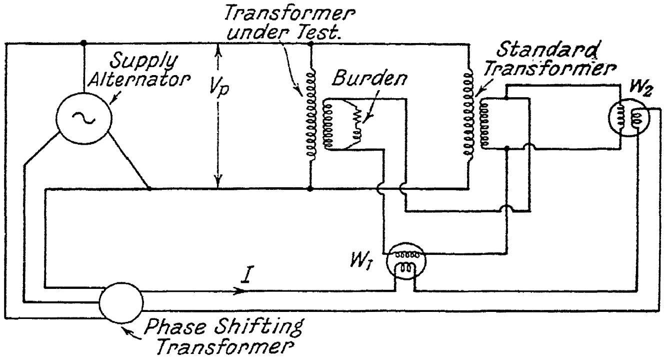


FIG. 431. COMPARISON METHOD OF TESTING POTENTIAL TRANSFORMER

$\frac{R_1}{R_2(R_1 + R_2)}$ may be written $\frac{1}{R_2} - \frac{1}{R_1 + R_2}$, and also $\omega^2 C^2 r^2$ may be neglected in comparison with unity.

Thus, $$\sin\theta = \omega[L - Cr^2]\left[\frac{1}{R_2} - \frac{1}{R_1 + R_2}\right]$$

Comparison Methods. The connections for one such method are given in Fig. 431 (Ref. (4)). Two wattmeters W_1 and W_2 are used in conjunction with a phase-shifting transformer. The current coils of the wattmeters are connected in series and are supplied with a current whose phase is variable from the rotor winding of the phase-shifting transformer. The pressure coil of wattmeter W_1 is supplied from the secondary windings of the two voltage transformers—one transformer being under test and the other a standard transformer of the same nominal ratio. The secondary windings are connected in opposition so that the voltage applied to the pressure coil of W_1 is the vector difference of the secondary voltages of the two transformers. The burden with which the test transformer is to be tested is connected across its secondary winding. The primary windings of the transformers are connected in parallel, and hence have the same primary voltage V_p. Wattmeter W_2 is for the purpose of checking the phase of the current I, in the current coils of the wattmeters, relative to the secondary voltage V_s of the standard transformer. The pressure coil of this wattmeter is, for this reason, connected across the secondary winding of this transformer.

As used in this test, wattmeter W_1 is essentially a voltmeter, and its deflection, per volt applied to the pressure coil, corresponding to some given current in the current coil, must be known. Let K be the volts per division for a current I in the current coil; this current being in phase with the applied voltage.

The operation of the method of testing consists in observing the reading D_1 of wattmeter W_1 when the current I in the current coil is in phase with the secondary voltage of the standard transformer. The phase of this current is altered, by adjusting the phase-shifting transformer, until wattmeter W_2 gives a maximum reading, when the current I in its current coil is in phase with the voltage V_s applied to its pressure coil.

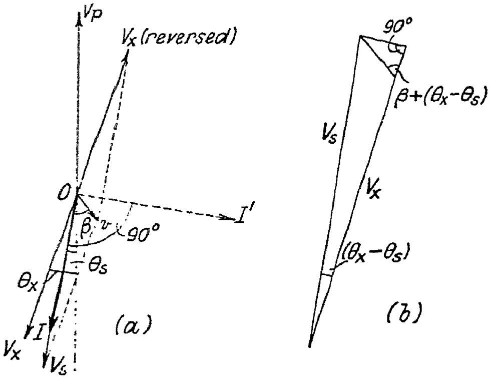


FIG. 432

The phase-shifter is then adjusted until wattmeter W_2 gives zero reading, when the current I must be 90° out of phase with the voltage V_s. The reading D_0 of wattmeter W_1 is then again observed.

Then if R_s and R_x are the ratios of the standard and test transformers respectively,

$$R_x = \frac{R_s \, . \, V_s}{V_s - KD_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (414)$$

and

$$\theta_x = \theta_s + \tan^{-1} \frac{KD_0}{V_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (415)$$

where θ_x and θ_s are the phase angles of the test and standard transformer respectively.

Theory. Referring to the vector diagram of Fig. 432 (*a*), vectors OV_p, OV_s, and OV_x represent the common primary voltage of the two transformers and their secondary voltages respectively, while angles θ_s and θ_x are their phase angles. Ov is the vector difference of V_s and V_x, and β is the phase angle between v and V_s.

Then, when the current I is in phase with V_s, we have, by projecting V_s on to V_x, as in Fig. 432 (*b*),

$$V_s \cos(\theta_x - \theta_s) = V_x + v \cos[\beta + (\theta_x - \theta_s)]$$

Now, the ratio R_s of the standard transformer is given by $\frac{V_p}{V_s}$, while the ratio of the transformer under test is $R_x = \frac{V_p}{V_x}$.

Thus, $$\frac{V_p}{R_s}\cos(\theta_x - \theta_s) = \frac{V_p}{R_x} + v\cos[\beta + (\theta_x - \theta_s)]$$

Again, when I is in phase with V_s, wattmeter W_1 reads D_1, and the power in this wattmeter is then $vI \cos\beta$.

$$\therefore \qquad vI\cos\beta = kD_1$$

where k is a constant.

Or $$v\cos\beta = KD_1$$

where K is the constant of the wattmeter in volts per division corresponding to a current I.

Substituting for v in the previous equation, we have

$$\frac{V_p}{R_s}\cos(\theta_x - \theta_s) = \frac{V_p}{R_x} + \frac{KD_1}{\cos\beta}\cos[\beta + (\theta_x - \theta_s)]$$

Now, $\theta_x - \theta_s$ is a very small angle, and its cosine may be taken as unity. Hence,

$$\frac{V_p}{R_s} = \frac{V_p}{R_x} + KD_1 \text{ (very nearly)}$$

or $$\frac{1}{R_s} - \frac{KD_1}{V_p} = \frac{1}{R_x}$$

Substituting R_sV_s for V_p, we have,

$$\frac{1}{R_x} = \frac{1}{R_s} - \frac{KD_1}{R_sV_s} = \frac{V_s - KD_1}{R_sV_s}$$

Thus, $$R_x = \frac{R_sV_s}{V_s - KD_1}$$

Again, from Fig. 432 (*b*), if $\theta_x - \theta_s$ is neglected in comparison with β,

$$\tan(\theta_x - \theta_s) = \frac{v\sin\beta}{V_s\cos(\theta_x - \theta_s)}$$
$$= \frac{v\sin\beta}{V_s} \text{ (very nearly)}$$

Now, when the current I is in quadrature with V_s (as represented by vector

OI'), the power measured by wattmeter W_1 is $vI \cos(90 - \beta)$. If the deflection is now D_0, then

$$KD_0 = v \cos(90 - \beta)$$
$$= v \sin \beta$$

$$\therefore \qquad \tan(\theta_x - \theta_s) = \frac{KD_0}{V_s}$$

i.e.

$$\theta_x - \theta_s = \tan^{-1} \frac{KD_0}{V_s}$$

or

$$\theta_x = \theta_s + \tan^{-1} \frac{KD_0}{V_s}$$

Other comparison methods are described in the works mentioned in Refs. (2), (3), (27).

BIBLIOGRAPHY AND REFERENCES

(1) *Electrical Measuring Instruments,* Vol. II, C. V. Drysdale and A. C. Jolley.

(2) *Commercial A.C. Measurements,* G. W. Stubbings.

(3) *Electrical Measurements,* F. A. Laws.

(4) *Dictionary of Applied Physics,* Vol. II, section on "Instrument Transformers," by R. S. J. Spilsbury.

(5) *The Essentials of Transformer Practice,* E. G. Reed.

(6) British Standards Institution, Standard Specification No. 81 (1936).

(7) "High Permeability and Low Loss Alloys for use in Current Transformers and Electrical Instruments," W. Phillips, *Proc. Meter Engineers Technical Association* (December, 1927).

(8) "Some Recent Advances in Alternating Current Measuring Instruments," K. Edgcumbe and F. E. J. Ockenden, *Jour. I.E.E.*, Vol. LXV, p. 553.

(9) "The Magnetization of Iron at Low Inductions," L. W. Wild, *Jour. I.E.E.*, Vol. LII, p. 96.

(10) "Testing Magnetic Sheet Steels," B. G. Churcher, *World Power,* Vol. IX, No. xlix.

(11) H. B. Brooks and F. C. Holtz, *Trans. Amer. I.E.E.*, 1922, Vol. XLI, p. 382.

(12) "Papers on Current Transformers," H. W. Price and C. Kent-Duff, *University of Toronto Bulletin No.* 2 (1921), Section No. 4.

(13) "Instrument Transformers," J. G. Wellings and C. G. Mayo, *Jour. I.E.E.*, Vol. LXVIII.

(14) "Accuracy of Measurement and Short Circuit Reliability of Current Transformers," *A.S.E.A. Journal,* Vol. VII, p. 55.

(15) "A Study of the Current Transformer with particular reference to Iron Loss," P. G. Agnew, *Bull. Bur. Stands.*, Vol. VII, No. 3 (1911).

(16) "Berücksichtigung der Beglaubigungsvorschriften bei der Vorausberechnung der Stromwandler," E. Wirz, *Bulletin No.* 1, *Schweiz. Elektrotechnischer Verein* (1919).

(17) "Siehe Theorie und Berechnung der Stromwandler," E. Wirz, *Bulletin Schweiz. Elektrotechnischer Verein* (1913), Vol. 11.

(18) "The Testing of Instrument Transformers," P. G. Agnew and F. B. Silsbee, *Trans. Amer. Inst.E.E.*, Vol. XXXI, p. 1635.

(19) "The Determination of the Constants of Instrument Transformers," P. G. Agnew and T. T. Fitch, *Bull. Bur. Stands.*, Vol. VI, p. 281, Scientific Paper No. 130.

(20) "Some Recent Developments in Exact Alternating-current Measurements," C. H. Sharp and W. W. Crawford, *Trans. Amer. I.E.E.*, Vol. XXIX p. 1517.

(21) "A Watt-hour Meter Method of Testing Instrument Transformers," P. G. Agnew, *Bull. Bur. Stands.*, Vol. II (1915), p. 347, Scientific Paper No. 233.

(22) E. Biffi, *L'Elettrotecnica* (1923), Vol. X, p. 508.

(23) "A Method of Measuring the Ratio and Phase Angle of a Current Transformer," W. I. Place, *Jour. I.E.E.*, Vol. LXVI, p. 657.

(24) "A Method for Testing Current Transformers," F. B. Silsbee, *Bull. Bur. Stands.*, Scientific Paper No. 309 (1917).

(25) "The Measurement of Ratio and Phase Displacement of High Alternating Voltages," B. G. Churcher, *Jour. I.E.E.*, Vol. LXV, p. 430.

(26) "Equipment for Ratio and Phase Angle Measurements of Voltage Transformers," C. Dannatt, *Metropolitan-Vickers Gazette*, No. 203, Vol. XII, p. 45.

(27) "Testing Potential Transformers," H. B. Brooks, *Bull. Bur. Stands.*, Vol. X, Scientific Paper No. 217 (1914).

(28) "Precision Testing of Current Transformers," A. H. M. Arnold, *Jour. I.E.E.*, Vol. LXVIII, p. 898.

(29) "Some Accessory Apparatus for Precise Measurements of Alternating Current," R. S. J. Spilsbury and A. H. M. Arnold, *Jour. I.E.E.*, Vol. LXVIII, p. 889.

(30) H. Schering and E. Alberti, *Archiv. für Electrotechnik* (1913), ii, p. 208.

(31) "Some Measurements on Phase Displacements in Resistances and Transformers," *Electrician* (1906), Vol. LVIII, pp. 160 and 169.

(32) "A New Null Method of Testing Instrument Transformers, and Its Application," G. F. Shotter, *Jour. I.E.E.*, Vol. LXVIII, p. 873.

(33) "A Precise Electrometer Method for Voltage-Transformer Testing," R. S. J. Spilsbury, *Jour. I.E.E.*, Vol. LXVII, p. 1143.

(34) "A Shielded Resistor for Voltage Transformer Testing," F. B. Silsbee, *Bull. Bur. Stands*, Scientific Paper No. 516.

(35) "Design, Construction, and Testing of Voltage Transformers," J. H. Buchanan, *Jour. I.E.E.*, Vol. LXXVIII, p. 292.

(36) "Capacitator Voltage Transformers," J. G. Wellings, J. R. Mortlock, and P. Matthews, *Jour. I.E.E.*, Vol. LXXIX, p. 577.

(37) *Instrument Transformers*, B. Hague.

(38) "Current-transformer Testing," A. H. M. Arnold, *Jour. I.E.E.*, Vol. LXXIV, p. 424.

(39) "Leakage Phenomena in Ring-type Current Transformers," A. H. M. Arnold, *Jour. I.E.E.*, Vol. LXXIV, p. 413.

(40) *The Magnetic Properties of the Nickel-iron Alloys.* (Published by the Bureau of Information on Nickel, the Mond Nickel Co., Ltd.)

(41) "C.T. Secondary Networks (Inter Connection in Three-phase Circuits)," G. W. Stubbings, *Elec. Rev.*, September 5th, 1947.

(42) "Voltage Transformers and Current Transformers Associated with Switchgear," W. Gray and A Wright, *Jour. I.E.E.*, Vol. C, Pt. II, No. 75, p. 223.

(43) "Some Special Characteristics of Soft Magnetic Materials used in Instrument Manufacture," G. A. V. Sowter, *Proc. I.E.E.*, Vol. 98, Pt. II, p. 714.

(44) "Performance Limits in Electrical Instruments, A. H. M. Arnold, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, p. 70.

CHAPTER XX

THE MEASUREMENT OF POWER

IN direct-current circuits power may be measured either by a wattmeter or by an ammeter and voltmeter, the product of whose readings gives the power in the circuit. In alternating current circuits the power at any instant is given by

$$p = ei \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (416)$$

where p = instantaneous power
e = instantaneous voltage
i = instantaneous current

Thus, if both the current and voltage waves are sinusiodal, the current lagging in phase by an angle ϕ, then

$$e = E_{max} \sin \omega t$$

and

$$i = I_{max} \sin (\omega t - \phi)$$

The instantaneous power p is therefore given by

$$\begin{aligned} p &= ei \\ &= E_{max} I_{max} \sin \omega t \sin (\omega t - \phi) \end{aligned}$$

or, writing θ for ωt,

$$p = E_{max} I_{max} \sin \theta \sin (\theta - \phi)$$

The mean power P

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} E_{max} I_{max} \sin \theta \sin (\theta - \phi) \, d\theta \quad . \qquad (417) \\ &= \frac{E_{max} I_{max}}{2\pi} \int_0^{2\pi} \frac{\cos \phi - \cos (2\theta - \phi)}{2} \, d\theta \\ &= \frac{E_{max} I_{max}}{4\pi} \left[\theta \cos \phi - \frac{\sin (2\theta - \phi)}{2} \right]_0^{2\pi} \\ &= \frac{E_{max} I_{max}}{2} \cos \phi \end{aligned}$$

$$\therefore \qquad P = EI \cos \phi \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (418)$$

where E and I are r.m.s. values of voltage and current.

The fact that the power factor ($\cos \phi$) is involved in the expression for the power means that a wattmeter must be used instead of merely an ammeter and voltmeter, since the latter method takes no account of power factor.

Wattmeter Measurements in Single-phase A.C. Circuits. Fig. 433 shows a wattmeter connected in such a circuit. The "current coil" of the instrument carries the load current, while the "pressure coil" carries a current proportional to, and in phase with, the voltage. The deflection of the wattmeter depends upon the currents in these two coils and upon the power factor. Inductance in the pressure coil circuit should be avoided as far as possible, since it causes the pressure-coil current to lag behind the applied voltage. A high non-inductive resistance is connected in series with the pressure coil in order that the reactance of the coil itself shall be small in comparison with the resistance of the whole pressure-coil circuit and also, of course, to ensure that the current taken by the pressure coil shall be small.

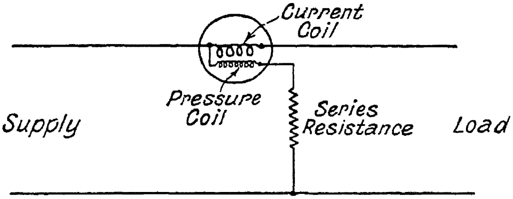


FIG. 433. WATTMETER CONNECTIONS

Wattmeter Errors. (*a*) PRESSURE-COIL INDUCTANCE

If r_p = resistance of pressure coil

l_p = inductance of pressure coil

R = resistance in series with pressure coil

V = voltage applied to the pressure-coil circuit

then, the current through the pressure coil

$$= \frac{V}{\sqrt{(r_p + R)^2 + \omega^2 l_p^2}} = i_p$$

The phase of this current relative to the voltage is lagging by a small angle β such that

$$\tan \beta = \frac{\omega l_p}{r_p + R}$$

By increasing the non-inductive resistance R, this angle is reduced, although the current i_p is reduced by such increase. The effect of variation of frequency is to increase β (since it increases ω), and to reduce the current i_p, slightly, by its effect in increasing the impedance of the pressure-coil circuit.

Thus, if α is the phase angle (lagging) of the load circuit, the wattmeter deflection is proportional to

$$Ii_p \cos(\alpha - \beta)$$

i.e. proportional to $I\frac{V}{Z_p}\cos(\alpha - \beta)$

where Z_p is the impedance of the pressure-coil circuit.

Now, $Z = r_p + R$

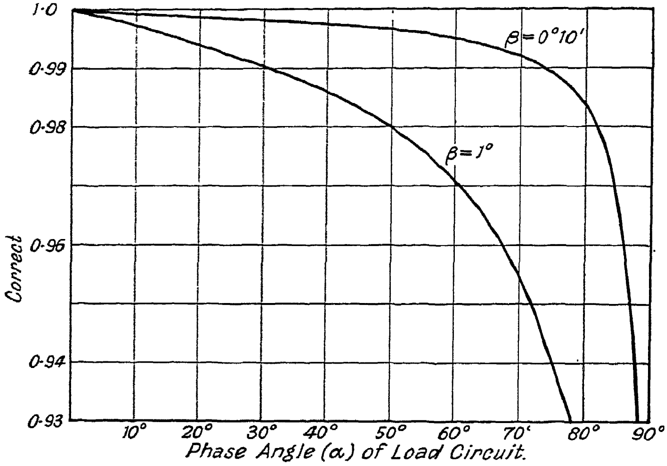


FIG. 434. WATTMETER CORRECTION FACTORS

Thus the deflection is proportional to

$$I\frac{V}{(r_p + R)}\cos\beta\cos(\alpha - \beta)$$

If the inductance of the pressure-coil circuit were zero, the deflection would be proportional to $\frac{IV}{(r_p + R)}\cos\alpha$, and the wattmeter would read correctly at all frequencies and power factors. The ratio of the true reading of the instrument to the actual reading is, therefore,

$$\frac{\frac{IV}{(r_p + R)}\cos\alpha}{\frac{IV}{(r_p + R)}\cos\beta\cos(\alpha - \beta)} = \frac{\cos\alpha}{\cos\beta\cos(\alpha - \beta)}$$

$$\text{True reading} = \frac{\cos \alpha}{\cos \beta \cos (\alpha - \beta)} \times \text{Actual reading} \quad . \quad (419)$$

Correction Factor. The "correction factor" by which the actual reading must be multiplied, in order to obtain the true reading, is $\frac{\cos \alpha}{\cos \beta \cos (\alpha - \beta)}$. The wattmeter will read high on lagging power factors of the load, since the effect of the inductance of the pressure coil circuit is to bring the current in it more nearly into phase with the load current than would be the case if this inductance were zero. If the power factor of the load is very low, a serious error may be introduced by pressure-coil inductance unless special precautions are taken to reduce its effect.

Fig. 434 gives curves showing the variation in the value of the correction factor as the power factor of the load varies, the phase angle β of the pressure-coil circuit being 1° in one case and 0° 10′ in the other.

The error, in terms of the actual instrument deflection, is

Actual reading − True reading

$$= \left[1 - \frac{\cos \alpha}{\cos \beta \cos (\alpha - \beta)}\right] \times \text{Actual reading}$$

$$= \left[1 - \frac{\cos \alpha}{\cos (\alpha - \beta)}\right] \times \text{Actual reading}$$

if $\cos \beta$ is assumed equal to unity.

$$\therefore \quad \text{Error} = \left[1 - \frac{\cos \alpha}{\cos \alpha + \sin \alpha \sin \beta}\right] \times \text{Actual reading}$$

$$= \left[\frac{\sin \alpha \sin \beta}{\cos \alpha + \sin \alpha \sin \beta}\right] \times \text{Actual reading}$$

or

$$\text{Error} = \frac{\sin \beta}{\cot \alpha + \sin \beta} \times \text{Actual reading} \quad . \quad . \quad (420)$$

(*b*) PRESSURE-COIL CAPACITANCE. The pressure-coil circuit may have capacitance as well as inductance, this being largely due to inter-turn capacitance in the high series resistance. The effect produced is similar to that of inductance in this circuit, except that the pressure-coil current tends to *lead* the applied voltage instead of to lag behind it. This causes the wattmeter to read low, on lagging power factors of the load, by increasing the angle between the load and pressure-coil currents.

The effect of frequency will be, of course, to vary the phase angle between V and i_p, the angle increasing with increase of frequency.

If the capacitance reactance of the pressure-coil circuit is equal to its inductive reactance, there will be no error due to these effects since the two individual errors will neutralize one another.

(c) EDDY CURRENTS. Eddy currents induced in the solid metal parts of the instrument, by the alternating magnetic field of the current coil, alter the magnitude and phase of this field, and so produce an error. The phase of the induced eddy e.m.f.'s will be 90° behind the inducing flux—i.e. rather more than 90° behind the main current in the current coil. The eddy current is practically in phase with its e.m.f., and this current sets up a magnetic field which, combined with that of the current coil, produces a resultant magnetic field which is less than that of the current coil alone and which also lags behind the current coil field by a small angle.

This eddy current error is not easily calculable, and may be serious if care is not taken to ensure that any solid metal parts (which should be avoided as far as possible) are well removed from the current coil. If the wattmeter current coil is designed for heavy currents, it should consist of stranded conductors in order to minimize the eddy currents flowing in the conductors of the current coil itself.

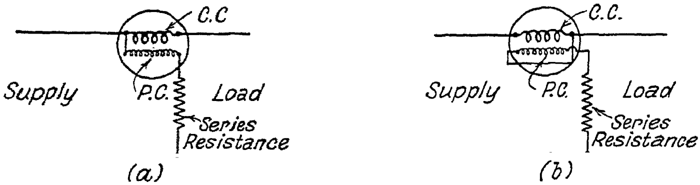


FIG. 435. ALTERNATIVE WATTMETER CONNECTIONS

Methods of Connection in the Circuit. There are two obvious methods of connecting a wattmeter in circuit, as shown in Fig. 435, in which C.C. and P.C. indicate current coil and pressure coil respectively. Neither measures the power in the load directly, without correction, even neglecting the errors discussed above.

In the method of diagram (a), in which the pressure coil is connected on the "supply" side of the current coil, the voltage applied to the pressure coil is higher than that of the load on account of the voltage drop in the current coil. In diagram (b) the current coil carries the small current taken by the pressure coil, in addition to the load current.

In the first case, the instrument measures the watts, I^2R_c, lost in the current coil, and in the second case the watts lost in the pressure coil, as well as the power in the load.

If the load current is small, the volt drop in the current coil is small, so that the first method of connection introduces a very small error. On the other hand, if the load current is large, the watts lost in the pressure coil will be small compared with the watts in the load, and the second method of connection is better.

Compensation for Power Loss in Pressure Coil. In some wattmeters a compensating coil is used to eliminate error due to the current coil carrying the pressure-coil current in addition to the load current, when the connections are as in Fig. 435 (*b*). This compensating coil is as nearly as possible identical and coincident with the current coil, so that if it were connected in series with the latter, and a current passed through the two coils—connected so that their magnetic effects are in opposition—the resultant magnetic field would be zero. Actually the compensating coil is connected in series with the pressure coil, but in such a way that its magnetic effect opposes that of the current coil and neutralizes the pressure-coil component of the current in the current coil. Thus, if no-load current flows in the instrument, the deflection should be zero, since the resultant current-coil field should be zero. The connections of this method of compensation are shown in Fig. 436.

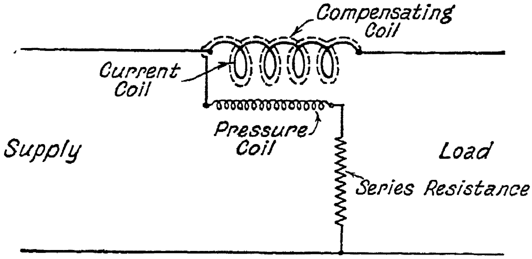


FIG. 436. CONNECTIONS OF COMPENSATING COIL

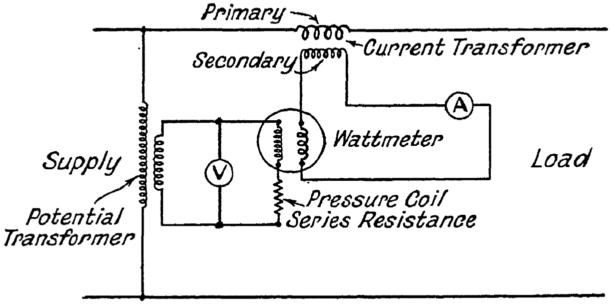


FIG. 437. USE OF INSTRUMENT TRANSFORMERS WITH A WATTMETER

The Use of Instrument Transformers with Wattmeters. Current and voltage transformers may be used with wattmeters just as they are used with ammeters and voltmeters. By using a number of current transformers, of different ratios, for the supply of the wattmeter current coil, and a number of voltage transformers for its pressure coil, the same wattmeter may be used to cover a very large range of power measurements. The connections of a wattmeter when so used are shown in Fig. 437, in which an ammeter and a voltmeter are connected in circuit, supplied from the same transformers as the wattmeter.

When such transformers are used corrections must be applied to allow for their ratio and phase angle errors. Fig. 438 gives the vector diagrams for the currents and voltages of the load, and in the wattmeter coils. Diagram (*a*) refers to a load with a lagging power factor, and diagram (*b*) to a load with a leading power factor. It is assumed that both current and potential transformers are used.

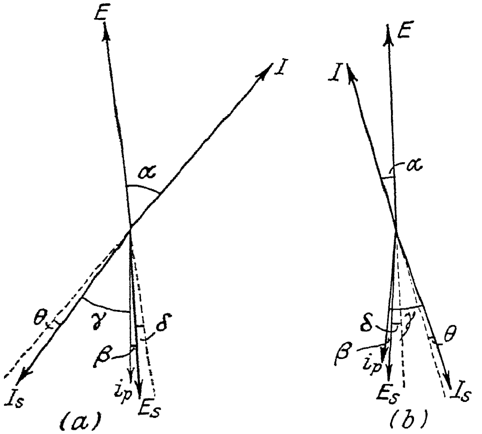


FIG. 438

In the diagrams—

E = voltage of the load
I = load current
α = phase angle between load current and voltage
γ = phase angle between the currents in the current and pressure coils of the wattmeter
I_s = current in wattmeter current coil
= secondary current of current transformer
E_s = voltage applied to wattmeter pressure coil
i_p = current in wattmeter pressure coil
β = angle by which i_p lags E_s on account of inductance of pressure coil
δ = phase angle of voltage transformer
θ = phase angle of current transformer

The vectors shown dotted are E and I reversed.

Obviously, when the load has a lagging phase angle α, we have

$$\alpha = \gamma + \theta + \delta + \beta$$

or, more strictly,

$$\alpha = \gamma + \theta \pm \delta + \beta$$

since the phase angle of the voltage transformer may be either lagging or leading.

In the case of a leading power factor of the load

$$\alpha = \gamma \pm \delta - \theta - \beta$$

Correction Factor. The correction factors—neglecting, for the present, the ratios of the transformers—become

$$\frac{\cos \alpha}{\cos \beta \cos (\alpha - \theta \pm \delta - \beta)}$$

in the case of lagging power factor of the load, and

$$\frac{\cos \alpha}{\cos \beta \cos (\alpha \pm \delta + \theta + \beta)}$$

in the case of leading power factor of the load.

Writing K for the correction factor, we have for the general expression for the power to be measured,

$$\text{Power} = K \times \underset{\text{reading}}{\text{Wattmeter}} \times \underset{\substack{\text{of current}\\ \text{transformer}}}{\text{Actual ratio}} \times \underset{\substack{\text{of voltage}\\ \text{transformer}}}{\text{Actual ratio}} \quad . \quad (421)$$

The transformer ratios to be used in the expression are the "actual" ratios as distinct from the nominal ratios. As was seen in the foregoing chapter, these ratios are not constant, but depend upon the load conditions, so that calibration curves of the transformers are necessary if accurate power measurements are to be made. It should be noted, also, that in the above the power losses in the instruments are not considered.

Measurement of Power without using a Wattmeter. It is possible to measure the power in a circuit without a wattmeter by using either three ammeters or three voltmeters, in conjunction with non-inductive resistances. These methods are not, however, of much practical importance.

THREE-VOLTMETER METHOD. The connections are as shown in Fig. 439, in which V_1, V_2, and V_3 are the three voltmeters and R a non-inductive resistance which is connected in series with the load. From the vector diagram of Fig. 439 (*b*) we have

$$V_1^2 = V_2^2 + V_3^2 + 2V_2V_3 \cos \phi$$

Neglecting the currents taken by voltmeters V_2 and V_3, the current in R is the same as the load current I. Thus, $V_2 = IR$.

Substituting, $V_1^2 = V_2^2 + V_3^2 + 2IRV_3 \cos \phi$

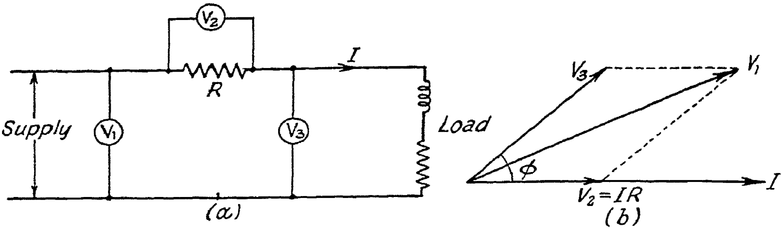
Now, $IV_3 \cos \phi$ is the power in the load, so that

$$\text{Power in load} = IV_3 \cos \phi = \frac{V_1^2 - V_2^2 - V_3^2}{2R} \quad . \quad . \quad (422)$$

The power factor also is given by

$$\cos \phi = \frac{V_1^2 - V_2^2 - V_3^2}{2V_2V_3} \quad . \quad . \quad . \quad . \quad . \quad (423)$$

The assumptions are made that the current in the resistance R is the same as the load current, and that this resistance is entirely non-inductive.



G. 439. THREE-VOLTMETER METHOD OF MEASURING SINGLE-PHASE POWER

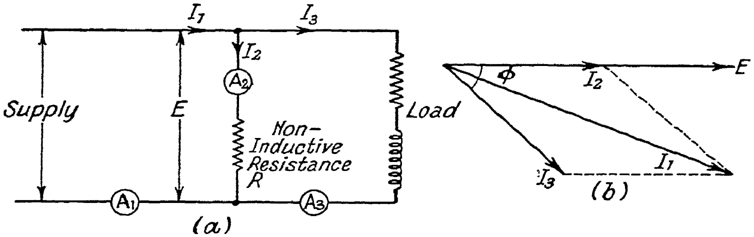


FIG. 440. THREE-AMMETER METHOD OF MEASURING SINGLE-PHASE POWER

THREE-AMMETER METHOD. This method is somewhat similar to the above. The necessary connections are shown in Fig. 440. The current measured by ammeter A_1 is the vector sum of the load current and that taken by the non-inductive resistance R (this latter being in phase with the voltage E). From the vector diagram,

$$I_1^2 = I_2^2 + I_3^2 + 2I_2I_3 \cos \phi$$

But

$$I_2 = \frac{E}{R}$$

$$\therefore \quad I_1^2 = I_2^2 + I_3^2 + 2\frac{E}{R} \cdot I_3 \cos \phi$$

Hence, the power $EI_3 \cos \phi$ is given by

$$EI_3 \cos \phi = \frac{(I_1^2 - I_2^2 - I_3^2) R}{2} \quad . \quad . \quad . \quad . \quad . \quad (424)$$

Also,

$$\cos \phi = \frac{(I_1^2 - I_2^2 - I_3^2)}{2I_2I_3} \quad . \quad . \quad . \quad . \quad . \quad (425)$$

Measurement of Three-phase Power. THREE-WATTMETER ME

The connections for this method are shown in Fig. 441, in which the load is star-connected. W_1, W_2, and W_3 are the three wattmeters, connected as shown. The arrows denote the directions of current and voltage which are conventionally considered positive. If the letters representing currents and voltages denote instantaneous values, then

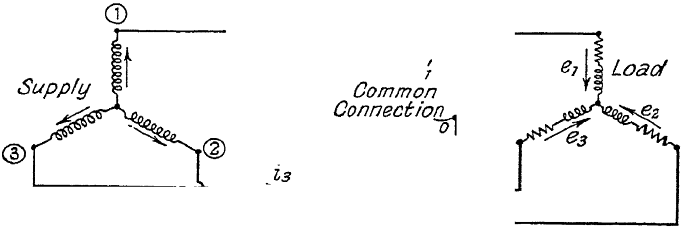


FIG. 441. THREE-WATTMETER METHOD OF MEASURING THREE-PHASE POWER

Instantaneous power in the load

$$= e_1 i_1 + e_2 i_2 + e_3 i_3$$

Let v be the potential difference between the star point of the load and the star point 0 of the wattmeter pressure coils. Then we have

$$e_1' + v = e_1$$
$$e_2' + v = e_2$$
$$e_3' + v = e_3$$

$\therefore$ total instantaneous power, by substitution, is

$$(e_1' + v)\, i_1 + (e_2' + v)\, i_2 + (e_3' + v)\, i_3$$
$$= e_1' i_1 + e_2' i_2 + e_3' i_3 + v\,(i_1 + i_2 + i_3)$$
$$= e_1' i_1 + e_2' i_2 + e_3' i_3$$

since $i_1 + i_2 + i_3 = 0$ in any three-phase system, whether balanced or not.

Now, $e_1' i_1 + e_2' i_2 + e_3' i_3$ is the total instantaneous power measured by the three wattmeters, and thus the sum of the readings of the wattmeters will give the mean value of the total power.

METER METHOD. This is the commonest method of g three-phase power. It is particularly useful when the load is unbalanced. The connections for the measurement of power in the case of a star-connected three-phase load are shown in Fig. 442. The current coils of the wattmeters are connected in lines (1) and (3), and their two pressure coils between lines (1) and (2) and (3) and (2) respectively.

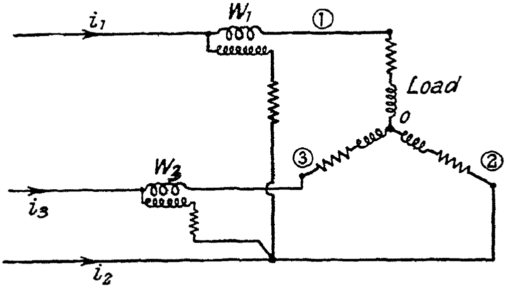


FIG. 442. TWO-WATTMETER METHOD OF MEASURING THREE-PHASE POWER

Fig. 443 gives the vector diagram for the load circuit, assuming a balanced load—i.e. the load currents and power factors are the same for all three phases. E_{10}, E_{20}, and E_{30} are the vectors representing the phase voltages, and are supposed to be equal, while I_1, I_2, and I_3 are vectors representing the line currents. The voltages applied to the pressure coil circuits of the wattmeters are E_{12} and E_{32}, which are the vector sums of the phase voltages as shown.

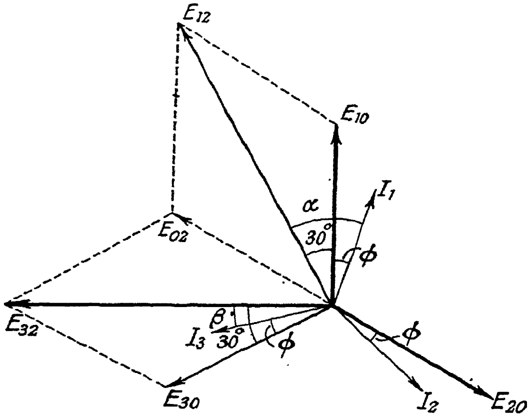


FIG. 443. VECTOR DIAGRAM, TWO-WATTMETER METHOD

Then, total instantaneous power in the load

$$= e_1 i_1 + e_2 i_2 + e_3 i_3$$

where e_1, e_2, e_3 are the instantaneous phase voltages and i_1, i_2, and i_3 are the instantaneous line currents.

Since $i_1 + i_2 + i_3 = 0$

$$i_2 = - i_1 - i_3$$

∴ Total instantaneous power

$$\begin{aligned} &= e_1 i_1 + e_2(- i_1 - i_3) + e_3 i_3 \\ &= i_1(e_1 - e_2) + i_3(e_3 - e_2) \end{aligned}$$

Now, $i_1(e_1 - e_2)$ is the instantaneous power deflecting wattmeter W_1, and $i_3(e_3 - e_2)$ is that deflecting wattmeter W_3. These wattmeters measure $I_1 E_{12} \cos \alpha$ and $I_3 E_{32} \cos \beta$ respectively, where α and β are the phase angles between I_1 and E_{12} and between I_3 and E_{32}. The sum of the wattmeter readings thus gives the mean value of the total power in the load.

Now $\alpha = 30 + \phi$

and $\beta = 30 - \phi$

Also, $E_{12} = E_{32} = \sqrt{3}E$

where E is the phase voltage.

∴ Sum of wattmeter readings,

$$W = \sqrt{3}\,IE \cos (30 + \phi) + \sqrt{3}\,IE \cos (30 - \phi)$$

If $I_1 = I_2 = I_3 = I$

$$\begin{aligned} W &= \sqrt{3}\,IE \ [\cos (30 + \phi) + \cos (30 - \phi)] \\ &= \sqrt{3}\,IE \ [2 \cos 30 \cos \phi] \\ &= 3\,IE \cos \phi \end{aligned}$$

which is, of course, the total power in the load.

It should be noted that if one of the voltages (such as E_{12}) is more than 90° out of phase with the current associated with this voltage in the wattmeter, the pressure-coil connections must be reversed in order that the instrument may give a forward reading. Under these circumstances the wattmeter reading must be reckoned as negative, and the algebraic sum of the readings of the two instruments gives the mean value of the total power.

Another important point is that, if the power factor of the load is 0·5—so that I_1 lags 60° behind E_{10} (cos 60° being 0·5)—then the phase angle between E_{12} and I_1 is 90° and wattmeter W_1 should read zero.

Power Factor. If W_1 and W_2 are the two wattmeter readings $W_1 + W_2$ gives the total power (as seen above).

$$W_1 - W_2 = \sqrt{3}\,IE\ [\cos(30 + \phi) - \cos(30 - \phi)]$$
$$= \sqrt{3}\,IE\ [-2 \sin 30 \sin \phi]$$
$$= -\sqrt{3}\,IE \sin \phi$$

$$\therefore \quad \frac{W_1 - W_2}{W_1 + W_2} = \frac{-\sqrt{3}\,IE \sin \phi}{3\,IE \cos \phi} = -\frac{\tan \phi}{\sqrt{3}}$$

or

$$\tan \phi = \frac{\sqrt{3}\,(W_2 - W_1)}{W_1 + W_2} . \qquad (426)$$

From which ϕ and the power factor $\cos \phi$ of the load may be found.

ONE-WATTMETER METHOD. This method can be used only when the load is balanced. The connections for a star-connected system are shown in Fig. 444. The current coil of the wattmeter is connected in one of the lines, and one end of the pressure coil is connected to the same line, the other end being connected alternately to first one and then the other of the remaining two lines by means of the switch S.

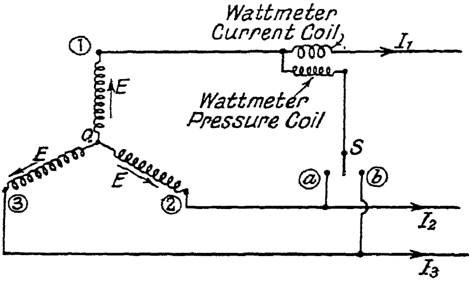


FIG. 444. ONE-WATTMETER METHOD FOR A BALANCED THREE-PHASE LOAD

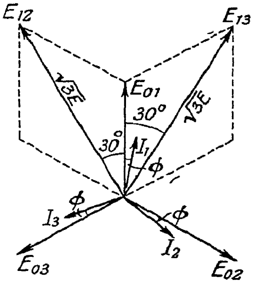


FIG. 445. VECTOR DIAGRAM, ONE-WATTMETER METHOD

The vector diagram for the method of measurement is given in Fig. 445, in which E_{01}, E_{02}, and E_{03} represent the three-phase voltages, and I_1, I_2, and I_3 the three-line currents. In a balanced system these three voltages are each equal to E, and the three currents are each equal to I. The phase angles are also all equal to ϕ. The vector E_{12} is the vector difference between E_{01} and E_{02}, and is the voltage applied to the pressure coil when the switch S is on contact "*a*." Similarly, E_{13} is the vector difference of E_{01} and E_{03}, and is applied to the wattmeter pressure coil when switch S is on contact "*b*."

Then $\quad E_{12} = E_{13} = \sqrt{3}\,E$

Wattmeter reading when switch S is on contact "a"

$$= \sqrt{3}\,EI\cos(30 + \phi)$$

$(30 + \phi)$ being the phase angle between E_{12} and I_1. When S is on contact "b" the wattmeter reading is

$$\sqrt{3}\,EI\cos(30 - \phi)$$

Thus, the sum of these readings is $3EI\cos\phi$, as shown in the analysis of the two wattmeter method, and this is the total power in the circuit.

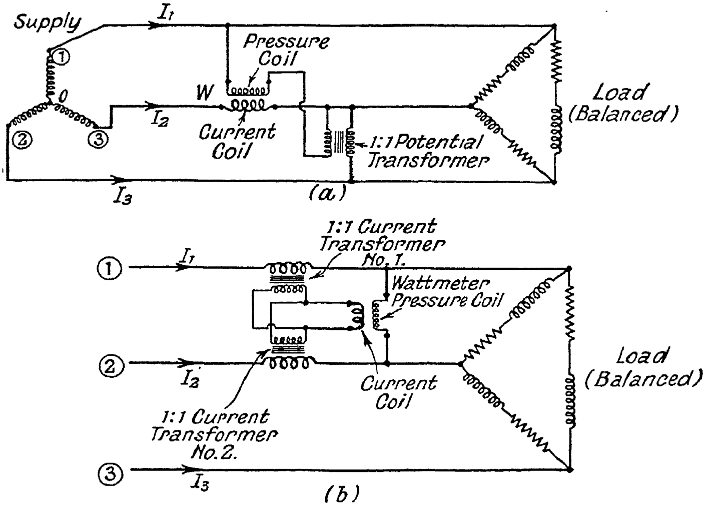


FIG. 446. BARLOW METHOD OF MEASURING THREE-PHASE POWER

In the same way the angle ϕ is given by

$$\tan\phi = \frac{\sqrt{3}\,(W_2 - W_1)}{W_1 + W_2}$$

and the power factor is $\cos\phi$ or $\cos\left(\tan^{-1}\dfrac{\sqrt{3}\,(W_2 - W_1)}{W_1 + W_2}\right)$

Method for a Balanced Three-phase Circuit using Transformers with One Wattmeter. Barlow (Ref. (11)) has described a method of measuring the power in a balanced three-phase circuit, with the load delta-connected, which does not involve the provision of an artificial neutral point and in which the wattmeter gives the total power directly without any multiplying factor.

There are two alternative methods of connection, one using a potential transformer of 1 : 1 ratio, and the other using two current transformers each of ratio 1 : 1. These connections are shown in Fig. 446, (*a*) and (*b*).

The vector diagram of Fig. 447 refers to the connections (*a*). $_{10}$, E_{20}, and E_{30} are the phase voltages of the supply, and I_1, I_2,

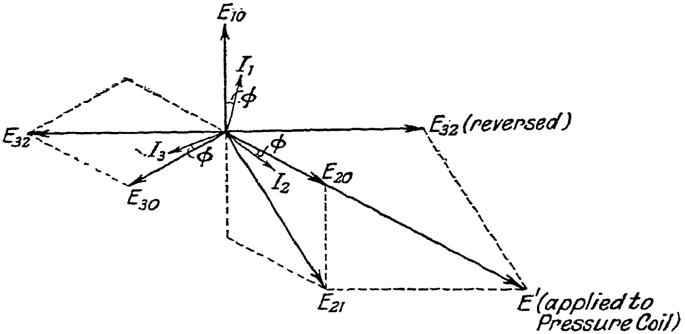


FIG. 447

and I_3 the line currents. The three voltages and three currents are equal in a balanced system, as are also the three phase angles ϕ. The current coil of the wattmeter carries the current I_2, while the voltage applied to the pressure coil is the vector difference of line voltages E_{21} and E_{32} (reversed). These two voltages are equal, and the phase angle between either of them and their vector difference E' is 30°, and is therefore in phase with E_{20}. The phase angle between E' and I_2 is thus ϕ.

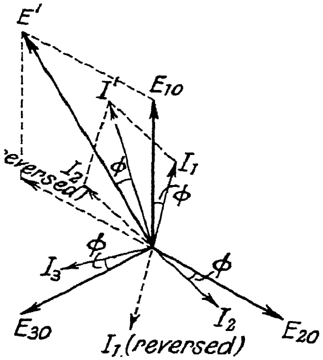


FIG. 448

The wattmeter measures $E'I_2 \cos\phi$. Now, if $E_{10} = E_{20} = E_{30} = E$ then $E_{21} = E_{32} = \sqrt{3}E$, and thus

The power measured by the wattmeter is $3EI\cos\phi$ (since $I_1 = I_2 = I_3 = I$), which is the mean value of the total power in the balanced circuit. The ratio error and phase angle of the potential transformer are, of course, neglected in the above reasoning.

Fig. 448 gives the vector diagram for the connections of Fig. 446 (*b*). Since the load is balanced,

$$I_1 = I_2 = I_3 = I$$

and
$$E_{10} = E_{20} = E_{30} = E$$

and the phase angles ϕ are all equal.

The vector E', which is the vector difference of the voltages ıd E_{20}, is the voltage applied to the pressure coil and equals . The currents in the secondary circuits of current transformers 1 and 2 are I_1 reversed and I_2 reversed. The connections to the current coil of the wattmeter are such that the current in it is the vector sum of I_1 and I_2 reversed, and is represented by the vector I'. The value of this current is $\sqrt{3}\,I$, and the phase angle between I' and E' is ϕ. Then,

$$\begin{aligned}\text{Power measured by wattmeter} &= E'I'\cos\phi\\ &= \sqrt{3}E\sqrt{3}I\cos\phi\\ &= 3EI\cos\phi\\ &= \text{total power in load}\end{aligned}$$

Again, the current transformer ratio and phase angle errors are neglected.

Wattmeters. Three types of wattmeter will be considered—

(*a*) Dynamometer.
(*b*) Induction.
(*c*) Electrostatic.

Of these, the first two are most commonly used.

(*a*) DYNAMOMETER TYPE. These instruments are similar in design and principle to the dynamometer ammeter already described (Chapter XVIII). The fixed coils carry the current in the circuit while the moving coil acts as the wattmeter pressure coil and carries a current proportional to the voltage of the circuit across which it is connected. A high non-inductive resistance is connected in series with the pressure coil.

Dynamometer wattmeters may be divided into two classes—

(i) Suspended coil, torsion instruments.
(ii) Pivoted coil, direct indicating instruments.

(i) *Torsion Wattmeters.* These instruments are used largely as *standard wattmeters.*

The moving, or pressure coil is suspended from a torsion head by a metallic suspension which serves as a lead to the coil. This coil is situated entirely inside the current, or fixed, coils and the winding is such that the system is astatic. Errors due to external magnetic fields are thus avoided. The torsion head carries a scale and, when in use, the moving coil is brought back to the zero position by turning this head, the number of divisions turned through, when multiplied by a constant for the instrument, giving the power.

Eddy current errors are eliminated as far as possible by winding the current coils of stranded wire and by using no metal parts within the region of the magnetic field of the instrument.

In the Drysdale single-phase astatic wattmeter, manufactured by Messrs. H. Tinsley, the moving coil, which is stitched to a flat strip of mica is divided into two equal portions wound so that the current (proportional to the applied voltage) circulates in a clockwise direction in one-half and in an anti-clockwise direction in the other. This coil is suspended by a silk fibre together with a spiral spring which gives the required torsion.

The fixed coil also is in two halves, which are wound so as to have opposite directions of current circulation in them. The cable used for the fixed coil consists of ten strands, insulated from one another. Thus, in effect there are ten current coils which run together and are thus as nearly as possible coincident in space. These ten coils, or cores, are brought out to a commutator so that a number of current ranges of the instrument may be obtained by grouping them all in parallel, all in series, or in a series-parallel combination.

The current is led into the moving coil by two fine phosphor-bronze ligaments, the spring, which is of German silver wire and is annealed, merely serving as a torsion control. The spring has a number of turns, and by carefully adjusting its length the constant of the instrument can be made an exact figure. The moving system carries a knife-edge pointer moving over a short scale at the front of the instrument, so that the zero position of the moving coil can be easily determined.

Damping is provided by the mica vane which carries the pressure coil. Drysdale states that, in order to reduce error due to the pressure-coil inductance to a negligible amount, the resistance of the pressure-coil circuit should be at least 3,000 ohms per millihenry of inductance.

Heavy-current Standard Wattmeters. The Drysdale wattmeter described above can be constructed to cover a current range up to 500 amp. Above this value of current, difficulties are encountered owing to the eddy currents in the heavy-section conductors required, and to non-uniform distribution of current over the cross-sections of these conductors.

Both A. E. Moore (Ref. (12)) and P. G. Agnew (Ref. (13)) have developed heavy-current standard wattmeters of the dynamometer type which are capable of dealing with currents up to about 5,000 amp.

The Agnew instrument has a double concentric tube as the "current coil," the tubes being joined at one end. The moving system consists of two astatic coils suspended, within the outer cylindrical tube, one above and one below the inner tube.

In the Moore instrument there is a tubular central conductor which can be water-cooled. This is surrounded by a cylindrical tube, or box, in two halves, bolted together (see Fig. 449), the two tubes being connected together at one end. The moving system, which is suspended within the outer tube as in the Agnew instrument, consists of two D-shaped coils with their straight sides horizontal, one above and one below the central tube.

In both instruments the magnetic field of the primary, or "current," conductor—in which field the moving system is situated—is independent of the distribution of current over the cross-sections of the tubular conductors.

(ii) *Direct-indicating Dynamometer Wattmeters.* Like the standard, or torsion, wattmeters above described, these instruments have a moving pressure coil which is almost entirely embraced by the fixed current coils. The moving coil is carried on a pivoted spindle and the movement is spring-controlled. The moving system carries a pointer and a damping vane, the latter moving in a sector-shaped box. The current coils are usually stranded or laminated, especially when heavy currents are to be carried. Currents up to about 200 amp. can be dealt with in direct-indicating wattmeters of suitable design. For currents above this, a low range wattmeter is usually employed, in conjunction with current transformers. Similarly, voltages up to about 600 volts are applied to the wattmeter pressure coil directly, but for higher voltages the pressure-coil circuit is designed for 110 volts, and a voltage transformer used to step down the voltage. Metal parts should be removed from the magnetic field of the instrument as far as possible, although care must be taken to ensure that this elimination of metal does not result in

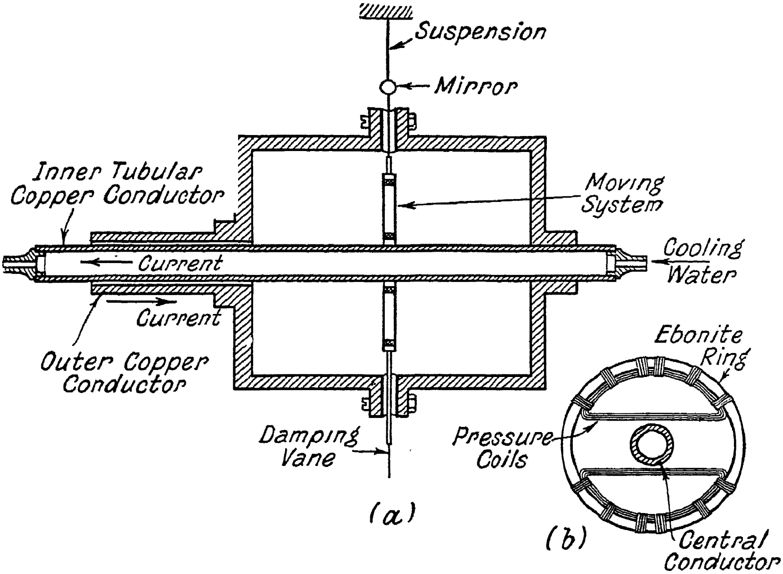
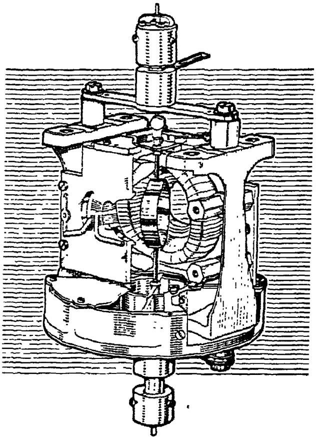


FIG. 449. A. E. MOORE'S HEAVY-CURRENT STANDARD WATTMETER

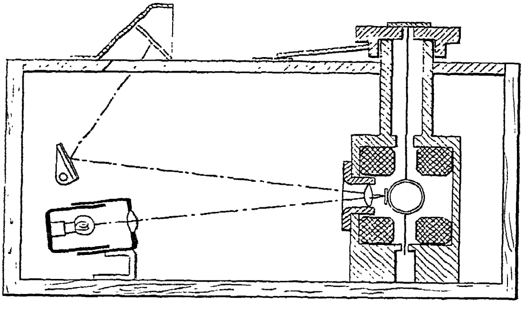
any relative movement of the working coils due to warping of the materials substituted for the metal.

Figs. 450, 451, 452 show the construction of some dynamometer wattmeters. In Fig. 450 the details of the suspended movement of a reflecting dynamometer wattmeter (Elliott Brothers (London), Ltd.) are shown. This instrument is intended for the measurement of power when the power consumption of the instrument itself must be small, e.g. when measuring the power taken by a low-power fluorescent lamp. In this particular measurement it is an advantage



(*Elliott Brothers* (*London*), *Ltd.*)

FIG. 450. MOVEMENT OF REFLECTING DYNAMOMETER WATTMETER

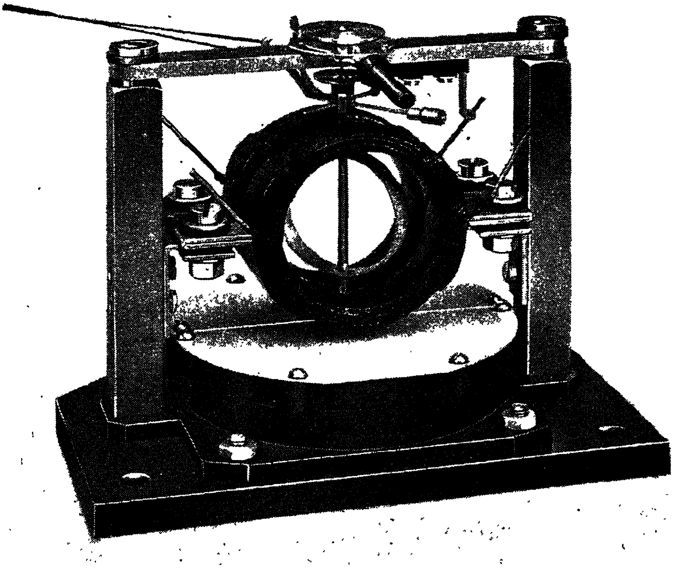


(*Everett Edgcumbe & Co., Ltd.*)

FIG. 451. INTERNAL ARRANGEMENT OF PORTABLE DYNAMOMETER WATTMETER

that the wattmeter has no iron in its magnetic circuit since the lamp voltage has a high harmonic content with an appreciable a.c. component. The wattmeter current coil takes 0·25 VA and the voltage winding is about 750 Ω per volt on 50 c/s.

Fig. 451 shows the internal arrangements of the Everett Edgcumbe wattmeter, with a moving system suspended, by a robust ligament, from a torsion head which carries a pointer moving over an evenly divided scale. The instrument is designed for measurements at very low power factors, e.g. iron loss and dielectric loss measurements. The power loss in the current element is very small, e.g. 5 microwatts when carrying a current corresponding to full scale deflection at unity power factor, the instrument being scaled to 2 watts at 12,000 volts.



(*Crompton Parkinson, Ltd.*)

FIG. 452. SINGLE ELEMENT INDICATING WATTMETER

In the deflectional type of dynamometer instrument the relative positions of current and pressure coils change with the deflection, whereas in the torsion type the relative positions are the same for all loads, since the moving coil is returned to the zero position in all cases.

Relation between the Torque and Mutual Inductance between Fixed and Moving Coils. Let the current in the current coil be given by

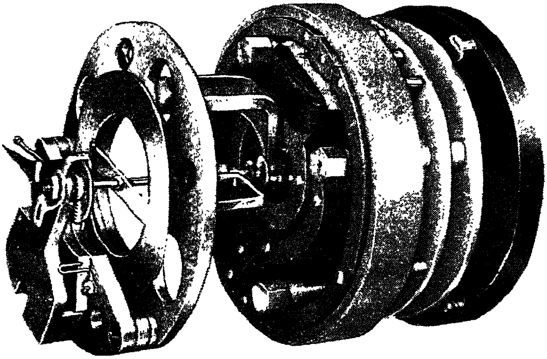
$$i_c = I_{c\,max} \sin (\omega t - \phi)$$

and let the voltage applied to the pressure coil be given by

$$e_p = E_{p\,max} \sin \omega t$$

Suppose that the resistance of the pressure coil circuit is R and its inductance negligible. Assume that there is no iron in the working magnetic circuit, and that eddy current effects in metal parts are absent.

Let S_p = Number of turns on pressure coil
S_c = ,, ,, current coil



(*Crompton Parkinson, Ltd.*)

FIG. 453. MOVEMENT OF SINGLE ELEMENT WATTMETER

Then pressure-coil current is

$$i_p = \frac{e_p}{R} = \frac{E_{p\,max}}{R} \sin \omega t$$

and is in phase with the applied voltage.

Then, referring to Fig. 455, which represents the wattmeter diagrammatically, the flux threading the pressure coil when placed with its plane parallel to those of the current coils is

$$\phi_{max} = k_1 A S_c i_c = k_2 S_c i_c$$

where A is the cross-sectional area of the pressure coil in the direction perpendicular to the magnetic axis of the current coils, uniform flux density being assumed.

The maximum value of the mutual inductance between current and pressure coils is given by

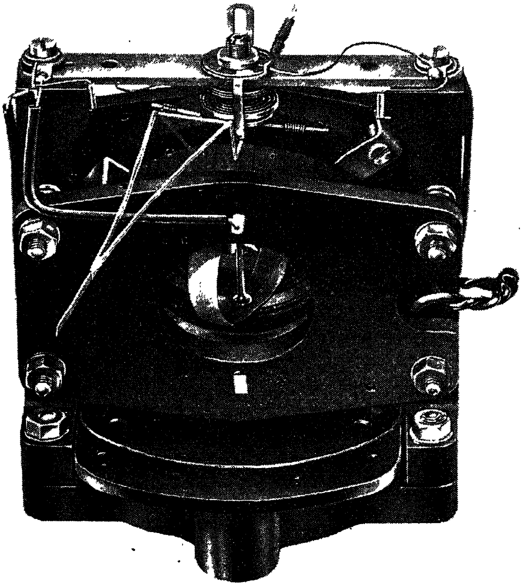
$$M_{max} = k_2 S_c S_p$$

Now the potential of a coil carrying a current i_p, and of S_p turns, when the flux threading it is ϕ is

$$i_p S_p \cdot \phi$$

If the pressure coil is turned through an angle θ from that shown in the figure, the flux threading it will be $\phi_{max}\cos\theta$ and its potential will thus be

$$\begin{aligned} V &= i_p S_p \phi_{max} \cos\theta \\ &= i_p \, . \, S_p \, . \, k_2 S_c i_c \cos\theta \\ &= M_{max}\, i_p i_c \cos\theta \quad . \quad . \quad . \quad . \quad . \quad (427) \end{aligned}$$



(*Sangamo Weston, Ltd.*)

FIG. 454. INTERNAL CONSTRUCTION OF DYNAMOMETER WATTMETER

If the torque is T when the pressure coil is in this position, the work done when the coil moves through a small angle $d\theta$ is $Td\theta$. If the corresponding change in potential is dV, we have

$$Td\theta = dV$$

$$\therefore \quad T = \frac{dV}{d\theta}$$

$$= \frac{d}{d\theta} \, . \, i_p i_c M_{max} \cos\theta$$

or

$$T = -i_p i_c M_{max} \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (428)$$

The torque will be in dyne-centimetres if i_p, i_c, and M_{max} are in e.m. units. Substituting for i_p and i_c we have for the instantaneous torque,

$$T = \frac{E_{p\,max}}{R} \sin\omega t \, . \, I_{c\,max} \sin(\omega t - \phi) \, . \, M_{max} \sin\theta$$

and for the mean torque,

$$T_M = \frac{1}{t'}\int_0^{t'} \frac{E_{p\,max} I_{c\,max} M_{max} \sin\theta}{R} \sin\omega t \sin(\omega t - \phi)\,dt$$

where t' is the periodic time of the alternating currents.

$$\therefore\ T_M = \frac{E_{p\,max} I_{c\,max} M_{max} \sin\theta}{R t'}\int_0^{t'} \sin\omega t \sin(\omega t - \phi)\,dt$$

$$= \frac{E_{p\,max} I_{c\,max} M_{max} \sin\theta}{R t'}\int_0^{t'} \frac{\cos\phi - \cos(2\omega t - \phi)}{2}\,dt$$

$$= \frac{E_{p\,max} I_{c\,max} M_{max} \sin\theta}{2R}\cos\phi$$

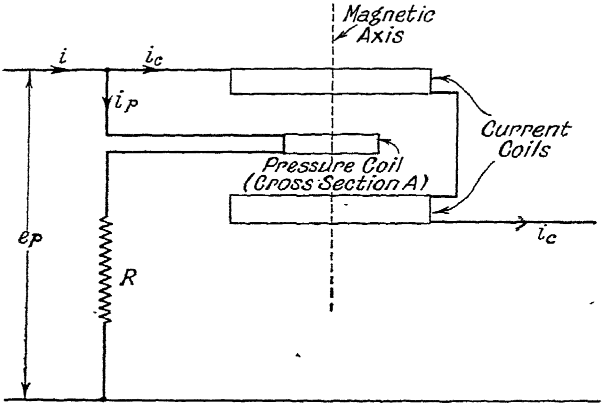


FIG. 455. CONNECTIONS OF DYNAMOMETER WATTMETER

or $$T_M = \frac{E_p I_c M_{max} \sin\theta}{R}\,.\cos\phi \qquad (429)$$

where E_p and I_c are r.m.s. values.

If the current taken by the pressure coil is negligibly small,

$$I_c = I = \text{the load current}$$

$$\therefore\quad T_M = \frac{M_{max}\sin\theta}{R} E_p I \cos\phi$$

or $$T_M = \frac{M_{max}\sin\theta}{R}\,.\,P \qquad (430)$$

where P = the power in the circuit.

Since the mutual inductance between the current and pressure coils for any angular deflection θ from the parallel position is given by

$$M = M_{max} \cos \theta$$

the mean torque may be written

$$T_M = \frac{P}{R} \cdot \frac{dM}{d\theta}$$

or, more generally,

$$T_M = K \cdot \frac{P}{R} \cdot \frac{dM}{d\theta} \qquad . \quad . \quad . \quad . \quad . \quad (431)$$

where K is a constant depending upon the units employed.

In the case of the torsion type of wattmeter, $\theta = 90°$ when the moving coil has been brought back to its zero position. The restoring torque also is proportional to the angle of twist of the torsion head. If α represents this angle of twist, we have, since the restoring torque must be equal to the deflecting torque,

$$\frac{M_{max} \sin 90°}{R} \cdot P = k'\alpha$$

or

$$P = \frac{R}{M_{max}} k'\alpha \qquad . \quad . \quad . \quad . \quad . \quad (432)$$

$$= K'\alpha$$

The errors due to inductance in the pressure-coil circuit and to other causes have already been considered in the previous pages.

A. H. M. Arnold (Ref. (16)) has discussed the performance of dynamometer wattmeters at audio frequencies (see also p. 710) and, in particular, nickel-iron cored instruments with a feed-back amplifier and current transformer.

Cambridge Reflecting Wattmeter. A laboratory-type dynamometer wattmeter suitable for the calibration of substandard instruments is made by the Cambridge Instrument Co. It has a high electromagnetic efficiency and is unaffected by stray magnetic fields. The nominal range of 500 watts is extended by the use of a range box giving four current ranges of 0·5, 1, 2·5, and 5 amp. and seven voltage ranges between 50 and 500 volts.

The scale is 2·5 metres long and consists of three parallel scales which are brought into use in turn by employing three separate lamps and optical systems. The makers state the accuracy of calibration as 0·05 per cent over the upper three-fourths of the scale.

J. de Gruchy (Ref. (19)) has pointed out the advantages of the phase-sensitive rectifier bridge as a wattmeter employing a dynamometer instrument, as a power-factor meter and as a rectifier-operated frequency meter.

(*b*) INDUCTION WATTMETERS. Induction wattmeters, the principle of which is the same as that of induction ammeters and voltmeters, can only be used on alternating current circuits: dynamometer wattmeters can be used on either a.c. or d.c. circuits.

Induction instruments are, however, only useful when the frequency and supply voltage are approximately constant.

Construction. These instruments have two laminated electromagnets; one is excited by the load current (or a definite fraction of it), and the other by a current proportional to the voltage of the circuit in which the power is to be measured. A thin aluminium disc is mounted so that it is cut by the flux from both of these magnets, and the deflecting torque is produced by the interactions between these fluxes and the eddy currents which they induce in the disc. One or more copper rings are fitted on one limb of the "shunt" magnet—i.e. the magnet excited by the pressure coil and its current—in order to cause the resultant flux in the magnet to lag in phase by exactly 90° behind the applied voltage.

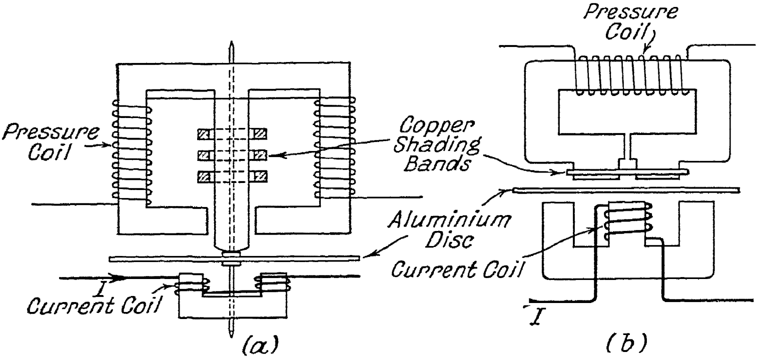


FIG. 456. INDUCTION WATTMETERS

Fig. 456 shows two common forms of magnets with their windings, the magnets being placed, in each case, one above and one below the moving disc of the instrument. The positions and shapes of the magnets are, in each case, such that the flux from both "shunt" and "series" magnets cuts the moving disc.

In the form of instrument shown in Fig. 456 (*a*), the two pressure coils, connected in series, are wound so that they both send flux through the centre limb. The series magnet in the instrument carries two small current coils in series, these being wound so that they both magnetize the core, upon which they are wound, in the same direction. The positions of the copper shading bands are adjustable in order that the correct phase displacement between the shunt and series magnet fluxes may be obtained.

In the instrument shown in Fig. 456 (*b*) there is only one pressure coil and one current coil. A copper shading band, whose position is adjustable, surrounds the two projecting pole pieces of the shunt magnet for the purpose of correcting the phase of the flux of this magnet.

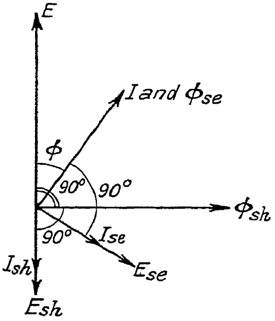


FIG. 457. VECTOR DIAGRAM FOR WATTMETER

Both types are spring-controlled and have the advantage of a long and uniform scale (up to 300°).

Currents up to about 100 amp. can be dealt with directly in such instruments. For currents above this current transformers are used in conjunction with the wattmeter. Unlike the dynamometer wattmeter, the pressure-coil circuit of the induction instrument is made as inductive as possible, in order that the flux of the shunt magnet may lag by nearly 90° behind the applied voltage.

Theory. Fig. 457 gives a simplified vector diagram for the wattmeter. The flux ϕ_{sh} of the shunt magnet is assumed to lag by exactly 90° behind the applied voltage. As previously stated, this is actually brought about by adjustment of the shading bands, since the angle of lag would be somewhat less than 90° unless such bands were used. The theory of the action of these shading bands in altering the phase of the resultant flux has already been dealt with in considering induction ammeters. It is assumed also that the flux ϕ_{se} of the series magnet is proportional to and in phase with the line current, and that hysteresis and saturation effects in the iron are negligible. Owing to the large air gap in the core these assumptions are justifiable.

Let vector E represent the applied voltage, and vector I, lagging ϕ behind E, the load current. ϕ_{se} is in phase with I.

e_{sh} = eddy e.m.f. induced in the disc by ϕ_{sh}

i_{sh} = eddy current due to and in phase with e_{sh} (the inductance of the eddy current path in the disc being neglected)

e_{se} = e.m.f. induced by the flux ϕ_{se}

i_{se} = eddy current due to and in phase with e_{se}

The instantaneous torque acting upon the disc is, from the theory of the induction ammeter (see page 666), proportional to $(\phi_{sh}i_{se} - \phi_{se}i_{sh})$ where these values of current and flux are instantaneous.

Let applied voltage be

$$e = E_{max} \sin \omega t,$$

then the current is

$$i = I_{max} \sin (\omega t - \varphi)$$

The flux

$$\phi_{se} = kI_{max} \sin (\omega t - \varphi)$$

and $$\phi_{sh} = -k'\int e\,dt = k' . \frac{E_{max}}{\omega}\cos\omega t$$
$\left(\text{since } e = -\frac{1}{k'}\frac{d\phi_{sh}}{dt}\right)$

where k and k' are constants.

The eddy e.m.f. induced by the flux ϕ_{se} is

$$e_{se} \propto \frac{d\phi_{se}}{dt} \propto I_{max}\,\omega\cos(\omega t - \phi)$$

and $$i_{se} \propto I_{max}\,\omega\cos(\omega t - \phi)$$

Also, $$e_{sh} \propto \frac{d\phi_{sh}}{dt} \propto \frac{E_{max}}{\omega} . \omega\sin\omega t$$
$$\propto E_{max}\sin\omega t$$

and $$i_{sh} \propto E_{max}\sin\omega t$$

The mean torque upon the disc

$$T_M \propto \Phi_{sh} . I_{se}\cos\phi - \Phi_{se}I_{sh}\cos(180 - \phi) \quad . \quad . \quad (433)$$

where Φ_{sh}, I_{se}, Φ_{se}, and I_{sh} are r.m.s. values and ϕ and $(180 - \phi)$ the phase angles between the interacting currents and fluxes.

By substitution we have

$$T_m \propto \frac{E}{\omega} . I\omega\cos\phi + IE\cos\phi$$

where I and E are r.m.s. values of current and voltage.

$$\therefore \quad T_m \propto EI\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (434)$$
$$\propto \text{the power in the circuit.}$$

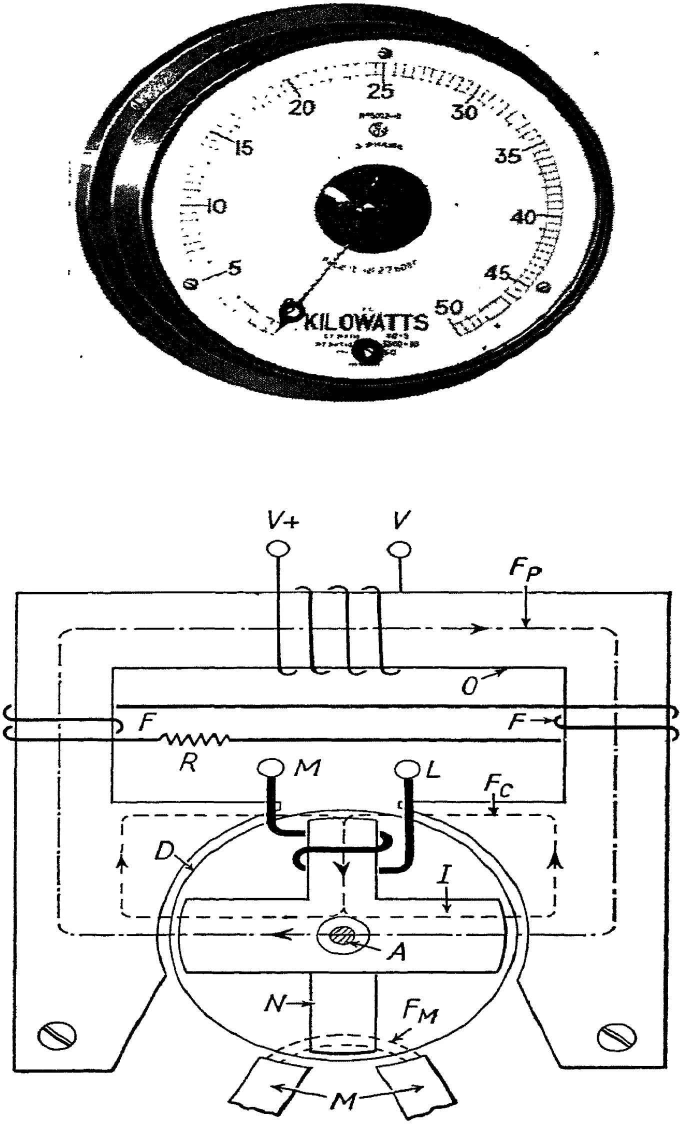
Comparison with Dynamometer Wattmeters. It appears from the above that the torque is independent of frequency. Actually, however, the torque is not quite independent of frequency, the above theory being simplified by assumptions, regarding the inductances of the various parts of the instruments, which are not justifiable under all conditions. It will, however, be sufficient to indicate the working principles of the instrument.

Since the deflecting torque is directly proportional to the power in the circuit, and the instrument is spring-controlled, the scale is uniform.

Compared with dynamometer wattmeters, these instruments have the advantages of a greater working torque and length of scale; but they suffer from the inherent disadvantages of less accuracy, greater weight of moving system, greater power consumption, and also that they can only be applied to power measurements on a.c. circuits.

Induction wattmeters are capable of first-grade accuracy only at a stated frequency and temperature. The requirements of wattmeters of the various grades of accuracy and the permissible limits of error are stated in British Standard Specification No. 89, "Indicating Ammeters, Voltmeters, Wattmeters, etc." (1937).

Fig. 458 (*b*) shows the arrangement of the circuits in the Lipman



(*Nalder Bros. & Thompson, Ltd.*)

G. 458. Induction Wattmeter, Lipman Type

type of induction wattmeter (Nalder Bros. and Thompson, Ltd.) which has a low VA consumption of its windings.

The potential winding *VV* is wound on the main magnet core which also carries coils *FF* for power-factor compensation, the resistance *R* being adjustable for calibration purposes to bring the operating fluxes, F_p and F_c, 90° out of phase with one another. Within the moving element, or rotor, *D*—an aluminium drum—is a fixed core, of cruciform shape, completing the magnetic circuit. The current coil *ML* is wound on one limb of this cruciform core. The potential flux passes horizontally through the rotor while the flux due to the current coil acts on the rotor in a vertical direction.

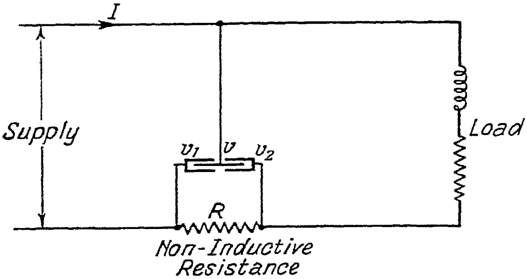


FIG. 459. CONNECTIONS OF ELECTROSTATIC WATTMETER

(c) ELECTROSTATIC WATTMETER. This form of wattmeter cannot be regarded as a commercial instrument like the dynamometer and induction forms already discussed. It is, however, a very useful instrument for the measurement of small amounts of power, especially when the voltage is high and the power factor low. The use of this instrument for the measurement of dielectric power loss was considered in Chapter IV. It is also useful for the calibration, in the laboratory, of commercial forms of wattmeter and watt-hour meter.

The electrostatic wattmeter consists of a quadrant electrometer used in conjunction with a non-inductive resistance, the essential connections to the load circuit being as shown in Fig. 459.

Let the load current be i (instantaneously), and let v, v_1, and v_2 be the instantaneous potentials of the needle and of the two pairs of quadrants, as shown in the figure.

It was shown when dealing with the theory of the quadrant electrometer (page 655) that the instantaneous torque

$$T \propto (v - v_1)^2 - (v - v_2)^2 \quad . \quad . \quad . \quad . \quad (435)$$

Now $(v - v_1)$ is the instantaneous value e of the supply voltage. Also $v - v_2 = (v - v_1) - (v_2 - v_1) = e - iR$, since the potential difference $v_2 - v_1 = iR$.

Hence the instantaneous torque

$$\begin{aligned} T &\propto e^2 - (e - iR)^2 \\ &\propto e^2 - e^2 + 2eiR - i^2R^2 \\ &\propto 2eiR - i^2R^2 \end{aligned}$$

or

$$T \propto 2R\left[ei - \frac{i^2R}{2}\right] \quad . \quad . \quad . \quad . \quad . \quad (436)$$

i.e. $T \propto$ the instantaneous power supplied *minus* half the watts lost in the non-inductive resistance R

If e = instantaneous value of the voltage on the load side of the instrument,

$$T \propto 2R\left[ei + \frac{i^2R}{2}\right] \quad . \quad . \quad . \quad . \quad (437)$$

$\propto$ the instantaneous power in the load *plus* half the watts lost in the non-inductive resistance

Other Methods of Use. The simple connections given in Fig. 459 have, as seen above, the disadvantage that the power measured will be different from the true power by an amount equal to half the loss in the resistance R.

Fig. 460 gives three methods of use whereby this disadvantage may be overcome.

Method 1. Diagram (*a*) shows connections which are essentially the same as those in Fig. 459, except that a two-way switch S is used so that the needle of the electrometer may be connected either to one side of the load or the other, as required.

Now, in the method discussed in the previous paragraph, the instantaneous torque

$$T \propto 2eiR + i^2R^2$$

if e is the instantaneous value of the voltage applied to the load.

If the controlling torque of the instrument is proportional to the deflection θ, we have for the steady deflection θ_D of the instrument

$$K\theta_D = \frac{1}{t'}\int_0^{t'} (2eiR + i^2R^2)\,dt \quad . \quad . \quad . \quad (438)$$

where the expression on the right-hand side is that for the mean value of the torque and K is a constant. t' is the periodic time of the voltage and current waves.

$$\begin{aligned} \therefore \quad \theta_D &= \frac{2R}{Kt'}\int_0^{t'} ei\,dt + \frac{1}{Kt'}\int^{t'} i^2R^2\,dt \\ &= \frac{2R}{K}P + \frac{1}{Kt'}\int_0^{t'} i^2R^2\,dt \end{aligned}$$

where P is the mean power in the load. Thus, when the switch S is on contact a, the deflection will be $\theta_D = \frac{2R}{K} P + \frac{1}{Kt'}\int_0^{t'} i^2R^2dt$.

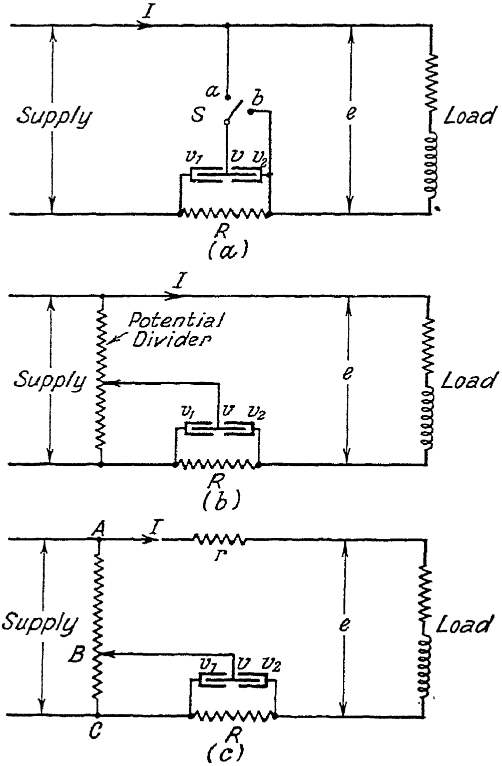


FIG. 460. ALTERNATIVE CONNECTIONS OF ELECTROSTATIC WATTMETER

Now, if the switch is thrown on to contact b, $v = v_2$, and we have, for the instantaneous torque,

$$\begin{aligned} T &\propto (v - v_1)^2 \\ &\propto (v_2 - v_1)^2 \\ &\propto i^2R^2 \end{aligned}$$

Thus, if θ_D' is the deflection under these conditions,

$$\theta_D' = \frac{1}{Kt'}\int_0^{t'} i^2R^2\,dt$$

$$\therefore \qquad \theta_D = \frac{2R}{K} \cdot P + \theta_D{}'$$

or

$$P = \frac{K(\theta_D - \theta_D{}')}{2R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (439)$$

So that the true power in the load can be obtained from the two readings corresponding to the two positions of switch S. This method was introduced by Potier.

Method 2. In the second method, a potential divider, consisting of a high non-inductive resistance, is used, and the electrometer needle is connected to its middle point.

The instantaneous voltage across the potential divider is

$$e + (v_2 - v_1)$$

and therefore that across half of it is

$$\frac{e + (v_2 - v_1)}{2}$$

Thus the instantaneous torque

$$\begin{aligned} T &\propto (v - v_1)^2 - (v - v_2)^2 \\ &\propto \left[\frac{e + (v_2 - v_1)}{2}\right]^2 - \left[\frac{e - (v_2 - v_1)}{2}\right]^2 \\ &\propto \tfrac{1}{4}\left[\{e + (v_2 - v_1) + e - (v_2 - v_1)\}\right. \\ &\qquad \left.\{e + (v_2 - v_1) - e + (v_2 - v_1)\}\right] \\ &\propto \tfrac{1}{4}[2e][2(v_2 - v_1)] \\ &\propto e(v_2 - v_1) \\ &\propto eiR \\ &\propto ei \end{aligned}$$

Thus with these connections the instrument measures the true power in the load.

If the use of the middle point of the potential divider as the point of connection of the needle is not feasible (possibly owing to the high voltage which would thereby be applied to the instrument), the method of connection shown in diagram (c) and suggested by Prof. Miles Walker, may be used.

Method 3. In this method a non-inductive resistance r is connected in series with the load, and the needle of the electrometer is connected to a suitable point on the potential divider.

$$\text{Let} \quad \frac{\text{Voltage } A \text{ to } C}{\text{Voltage } B \text{ to } C} = n$$

Then, instantaneous torque

$$T \propto (v - v_1)^2 - (v - v_2)^2$$

Now $(v - v_1) = \dfrac{e + ir + iR}{n}$

and $(v - v_2) = v - v_1 - (v_2 - v_1) = (v - v_1) - iR = \dfrac{e + ir + iR}{n} - iR$

$$\therefore \quad T \propto \left(\frac{e + ir + iR}{n}\right)^2 - \left(\frac{e + ir + iR}{n} - iR\right)^2 \quad . \quad (440)$$

$$\propto 2iR\left(\frac{e + ir + iR}{n}\right) - i^2R^2$$

or

$$T \propto \frac{2iRe}{n} + \frac{2i^2rR}{n} + \frac{2i^2R^2 - ni^2R^2}{n} \quad . \quad . \quad . \quad (441)$$

Now, if the value of r is made equal to $\dfrac{R}{2}(n - 2)$, we have, by substitution,

$$T \propto \frac{2iRe}{n} + 2i^2\frac{R^2(n - 2)}{2n} + \frac{i^2R^2(2 - n)}{n} \quad . \quad . \quad (442)$$

$$\propto \frac{2iRe}{n}$$

$$\propto ie$$

$\propto$ true power in the load

It should be noted that in the above discussions the current taken by the electrometer (as a capacitor) is neglected. The potential divider also, although spoken of as a non-inductive resistance, may be the high voltage winding of the supply transformer from which tappings are brought out.

Use for Calibration of Commercial Wattmeters or Watt-hour Meters. Fig. 461 shows the connections for the calibration of a commercial instrument by means of the electrostatic wattmeter. The load is, in this case, fictitious. The voltage for the pressure coil of the wattmeter under test is supplied from one alternator (A_2)—possibly through a step-up transformer (not shown in the figure)—while the current coil of the instrument is supplied from another alternator (A_1). If the required current is large, a step-down transformer may be used. The two alternators, which are driven by a motor M, are mechanically coupled, and arrangements are made for rotating the stator of one of them so that any required phase displacement between current and voltage may be obtained. In this way the power factor of the load is altered.

The needle of the electrostatic instrument is supplied from a tapping point on a potential divider across which the voltage from alternator A_2 is connected. The two pairs of quadrants are connected to the two ends of a non-inductive resistance R, through which the load current flows. The mid-point of this resistance is connected to one end of the potential divider, as shown.

$$\text{Let} \quad \frac{\text{Voltage } A \text{ to } C}{\text{Voltage } B \text{ to } C} = n$$

Then, if e is the instantaneous value of the load voltage, the instantaneous value of the voltage B to C is $\frac{e}{n}$, and hence

$$v - v_2 = - -$$

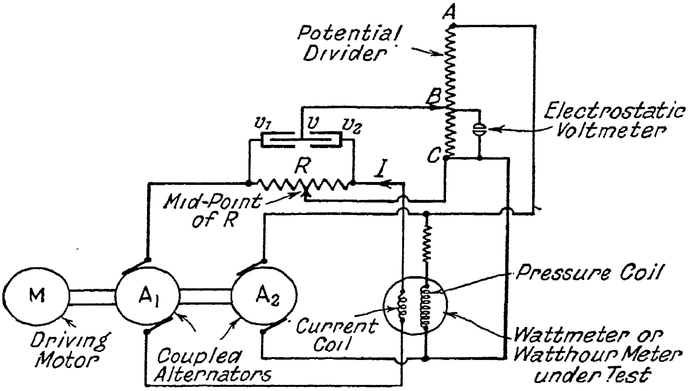


FIG. 461. USE OF ELECTROSTATIC WATTMETER FOR WATTMETER TESTING

where i is the instantaneous value of the load current

$$\text{and} \quad v - v_1 = \frac{e}{n} + \tfrac{1}{2}\,iR$$

Thus, the instantaneous torque

$$T \propto (v - v_1)^2 - (v - v_2)^2$$

$$\text{or} \quad T \propto \left(\frac{e}{n} + \frac{iR}{2}\right)^2 - \left(\frac{e}{n} - \frac{iR}{2}\right)^2 \quad . \quad . \quad . \quad . \quad (443)$$

$$\propto \left(\frac{e}{n} + \frac{iR}{2} + \frac{e}{n} - \frac{iR}{2}\right)\left(\frac{e}{n} + \frac{iR}{2} - \frac{e}{n} + \frac{iR}{2}\right)$$

$$\propto \left(\frac{2e}{n}\right)(iR)$$

$$\propto ei$$

$\propto$ true instantaneous power in the load.

The mean value of the torque

$$T_m \propto \frac{2R}{nt'}\int_0^{t'} eidt$$

$$\frac{2R}{n} \text{ (mean power in the load)}$$

t' being the periodic time of the current and voltage waves.

If θ_D is the steady deflection of the electrostatic instrument, which is spring-controlled, we have that

$$k\theta_D = T_m \text{ where } k \text{ is a constant}$$

or mean power $= \frac{nk'\theta_D}{2R}$

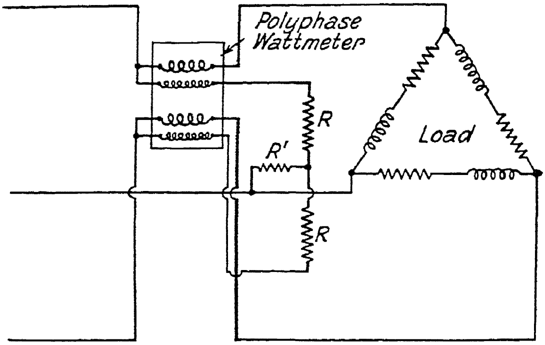


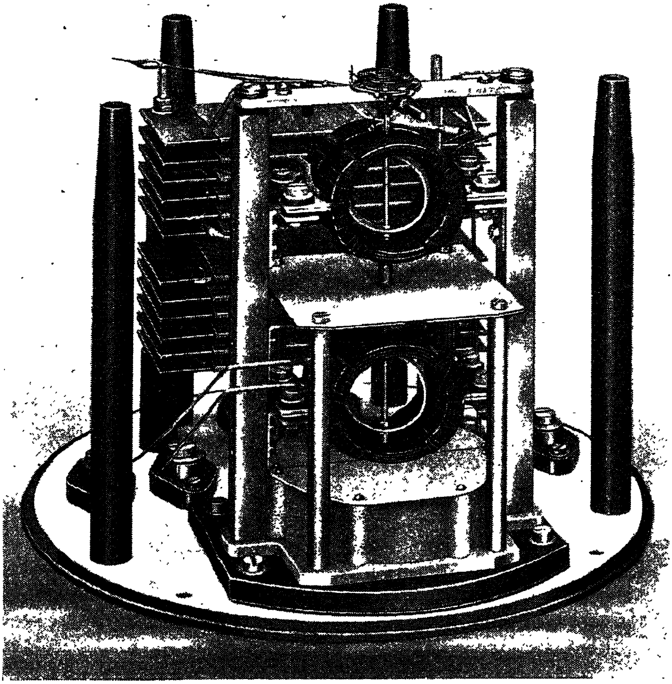
FIG. 462. POLYPHASE WATTMETER CONNECTIONS

where k' is the constant of the electrostatic instrument, and must be determined experimentally by the use of standard resistors and an electrostatic voltmeter.

F. R. Axworthy and J. K. Choudhury (Ref. (17)) have discussed the use of the electrostatic wattmeter, particularly for iron-loss measurements with mixed frequency inductions, and to measure the power absorbed by fluorescent lamps. They describe the construction of a portable electrostatic wattmeter of the quadrant type with a self-contained lamp and scale.

POLYPHASE WATTMETERS. In order that the power in a polyphase circuit may be measured without the use of more than one wattmeter, polyphase wattmeters have been developed. These consist of two separate wattmeter movements mounted together in one case with the two moving coils (assuming the instrument to be of the dynamometer type) mounted on the same spindle, so that the total deflecting torque acting on the moving system shall be the sum of the torques produced by the two component wattmeter working systems. The readings of the instrument thus give the total power in the circuit directly, the addition being carried out by the instrument itself.

Thus, for example, in measuring three-phase power, the connections to the polyphase wattmeter are the same as those for the measurement of power by the two-wattmeter method, using two single-phase instruments. The only difference is that the two single-phase instruments are combined in the polyphase wattmeter and operate a single moving system whose deflection gives the total power directly.



(Crompton Parkinson, Ltd.)

FIG. 463. DOUBLE ELEMENT WATTMETER

An important point in connection with such double element wattmeters is that there shall be no mutual interference between the two elements. Thus, the fixed coils of one of the elements must not produce a torque by interacting with the field of the moving coil of the other element. In order to eliminate any such action and ensure that the total torque shall be merely the sum of the torques produced within the two component elements themselves, a laminated iron shield is placed between the two elements to provide magnetic screening.

Compensation for mutual interference can be made by a method due to Weston, the connections for which are shown in Fig. 462. Instead of connecting the two pressure coils—each of resistance R (including their series resistances)—directly to the line which does not contain a current coil, they are each connected to one end of a resistance R' whose other end is connected to the line, as shown. By suitably adjusting R' the currents in the pressure-coil circuits can be altered to compensate for any mutual interference between the wattmeter elements (Ref. (2)).

Fig. 463 illustrates the internal construction of a commercial type of double-element wattmeter. A nickel-iron plate shields the two elements from one another. To reduce the weight of the moving system, aluminium is used for many of the parts. The total weight of the double-element movement is 5·25 grammes and the torque/weight ratio 0·09 (average for 100° deflection).

Fig. 464 shows the construction of the Drysdale standard polyphase wattmeter, manufactured by Messrs. H. Tinsley & Co.

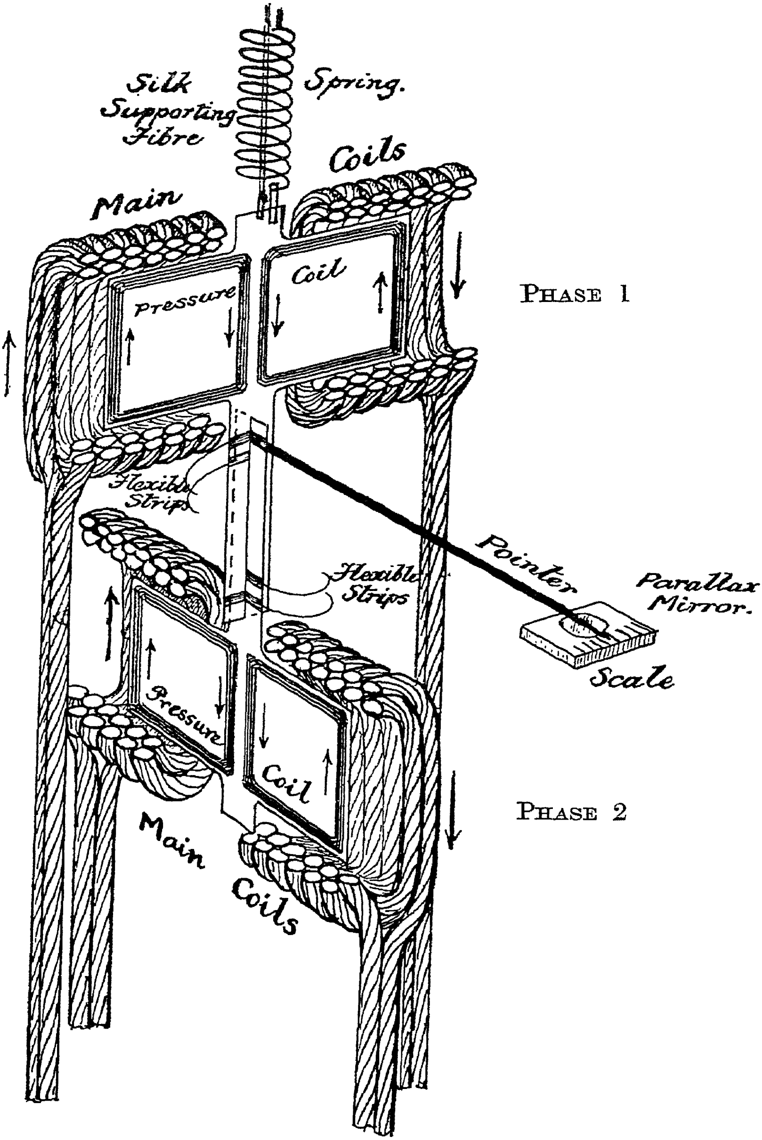
These wattmeters have current ranges of 0·5, 1·0, 2·5, 5·0 amperes (the range can be extended to 2,400 A by the use of Class AL current transformers). Voltages up to 1,000 V can be used by employing voltage resistance boxes.

Summation Methods. It is often necessary to measure the total power in a number of separate circuits, and it is convenient to have this total indicated upon a single instrument. This is a simple matter if the circuits are interconnected so that one portion of the system carries the whole load. If, however, there is no part of the system carrying the aggregate load, more complicated methods of measurement must be employed.

A four-element wattmeter, which consists essentially of two polyphase wattmeters like those described in the preceding paragraph with all four moving systems (two in each polyphase wattmeter) mounted on a common spindle, can be used for the measurement of the total power in two three-phase systems, but for more than two systems current transformers are used in conjunction with a double-element "summation" wattmeter.

The principle of such summation measurements is illustrated in Fig. 465, in which the total power supplied to three circuits A, B, and C is measured by a two-element summation wattmeter, in conjunction with six current transformers, all of whose ratios are the same. Actually voltage transformers may also be necessary if the line voltage is too great to be applied directly to the pressure coils of the wattmeter, but these have been omitted for simplicity.

It will be observed that the principle of measurement is essentially



(H. Tinsley & Co., Ltd.)

FIG. 464. DRYSDALE STANDARD POLYPHASE WATTMETER

that of the two-wattmeter method of measurement of three-phase power.

Current transformers a, b, and c, together supply current coil F_1, while transformers a', b', and c' supply current coil F_2. Neglecting the small phase angles of the current transformers, the currents in their secondary circuits are each 180° out of phase with their primary currents, and hence the vector sum of the former will be equal to the vector sum of the primary currents divided by the ratio of the transformer (the same in all cases), the ratio errors of the transformers also being neglected. It can be seen, therefore, that the current supplied to current coil F_1 is proportional to the vector sum of the currents supplied to the three circuits A, B, and C by line (3), and the current supplied to current coil F_2 is proportional to the vector sum of the currents supplied by line (1).

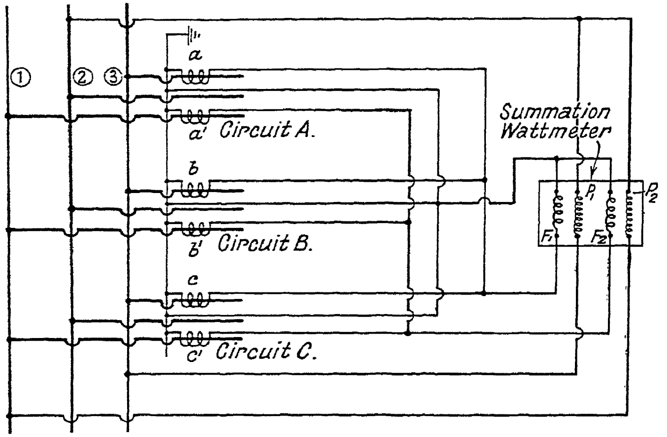


FIG. 465. CONNECTIONS FOR SUMMATION METHOD OF POWER

The pressure coils P_1 and P_2 are connected between lines (3) and (2) and between (1) and (2) respectively.

The reading of the summation wattmeter is thus equal to the total power supplied to the three circuits A, B, and C, divided by the ratio of the current transformers, the current transformer errors being neglected.

The subject of summation metering is discussed further in Ch. XXI (p. 808).

Measurement of Reactive Volt-amps. If E and I are the r.m.s. values of the voltage and current in a single-phase circuit and ϕ is the phase angle between them, then the actual power in the circuit is, of course, $EI \cos \phi$.

The power is equal to the voltage multiplied by the component of the current which is in phase with it (i.e. $I \cos \phi$). The component of the current, which is 90° out of phase with the voltage, is $I \sin \phi$, and the product $EI \sin \phi$ is the "wattless" or "reactive" component of the volt-amperes. The measurement of this reactive component is, however, useful, since the phase angle ϕ of the circuit can be obtained from the ratio $\frac{\text{Reactive component}}{\text{True power}}$, which equals $\frac{EI \sin \phi}{EI \cos \phi}$, or $\tan \phi$.

It should first be observed that $\sin \phi = \cos (90 - \phi)$, and therefore a wattmeter may be used for the measurement if the current coil carries the load current I and the voltage applied to the pressure coil is such that its phase displacement from the actual voltage of the circuit is 90°. Under these circumstances the wattmeter will read $IE \cos (90 - \phi)$ or $IE \sin \phi$.

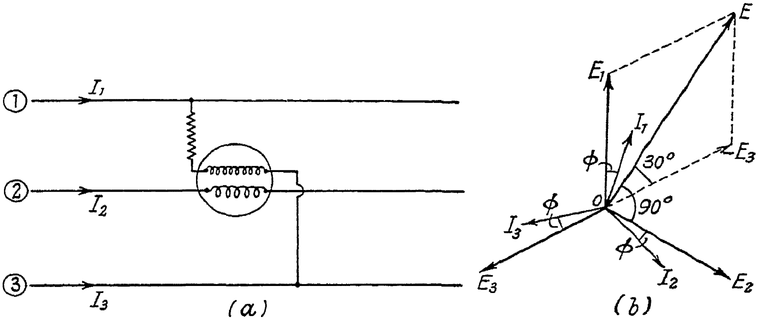


FIG. 466. MEASUREMENT OF REACTIVE POWER

It may be more convenient in single-phase measurements to compensate the wattmeter so that the field of its pressure coil lags 90° in phase behind the phase of this field when the wattmeter is to be used for power measurement.

In the case of a balanced three-phase load, when the power is being measured by one wattmeter, it is a simple matter to use this wattmeter for the measurement of the reactive volt-amps by connecting its current and pressure coils, as shown in Fig. 466. The current coil is connected in one line and the pressure coil is connected across the other two lines.

Referring to the vector diagram of Fig. 466 (*b*), the vector OI_2 represents the current through the wattmeter current coil. The voltage applied to the pressure coil is the vector difference of OE_1 and OE_3, i.e. the vector OE. Now the angle between OE_2 and OE_3 reversed is 60°, and since the three-phase voltages E_1, E_2, and E_3 are equal, the angle between OE_3 reversed and vector OE is 30°.

Hence, the angle between OE_2 and OE is 90°, and between OI_2 and OE the angle is $(90 + \phi)$. This means that the wattmeter reads

$$\begin{aligned} & OE \times OI_2 \times \cos(90 + \phi) \\ &= \sqrt{3}EI \cos(90 + \phi) \\ &= -\sqrt{3}EI \sin\phi \\ &= -W_r \end{aligned}$$

where E is the phase voltage and I the line current of the system.

The total reactive volt-amperes of the circuit is

$$3EI \sin\phi = -\sqrt{3}\,W_r$$

and hence, if W is the measured value of the total real power, the phase angle ϕ of the load is

$$\tan^{-1}\left(\frac{\sqrt{3}\,W_r}{W}\right)$$

BIBLIOGRAPHY AND REFERENCES

(1) *Commercial A.C. Measurements*, G. W. Stubbings.

(2) *Electrical Measurements*, F. A. Laws.

(3) *Alternating Current Electrical Engineering*, W. T. Maccall.

(4) *Theory and Practice of Alternating Currents*, A. T. Dover.

(5) *Electrical Measuring Instruments*, C. V. Drysdale and A. C. Jolley.

(6) *Theory and Calculation of Electric Currents*, J. L. La Cour and O. S. Bragstad.

(7) British Standard Specification, No. 89 (1929).

(8) *Alternating Currents*, A. Russell.

(9) "Recent Developments in Electricity Meters, with particular reference to those for Special Purposes," J. L. Carr, *Jour. I.E.E.*, Vol. LXVII, p. 859.

(10) "The Metering Arrangements for the 'Grid' Transmission System in Great Britain," C. W. Marshall, *Jour. I.E.E.*, Vol. LXVIII, p. 1497.

(11) "A New Method of Measuring the Total Power in a Balanced Three-phase Circuit, employing only One Wattmeter," H. Monteagle Barlow, *Jour. I.E.E.*, Vol. LXVI, p. 1001.

(12) "A New Concentric Standard Dynamometer Wattmeter for Heavy Currents; and Concentric Non-inductive Standards of Low Resistance," A. E. Moore, *Jour. I.E.E.*, Vol. LV, p. 380.

(13) P. G. Agnew, *Bull. Bur. Sta.*, Vol. VIII (July, 1912).

(14) "A Valve Wattmeter," E. Mallett, *Jour. I.E.E.*, Vol. LXXIII, p. 295.

(15) "The Development of a Sensitive Precision Wattmeter for the Measurement of very Small Powers." N. H. Searby, *Jour. I.E.E.*, Vol. LXXVI, p. 205.

(16) "Performance Limits in Electrical Instruments," A. H. M. Arnold, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, p. 701.

(17) "The Design and Application of a Portable Electrostatic Wattmeter," F. R. Axworthy and J. K. Choudhury, *Trans. Soc. Inst. Tech.*, Vol. IV, No. 2, June, 1952.

(18) "The Use of an Electrostatic Wattmeter for Magnetic-loss Measurements," J. K. Choudhury and A. Glynne, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, p. 539; and Vol. XCIX, Pt. II, p. 38.

(19) "Some Applications of the Phase-sensitive Rectifier Bridge to Electrical Measurements," J. de Gruchy, *Trans. Soc. Inst. Tech.*, Vol. II, No. 3, 1950.

CHAPTER XXI

THE MEASUREMENT OF ENERGY

It may be stated generally that the measurement of energy is essentially the same process as the measurement of power, except that the instrument used must not merely indicate the power, or rate of supply of energy, but must take into account also the length of time for which this rate of supply is continued. Actually, energy, or "supply" meters do not indicate power directly. For a given amount of energy supplied to a circuit, their registrations should always be the same, no matter what the instantaneous values of the power during the time in which the energy is supplied.

Types of Energy, or Supply, Meters. There are three general types of energy, or supply, meters—

(*a*) Electrolytic meters.
(*b*) Motor meters.
(*c*) Clock meters.

The first type is essentially a d.c. instrument although it can be made to function in an a.c. circuit for the measurement of kilovolt-ampere hours by using a rectifier unit and a current transformer (see Fig. 467); the other two types may be used on either d.c. or a.c., according to their construction. The induction form of motor meter can, of course, only be used on a.c. circuits, since its principle is the same as that of the induction wattmeter. As already mentioned in Chapter XVII, supply meters used on d.c. circuits may be either ampere-hour, or watt-hour, meters. In the former case, the registrations of the meter are converted to watt-hours by multiplying by the voltage (assumed constant) of the circuit in which it is used. Usually such meters are calibrated to read direct in kilowatt-hours at the declared voltage, thus rendering the readings incorrect when used on any other voltage.

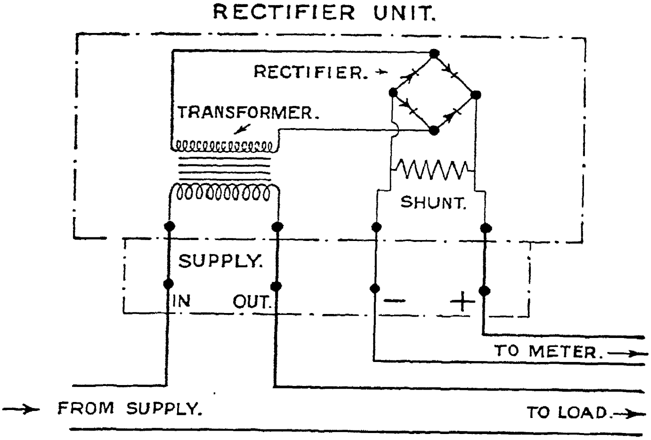
In the same way the question of power factor obviously prevents the use of such ampere-hour meters on a.c.

The advantages of simplicity and cheapness in the case of ampere-hour meters is largely discounted by the fact that variations of circuit voltage are not taken into account by them. For example, suppose that the voltage of a supply, whose nominal voltage is 220, has an average value of only 215 volts for a period of 1 hour, during which a current of 100 amp. is being taken by a consumer. Then, if an ampere-hour meter, calibrated for 220 volts, is used to measure the energy supplied to the consumer, the measured quantity of energy will be 220×100 watt-hours, or 22 kilowatts; and it is for this amount that the consumer will be charged.

Actually, however, the energy supplied is only 215 × 100 watt-hours or 21½ kilowatt-hours, so that the consumer loses the cost of ½ kWh. in one hour under these conditions.

A watt-hour meter would have taken into account the fall in the supply voltage and would therefore have meant a saving in cost to the consumer. The converse of the above occurs, of course, if the supply voltage is higher than the nominal when the supply company loses by the use of an ampere-hour meter.

Electrolytic Meters. These are inherently ampere-hour meters, s are proportional to the weight of metal deposited, or of gas liberated, from an electrolytic solution. This means that the readings are merely proportional to the number of coulombs, or of ampere-hours, passed through the meter.



(*Reason Manufacturing Co., Ltd.*)

FIG. 467. REASON RECTIFIER UNIT FOR USE WITH ELECTROLYTIC METERS

In addition to cheapness and simplicity, already mentioned, these meters have the advantages that they are accurate even at very small loads; they are unaffected by stray magnetic fields, since they do not depend in any way upon the magnetic effect of the current; and, since there are no moving parts, friction errors are absent. Their general disadvantages are that the potential drop across their terminals is from 1 to 2 volts; the considerable amount of glass used in their construction renders them somewhat fragile; the necessity for frequent inspection to ensure that they do not stand in need of refilling or resetting; and the destruction of the old record of energy supplied when the meter is reset.

For an electrolytic meter to be shunted successfully, the back e.m.f. of polarization must be small.

This can be seen from the following—

Let the back e.m.f. of the meter $= e$, and let the resistance of the meter circuit $= r$. Resistance of the shunt $= R_s$. Then, referring to Fig. 468,

I = the load current

I_m = the meter current

I_s = the current through the shunt

$$I_sR_s = I_mr + e$$

But

$$I = I_s + I_m$$

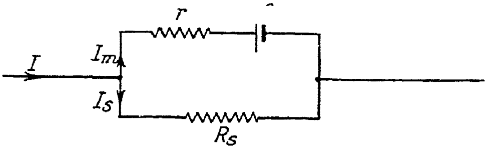


FIG. 468

$$\therefore \quad (I - I_m) R_s = I_mr + e$$

$$IR_s = I_m(r + R_s) + e$$

$$I_m = \frac{IR_s}{(r + R_s)} - \frac{e}{(r + R_s)}$$

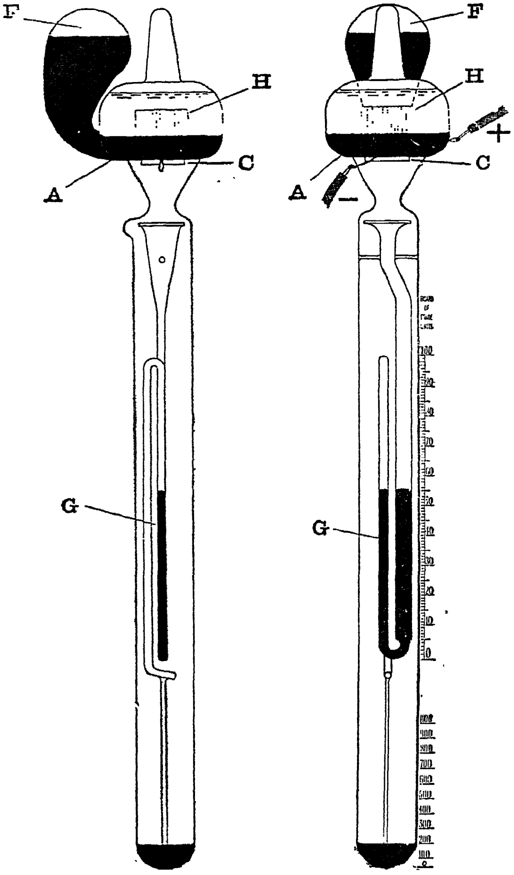
or

$$\frac{I_m}{I} = \frac{R_s}{r + R_s} - \frac{e}{I(r + R_s)} \qquad (444)$$

Thus it can be seen that, if e is constant, the meter current is not a constant proportion of the load current, but is reduced as I is reduced.

THE WRIGHT, OR REASON, METER. The construction of this meter is shown in Fig. 469. It has been very generally used on d.c. supply systems, and has been perhaps the most successful of all electrolytic meters.

The anode *A* consists of an annular ring of mercury, contained in a shallow trough at the top of the tube. The loss of mercury by the anode, owing to its transference to the cathode during the electrolytic action, is compensated for by a reservoir of mercury *F* projecting from the side of the part of the tube at which the anode ring is situated. This reservoir maintains the level of the mercury in the anode constant. The cathode *C* consists of a ring of sand-blasted iridium, and the electrolyte, which fills the whole of the tube of the meter with the exception of the space occupied by mercury, is a saturated solution of mercury and potassium iodides. The tube is hermetically sealed and external conditions have therefore no effect upon the action of the meter. Standing up from the inner circumference of the annular anode trough is a grid or fence of glass rods to prevent mercury from the anode spilling over into the measuring tube *T*, owing to mechanical vibration. This fence does not, however, interfere with the full circulation of the electrolyte. The measuring tube has a funnel-shaped mouth, and is situated immediately below the cathode ring in order to catch the drops of mercury which fall from the latter after being deposited from the electrolyte by the electrolytic action of the current. Alongside this measuring tube is a scale calibrated in kW-hours, and corresponding, of course, to the voltage of the circuit in which the meter is to be used. The measuring tube is bent back on itself twice,



(*Reason Manufacturing Co., Ltd.*)

FIG. 469. REASON ELECTROLYTIC METER

so that, upon the mercury in it reaching a certain level, it siphons over into a larger measuring tube below. This siphoning occurs after the passage of 100 units through the meter, and the lower (and larger) measuring tube is thus graduated in hundreds of units. The electrolytic action is as follows. The current is led into and out of the meter by platinum wire leads, one dipping into the mercury anode and the other being welded to the iridium cathode. The passage of the current results in a chemical action which removes mercury from the anode and deposits an equal quantity of mercury upon the cathode, the electrolyte itself being left unchanged. The weight of mercury so deposited is obviously directly proportional to the quantity of electricity passed through the meter; and this results in a strictly uniform scale (assuming uniformity of the sectional area of the measuring tube) The meter is reset by inverting it and allowing the mercury to run back into the upper receptacle.

This type of meter is almost always shunted in order to increase its capacity. Its polarization back e.m.f. is only about 0·0001 volt, and thus does not introduce any appreciable error at low loads due to departure from the correct division of current between meter and shunt. Since the meter has a negative temperature coefficient, a resistance, *H*, having a fairly large positive temperature coefficient is connected in series with the meter before the connections are made to the shunt which is usually of manganin and has therefore a very small temperature coefficient. The series resistance takes the form of a coil of wire, part of which is tinned iron and the remainder constantan, and its resistance is adjusted so as to maintain the total resistance of itself and the meter together, as nearly constant as possible under all temperature conditions.

Motor Meters. These meters may be for use on direct current or alternating current circuits; and in the former case they may be either ampere-hour or watt-hour meters. In this class of meter the moving system is allowed to revolve continuously instead of being allowed merely to rotate through a fraction of one revolution as in an indicating instrument. The speed of revolution is proportional to the current in the circuit in the case of an ampere-hour meter; and to the power in the circuit in the case of a watt-hour meter. It follows, therefore, that the number of revolutions made by the revolving portion in any given time is proportional, in the ampere-hour meter, to the quantity of electricity supplied in that time; and in the case of the watt-hour meter to the energy supplied. The number of revolutions made by the meter is recorded by a counting mechanism consisting of a train of wheels, to which the spindle of the rotating system is geared.

The control of speed is brought about by a permanent magnet (brake magnet), so placed that it induces currents in some part of the rotating system, these currents producing a retarding torque proportional to their magnitude, which latter is proportional to the speed of the rotating system. This system attains a steady speed when the retarding torque exactly balances the driving torque produced by the current or power in the circuit.

ERRORS IN MOTOR METERS. The two principal errors common to all motor meters are friction errors and braking errors. The friction error is considerably more important than the corresponding error in most indicating instruments, since it is continuously operative and affects the speed of the rotor for any given value of current or power. The braking action in these meters corresponds to the damping in an indicating instrument. The braking torque in an integrating meter directly affects the speed, for a given driving torque, and hence affects the number of revolutions made in a given time.

Friction forces which exist when the rotor is just starting to revolve (static friction) may prevent it from starting at all if the load is small, and will cause its registration to be low at small loads. This part of the friction torque may be assumed to remain constant when the moving part of the meter is rotating, and may be compensated for by arranging for a small constant driving torque to be applied to the moving system independent of the load. When the meter is running normally, a friction torque is exerted which is proportional to the speed, but this is not of great importance, since it merely adds to the braking action.

In some meters, however, such as those of the mercury motor type, a friction torque proportional to the square of the speed exists, and has to be compensated for. Since the friction torque is proportional to the load on the bearings the weight of the rotating system should be as small as possible.

As regards errors due to variation in the braking action, it can be seen that the steady speed of the meter is such that the braking torque—proportional to the speed—is equal to the driving torque. The braking torque is also proportional to the strength of the brake magnet. Employing symbols, we have,

$$T_B \propto \phi i$$

where T_B = braking torque
ϕ = flux of brake magnet
i = current induced by the rotation of the moving system in the field of the brake magnet
Let e = induced voltage
r = resistance of the path of the current i in the rotating disc

The induced voltage $e \propto \phi n$

where n is the speed of the revolving part of the meter.

$$T_B \propto \frac{\phi^2 n}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (445)$$

This braking torque equals the constant driving torque T_D when a steady speed of revolution is attained. Thus, if N is the steady speed of the meter,

$$T_B' \propto \frac{\phi^2 N}{r}$$

and $$T_B' = T_D$$

where T_B' is the braking torque at speed N.

$$\therefore \qquad T_D \propto \frac{\phi^2 N}{r}$$

or $$N \propto \frac{r}{\phi^2} . T_D \quad . \quad . \quad . \quad . \quad (446)$$

Hence the steady speed attained by the meter for a constant driving torque T_D is directly proportional to the resistance of the path of the induced (or eddy) current, and inversely proportional to the square of the flux of the brake magnets.

It will be realized from this how very important it is that the strength of the brake magnet shall remain constant throughout the time that the meter is in service. Careful design and treatment during manufacture are necessary to ensure this constancy. In general, the braking torque will be reduced by increase of temperature, since this will increase the resistance r. It is somewhat difficult to compensate completely for this reduction, but in some meters the driving torque is also reduced by increase of temperature, and this brings about automatic partial compensation.

In one ampere-hour meter (see Ref. (23)) two very small, circular, cobalt-steel magnets are used, mounted in opposite polarity. A magnetic shunt is provided, the permeability of which varies with temperature so that an increase of speed of the meter-disc of only 1 per cent is caused by a change of temperature of 30° F.

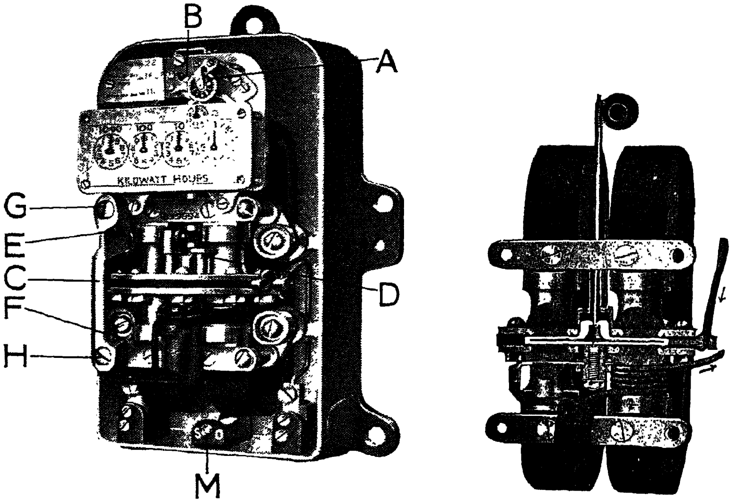
MOTOR METERS FOR D.C. CIRCUITS. There are two classes of such

(*a*) Mercury motor meters.
(*b*) Commutator motor meters.

The most important difference between the two types is in the method of leading current into the armature, or rotating part, of the meter. In type (*a*), the armature usually consists of a thin metal disc rotating in a bath of mercury, this mercury being used to lead the current into and out of the disc instead of the commutator as used in type (*b*).

Mercury Motor Meters. (i) *Ampere-hour Type.* A form of y An meter which has been commonly ed for d.c. circuits is the Ferranti meter shown in Fig. 470, the principle of operation being illustrated by the simplified diagram of Fig. 471.

There are two link-shaped permanent magnets—one for driving purposes and the other for braking. These magnets have mild steel pole-pieces fitting into circular brass plates between which is a fibre ring of the same external diameter as the plates, which are faced with Presspahn on their inner sides. The plates and fibre ring together form a shallow circular box, or chamber, which contains a thin amalgamated copper disc, the latter being the armature of the meter. The remainder of the space inside this chamber is filled with mercury, which exerts a considerable upthrust on the disc and reduces the pressure on the bearings. The disc is mounted at the base of a spindle, pivoted and working in jewelled cup bearings; the upper part of the spindle has a worm cut on it to engage in the gear wheels of the recording train.



(*Ferranti, Ltd.*)

FIG. 470. FERRANTI D.C. AMPERE-HOUR METER

The current is led into the disc, through the mercury, at its circumference on the right-hand side, and flows radially to the centre where it passes out through the spindle *via* the mercury to the jewel bearing, to which external contact is made.

A torque is produced owing to the presence of this current in the field of the right-hand magnet, and the disc rotates as shown in Fig. 471. In its rotation the disc cuts through the field of the left-hand magnet and an eddy current is induced in it which results in a braking torque as shown, this torque controlling the speed of rotation of the disc.

As previously shown, the speed

$$N \propto \frac{r}{\phi^2} \cdot T_D$$

where T_D is the driving torque.

Now, obviously, $T_D \propto$ the current

$$\propto I$$

Hence, if r and ϕ are constant,

$$N \propto I$$

i.e. the speed of the disc is proportional to the current. Thus, the number of revolutions in a given time will be proportional to the quantity of electricity passed in that time, i.e. proportional to $\int I dt$.

In order to compensate for the mercury friction when the disc is rotating, two iron bars are placed across the two permanent magnets, one above and one below the mercury chamber. The lower bar carries a small compensating coil of a few turns through which the load current passes. This coil sets up a local magnetic field, which strengthens the right-hand (or driving) magnet field and weakens the left-hand magnet field. Fluid friction is thereby compensated for.

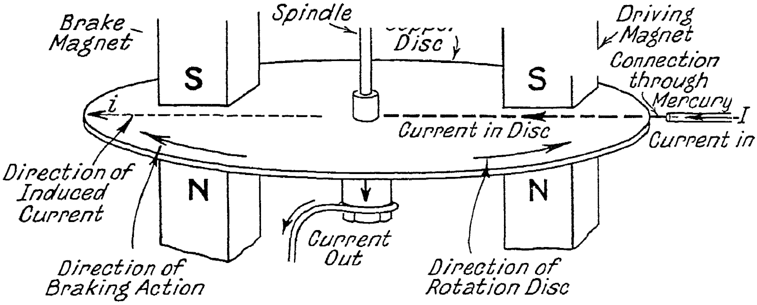

Brake Magnet
Spindle
Disc
Driving Magnet
Connection through Mercury
S
S
i
I
Current in Disc
Current in
Direction of Induced Current
N
N
Current Out
Direction of Braking Action
Direction of Rotation Disc

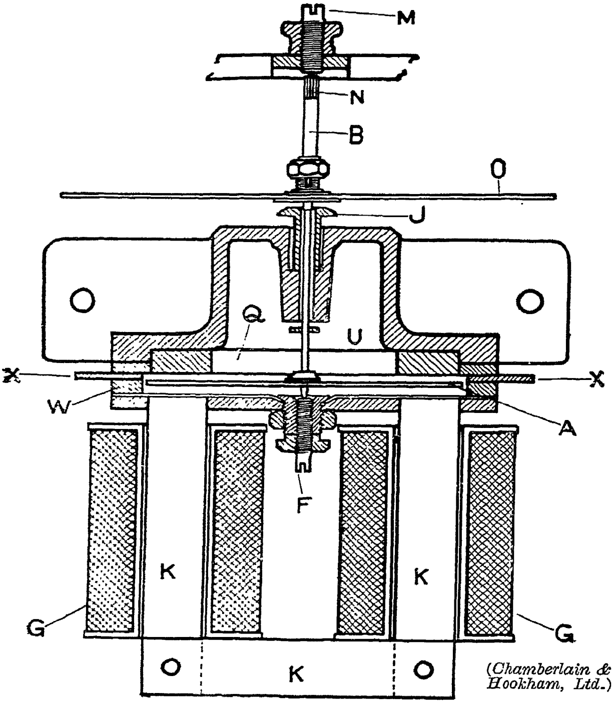
FIG. 471. ILLUSTRATING THE PRINCIPLE OF THE FERRANTI D.C. AMPERE-HOUR METER

No compensation is necessary for bearing friction, since, owing to the upthrust of the mercury on the disc, the bearing pressure is very small.

The Chamberlain and Hookham ampere-hour mercury meter, has the same principle as that of the Ferranti meter, except that only one permanent magnet is used; this produces both the driving and braking torque.

Compensation for fluid friction is provided by an iron bar, wound with a coil in series with the load, the bar being in parallel with the air gap of the permanent magnet. The effect is to reduce the flux of the latter on heavy loads and, since the braking torque is proportional to the square of the flux, while the driving torque is directly proportional to the flux, the result is an increase in the driving torque relative to the braking torque.

(ii) *Watt-hour Mercury Meters.* The construction of the Chamberlain and Hookham type of mercury watt-hour meter is shown in Fig. 472. In this meter there is an electromagnet *K* which carries two magnetizing coils connected—in series with a resistance—across the supply mains, so that the current through them is proportional to the voltage. This magnet is situated under the mercury chamber containing the aluminium disc-armature *A*. Above this disc is an iron ring *Q* which completes the magnetic circuit of the electromagnet and causes its magnetic field to be perpendicular to the disc. This disc has radial slots cut in it in order to ensure radial flow of the current in it, this current being led into and out of the disc through mercury contacts at diametrically opposite points. These slots prevent the same disc being used



(*Chamberlain & Hookham, Ltd.*)

FIG. 472. CHAMBERLAIN AND HOOKHAM MERCURY WATT-HOUR METER

for braking purposes, so that another aluminium disc O is used—mounted on the same spindle—in conjunction with a permanent magnet for this purpose.

Within the small limits of variation of voltage to be expected on an ordinary supply system, the flux of the electromagnet may be assumed to be strictly proportional to the current through the magnetizing coils, i.e. to the voltage. Hence, since the torque driving the armature is proportional to the product of the current through it and the flux of the electromagnet, this torque is directly proportional to the watts in the circuit. The braking torque is obviously proportional merely to the speed of the armature, and so the steady speed of rotation is proportional to the watts, and the number of revolutions to the watt-hours supplied—i.e. to $\int EIdt$.

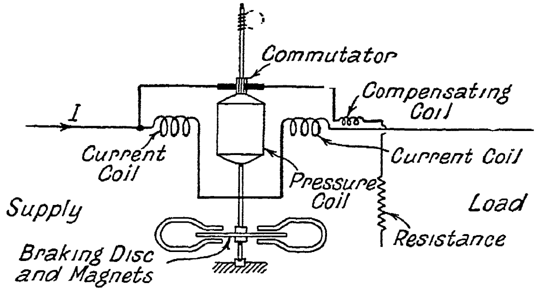


FIG. 473. ELIHU-THOMSON COMMUTATOR WATT-HOUR METER

Compensation for fluid friction at high speeds of revolution is provided by taking one or two turns of the current lead round the poles of the electromagnet, so that its field is strengthened thereby when the load is heavy.

In calibrating meters of the above type, there are several adjustments which can be made. Adjustable magnetic shunts are often fitted in the case of ampere-hour meters, so that the field strengths of the permanent magnets may be varied. The calibration of watt-hour meters may be varied by adjustment of the position of the brake magnet to give a greater or less braking torque and also by variation of the position of the electromagnet.

Large alterations of the calibration can be made by varying the gear ratios in the recording train, a large number of spare gear wheels being usually stocked in meter-testing laboratories for this purpose.

(*b*) *Commutator Meters.* Fig. 473 illustrates the principle of a commutator watt-hour meter, namely, the Elihu-Thomson meter though this type of meter is now mainly of historical interest.

There are two fixed current coils, each consisting of a few turns of heavy copper strip. These produce a magnetic field whose strength is proportional to the load current; in this field rotates the armature, which carries a number of coils connected to the segments of a small commutator. The armature coils are wound upon a non-magnetic former, and are connected, through brushes pressing on the commutator, and in series with a suitable resistance, across the supply. The commutator is of silver and the brushes are silver-tipped, the object being to reduce friction.

A compensating coil is also connected in series with the armature, and is so placed that it strengthens the magnetic field of the current coils when the pressure-coil, or armature, current flows through it. The object of this coil is to compensate for friction, and its position is adjusted so that the armature just fails to revolve when no load current is flowing, the shunt coils being energized.

The armature carries a current proportional to the voltage of the circuit, and the torque, which is proportional to the product of this current and the flux produced by the current coils, is thus proportional to the watts in the load.

The braking torque is provided by an aluminium disc and two permanent magnets, as shown. As in the mercury meters, the braking torque is proportional to the speed of the disc. Hence, the steady speed attained by the revolving system of the meter is proportional to the watts in the load.

Commutator meters are far less commonly used than the mercury type. The advantages of the latter are greater simplicity in construction; smaller voltage drop across the meter; the capacity for carrying a considerably greater current without shunting; and also small starting friction, owing to the very small pressure on the bearings as a result of the upthrust of the mercury on the rotating system.

The Grassot Fluxmeter as a Quantity Meter. The author has described (Ref. 25) the use of the Grassot fluxmeter for the measurement of a small fraction of an ampere-hour passing through a circuit in (say) a few seconds. In such a case neither an ampere-hour meter nor a ballistic galvanometer can be used successfully. The fluxmeter is used in conjunction with a suitable shunt in the current circuit as shown in Fig. 474. When the key is closed on contact b the fluxmeter is short-circuited and the movement is brought quickly to rest owing to the electromagnetic damping action.

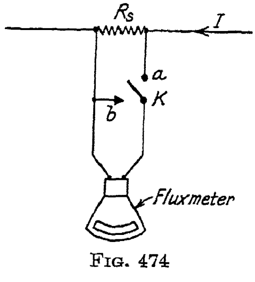


FIG. 474

At the beginning of the time interval T during which the quantity of electricity to be measured is passed through R_s, the key K is moved from b to a and is returned to b

at the end of the interval. The quantity of electricity is determined by multiplying the fluxmeter deflection which ensues between the two movements of the key K by a constant whose value depends upon that of R_s and upon the constants of the fluxmeter itself.

THEORY. Let I = current in the main circuit; i = current in the fluxmeter circuit at any time during the measuring period; L_s = inductance of the shunt R_s; R and L = resistance and inductance of the fluxmeter circuit. e_s and e_f are as defined on page 366.

Then, $e_s = R_s(I - i) + L_s \dfrac{d(I-i)}{dt}$

$e_f = K\dfrac{d\theta}{dt}$, where K is the fluxmeter constant and $\dfrac{d\theta}{dt}$ the angular velocity of its coil.

Now, $e_s = e_f + L\dfrac{di}{dt} + Ri$

or $$R_s(I-i) + L_s\frac{d(I-i)}{dt} = K\frac{d\theta}{dt} + L\frac{di}{dt} + Ri$$

Since i is very small Ri is negligible and $I - i = I$ (very nearly).

$$\therefore \quad R_sI + L_s\frac{dI}{dt} = K\frac{d\theta}{dt} + L\frac{di}{dt}$$

Integrating with respect to t we have

$$\int_0^T\left(R_sI + L_s\frac{dI}{dt}\right)dt = \int_0^T K\frac{d\theta}{dt}\,.\,dt + \int_0^T L\frac{di}{dt}\,.\,dt,$$

$$\therefore \quad R_s\int_0^T Idt + L_s\int_{I_1}^{I_2} dI = K\int_{\theta_1}^{\theta_2} d\theta + L\int_{i_1}^{i_2} di$$

I_2 and I_1 are final and initial values of the current in the main circuit, i_2 and i_1 (both zero) are corresponding values for the fluxmeter circuit, while θ_2 and θ_1 are the final and initial values of the fluxmeter deflection. If L_s is small we may write

$$R_s\int_0^T Idt = K(\theta_2 - \theta_1)$$

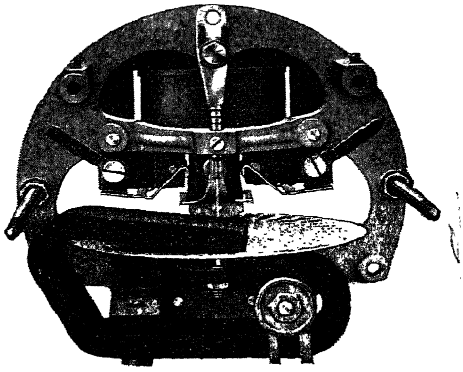
or, $\int_0^T Idt$, which is the quantity to be measured, is given by

$$\frac{K}{R_s}(\theta_2 - \theta_1)$$

It can be shown (Ref. 25) that $K = \dfrac{k}{10^8}$ where k is the number of flux linkages per unit deflection of the fluxmeter.

MOTOR METERS FOR A.C. CIRCUITS. For a.c. circuits the commutator type of meter could be used, but not the mercury type. The errors to which the commutator meter would be subject, if so used, would be the same as those of the dynamometer wattmeter, since their principles of operation are essentially the same.

Induction type meters are, however, almost universally used for a.c. energy measurements, since they possess the definite advantages, as compared with the commutator type, of a higher $\frac{\text{Torque}}{\text{Weight}}$ ratio, and the absence of a commutator with its accompanying friction. The induction type is therefore more accurate than the commutator type on light loads. The principle of these meters is almost exactly the same as that of the induction wattmeter. The construction also is very similar, except that the spring control and pointer of the wattmeter are replaced, in the energy meter, by a brake magnet which induces eddy currents in the disc (which now revolves, instead of rotating through only a fraction of a revolution as in the wattmeter), and by a recording train of wheels driven by a worm on the spindle of the meter.



(*Metropolitan-Vickers Elec. Co., Ltd.*)
FIG. 475. METROPOLITAN-VICKERS TYPE NA SINGLE-PHASE WATT-HOUR METER

Single-phase Induction Type Watt-hour Meter. The construction of a typical meter of this type is shown in Fig. 475, the brake-magnet and recording wheel-train being omitted for clearness. It can be seen that there is little difference in construction between this instrument and the induction wattmeter illustrated in Fig. 456 (previous chapter). The chief alterations are the provision of only one pressure coil, upon the centre limb of the "shunt" magnet, and only one copper shading band upon this limb. In addition, there are two copper bands placed obliquely on the other two limbs of this magnet, their purpose being to compensate for friction in the meter.

Fig. 476 shows the construction of the moving element of a single-phase watt-hour meter manufactured by Aron Electricity Meter, Ltd.

It should be observed that, since the disc is revolving continuously when on load, e.m.f's will be induced in it *dynamically*, as it cuts through the flux of the two electromagnets, in addition to the *statically* induced e.m.f's due to the alternating flux in these magnets. The full-load speed of rotation in most meters is, however, only about 40 r.p.m., and as this is small compared with the speed of alternation of the static flux (corresponding—say—to a frequency of 50 cycles per second), the torque corresponding to the eddy currents dynamically induced in the disc will be very small compared with the operating torque corresponding to the statically induced eddy currents.

Neglecting, for the present, friction in the meter, and assuming that the flux of the "shunt" magnet lags by exactly 90° in phase behind the applied voltage, we have, following the theory of the induction wattmeter,

$$\text{Operating (or driving) torque} \propto EI \cos \phi$$

where E and I are the r.m.s. values of voltage and current in the circuit respectively, and ϕ is the phase angle between them. This torque is thus proportional to the power in the circuit.

Now, the retarding torque of the eddy-current brake has been shown to be proportional to the speed of revolution of the disc, i.e.

$$T_B \propto N$$

where N = speed of revolution.

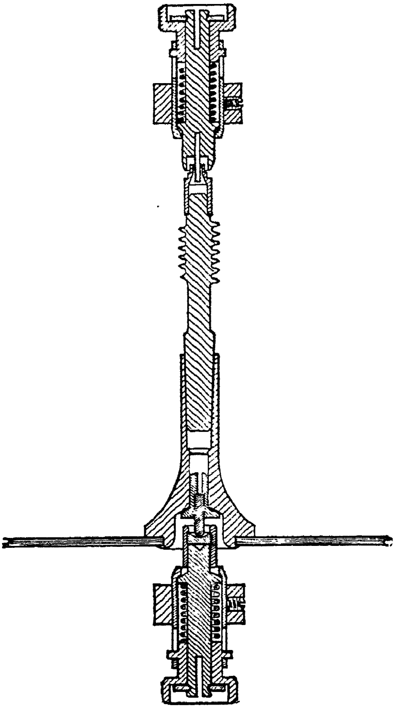
Hence, since for a steady speed the driving torque T_D is equal to T_B, we have

$$N \propto EI \cos \phi$$

or

$$\text{Power} \propto \text{speed of revolution}$$

Thus the total number of revolutions, which equals $\int N dt$, is proportional to $\int EI \cos \phi \,.\, dt$, i.e. proportional to the energy supplied.



(*Aron Electricity Meter, Ltd.*)

FIG. 476. MOVING ELEMENT OF SINGLE-PHASE METER

The voltage and frequency of the supply circuits upon which these meters are principally used are sufficiently constant for errors due to variations of these quantities to be negligible.

METER PHASE ERROR. As pointed out when considering the induction wattmeter, the object of the copper shading band on the shunt magnet is to produce a phase displacement of exactly 90° (assuming the flux of the series magnet to be in phase with the load current) between the flux of this magnet and the applied voltage.

An error due to incorrect adjustment of the position of this shading band (resulting in an incorrect phase displacement between the voltage and shunt magnet flux) will be evident when the meter is tested on a load whose power factor is less than unity. An error on the "fast" side under these conditions can be eliminated by bringing the shading band farther down the limb of the shunt magnet (i.e. nearer to the disc).

An error in the speed of the meter when tested on non-inductive load may be eliminated by adjustment of the position of the brake magnet, a movement of the poles of this magnet towards the centre of the disc reducing the braking torque, and *vice versa.*

The full load current, the supply voltage, and correct number of revolutions of the disc per kilowatt-hour, are marked on the name-plate of the meter.

Example. For a 20 amp. 230 volt meter, the number of revolutions per kWh. is 480. If, upon test at the full load (4,600 watts), the disc makes 40 revolutions in 66 sec., calculate the error.

In 1 hr. at full load, the meter should make $4\cdot6 \times 480$ revolutions.

This corresponds to a speed of $\dfrac{4\cdot6 \times 480}{60} = 36\cdot8$ r.p.m.

Thus the correct time for 40 revolutions $\dfrac{40}{36\cdot8} \times 60 = 65\cdot2$ sec.

$\therefore$ Time taken is $0\cdot8$ sec. too long, and the meter is $\dfrac{0\cdot8}{65\cdot2} \times 100$ $1\cdot2$ per cent slow.

FRICTION COMPENSATION. The two shading bands embrace the flux contained in the two outer limbs of the shunt magnet, and thus eddy currents are induced in them which cause a phase displacement between the enclosed flux and the main gap-flux. As a result, a small driving torque is exerted on the disc, this torque being adjusted, by variation of the positions of these bands, to compensate for friction torque in the instrument.

To check the correctness of this friction compensation, the meter is tested on a light load, under which conditions it should run at the correct speed if the compensation is correct. Care must be taken to ensure that the compensating torque is not so great as to cause the meter to run when no load current is flowing, i.e. when only the shunt magnet flux exists.

CREEP. In some meters a slow, but continuous, rotation is obtained when the potential coils are excited but with no-load current flowing. This may be due to incorrect friction compensation, to vibration, to stray magnetic fields, or to the fact that the voltage of the supply circuit is in excess of the normal.

To prevent such "creeping," two holes, or slots, are cut in the disc on opposite sides of the spindle. The disc tends to remain stationary when one of the holes comes under one of the poles of the shunt magnet.

In some cases, a small piece of iron wire is attached to the edge of the disc. The force of attraction of the brake magnet upon this iron wire is sufficient to prevent continuous rotation of the disc under no-load conditions.

Effect of Temperature Variation. The errors introduced by variations of temperature of an induction meter are usually small, since the various effects produced tend to neutralize one another.

Thus, increase of temperature increases the resistance of the eddy-current paths in the disc, and this reduces both the driving and braking torques. Again, the resistance of the pressure coil is increased with increase of temperature, so that it takes less current, resulting in a reduced flux in the shunt magnet. This reduces the driving torque, but the flux from the brake magnet is also reduced by increase of temperature, and so these effects tend to neutralize one another.

On non-inductive loads the temperature error is certainly negligible; on inductive loads, however, especially in the case of large whole-current meters, the error may be of considerable importance.

Before leaving the subject of single-phase watt-hour meters, mention should be made of an interesting article by A. Salzmann (Ref. (41)) describing their use (when special instruments for the purposes (see Chapter XXII) are not available) for power, power factor, polarity and phase-sequence tests.

Polyphase Watt-hour Meters of the Induction Type. As in the case power measurement with a two-element wattmeter, the energy supplied to a three-phase circuit may be measured by a double-element meter of the induction type, each element being similar in construction to the single-phase meter.

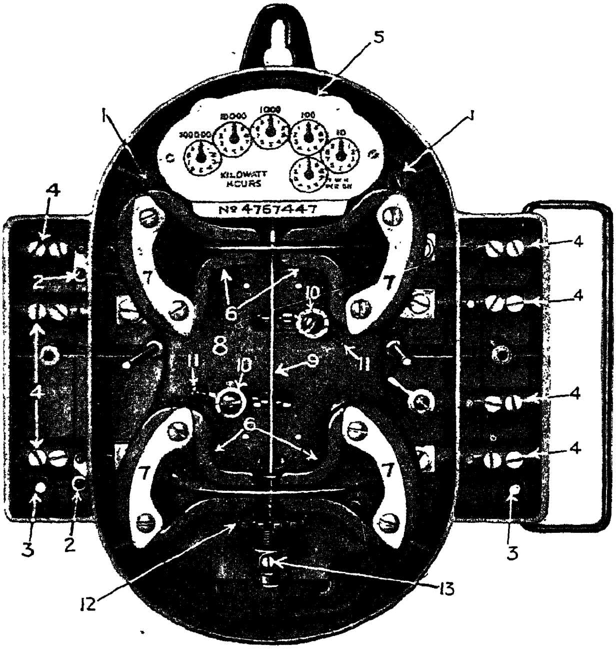
There are two discs mounted on the same spindle, and two separate brake magnets; the spindle drives a single counting train. In addition to the phase adjustment and friction compensating device in each element, one of the elements has an adjustable magnetic shunt across its shunt magnet, in order that the driving torque for the same watts may be made the same in the two instruments. In carrying out this adjustment, the two current coils of the meter are connected in series, and the two pressure coils in parallel, the polarities being such that the two driving torques are in opposition. Adjustment of the magnetic shunt is continued until the meter ceases to rotate with full-rated watts supplied to both elements.

The adjustments for phase displacement, friction compensation, and for the correct position of the brake magnets are made upon the two elements separately, a single-phase circuit being used. The adjustment of the brake magnet positions must be carried out on the two elements alternately, since both magnets are operative in producing the braking torque. Both shunt elements should be energized meanwhile, to allow for interaction.

If the phase compensation is not carried out exactly in each element it may happen that, taken together, the two errors neutralize one another when the load is balanced, giving a net error of zero.

Since, however, this condition does not hold for all loads, whether balanced or not, it is essential to observe certain conventional rules with regard to the connections of the meter so that it may, when in service, be connected in the same way as when under test.

In the British Standard Specification No. 37, for electricity meters, a standard system of marking the terminals is laid down, in order that the correct connections may be made. The three lines of the three-phase system are designated the "white," "blue," and "red" lines respectively, the white phase leading the blue by 120°, and the blue leading the red by the same angle. The specification states that in the upper element the current coil shall be connected in the blue phase and the current coil of the lower element in the red phase. The current in the upper current coil therefore leads that in the current coil of the lower element by 120°.



(*Sangamo Weston, Ltd.*)

FIG. 477. THE CONSTRUCTION OF A "SANGAMO" TYPE H.E. POLYPHASE METER

The connections to these polyphase meters are the same as those employed in the two-wattmeter method of measuring three-phase power. Fig. 477 shows the construction of a polyphase meter.

The most important meter of this type is the Aron clock meter. It has the advantages of being unaffected by external magnetic fields and of being comparatively free from frequency and wave-form errors, especially if its pressure-coil circuit is made as far as possible non-inductive. It is especially useful when the power factor of the circuit in which the energy is to be measured is high. The meter will register accurately on very low loads, but its cost and somewhat complicated construction have prevented its use as a house service meter. It can be used on either a.c. or d.c. circuits.

Although many of these clock meters are still in service on d.c. circuits, their cost of manufacture would now be so high that they are no longer being produced. Their novel principle of operation, and their good record of service deserve, however, to be recognized by a brief description.

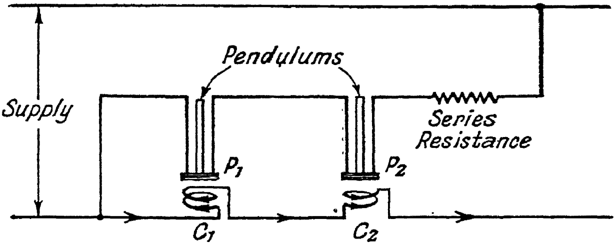


FIG. 478. CONNECTIONS OF ARON CLOCK TYPE METER

PRINCIPLE OF OPERATION. The connection diagram of Fig. 478 illustrates the principle of the meter. There are two similar flat circular coils P_1 and P_2 carried at the bottom ends of two pendulums which are continuously driven by clockwork, the latter being wound electrically. These coils are connected in series with one another, and with a high resistance, having a very small temperature coefficient across the line. They thus carry a current proportional to the line voltage. Obviously this circuit must be non-inductive, and must not vary in resistance appreciably with varying temperature, if this proportionality is to remain constant under all conditions of frequency and temperature.

Beneath each of these pendulum coils is a fixed current coil (C_1 and C_2), these being connected in series with the line and being wound so that their magnetic fields are in opposite directions—i.e. one upwards and one downwards.

When no line current is flowing, the two pendulums, which are of exactly the same length, swing at the same rate; but if a current flows through C_1 and C_2 one of these coils exerts a force upon its adjacent pendulum which reduces the speed of the latter and the other coil causes the speed of its adjacent pendulum to increase. This can be seen from the following.

The time of swing of a pendulum, whether simple or compound, may be written

$$T = \frac{k}{\sqrt{g}} \quad . \quad . \quad . \quad . \quad . \quad (447)$$

where k is a constant depending upon the dimensions of the pendulum and g is the acceleration due to gravity.

Thus, the number of swings per second, N, for each of the pendulums when swinging freely is given by

$$N = K\sqrt{g} \text{ where } K = \frac{1}{k}$$

The effects of the magnetic fields of coils C_1 and C_2 are to produce forces $+$ F and $-$ F acting, one up and one down, upon the pendulums adjacent to the coils C_1C_2. Let $+\ a$ and $-\ a$ be the corresponding accelerations. Then the rates of swing of the two pendulums will be

$$N_1 = K\sqrt{g + a} = K\sqrt{g}\sqrt{1 + \frac{a}{g}}$$

and $$N_2 = K\sqrt{g - a} = K\sqrt{g}\sqrt{1 - \frac{a}{g}}$$

It can be shown that

$$N_1 - N_2 = K\sqrt{g} \,.\, \frac{a}{g} = \frac{Ka}{\sqrt{g}} \quad . \quad . \quad . \quad . \quad . \quad (448)$$

Now the acceleration "a" is dependent upon the ampere-turns on the coils P_1P_2 and C_1C_2, and upon the dimensions and relative positions of the coils, so that we may write

$$a = k' \,.\, \overline{A}_p \,.\, \overline{A}_c$$

where k' is a constant and $\overline{A}_p$ and $\overline{A}_c$ are the ampere-turns on the pressure coils and upon the current coils respectively.

Thus, $$N_1 - N_2 = \frac{Kk'}{\sqrt{g}} \overline{A}_p\overline{A}_c = K' \overline{A}_p\overline{A}_c$$
$$= K''i_pi_c \quad . \quad . \quad . \quad . \quad (449)$$

where K' and K'' are constants, the latter including the product of the number of turns on a pressure coil and of a current coil. i_p and i_c are the currents in the pressure and current coils respectively. The product i_pi_c is proportional to the instantaneous power in the circuit, if these currents are instantaneous, the assumption being made that i_p is directly proportional to the voltage of the circuit. The *difference* between the rates of swing of the two pendulums is therefore proportional to the power in the circuit. By means of a differential gear this difference in rate is integrated so that the energy supplied to the circuit is recorded by a train of wheels and dials in the usual way. It should be noted that in the above discussion the natural periods of swing of the two pendulums are assumed to be exactly equal. Actually such exact equality is difficult to obtain, and arrangements are made to eliminate error due to lack of such equality, as described later.

CONSTRUCTIONAL DETAILS. The two pendulums drive two trains of wheels each ending in a bevel wheel. These two bevel wheels run loose on a spindle and their bevelled sides face one another. Between, and engaging with these two wheels, is a small planet wheel which is carried on an axle rigidly attached to and perpendicular to, the spindle upon which the bevel wheels are loosely mounted (see Fig. 479). This spindle drives the recording train of wheels, so that if the axle bearing the planet wheel is caused to rotate the spindle rotates and the recording train is driven.

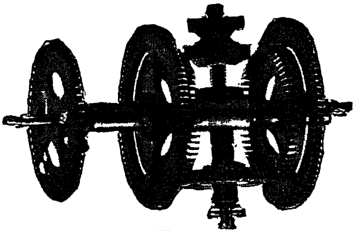

(Aron Electricity Meter, Ltd.

FIG. 479. DIFFERENTIAL GEAR OF ARON CLOCK METER

If the two pendulums swing at the same speed, the two bevel wheels rotate in opposite directions at the same speed; the planet wheel rotates on its axle, but the axle itself remains stationary. If, however, the pendulums swing at different rates, the bevel wheels rotate at different rates and the axle of the planet wheel is driven round, producing a rotation of the spindle and a movement of the recording train of wheels. The speed of rotation of this spindle—and hence the speed of the recording wheels—is proportional to the *difference* in speed of the bevel wheels, i.e. to the difference in the rates of swing of the pendulums and, therefore, to the power in the circuit.

Because of the difficulty of obtaining exactly equal natural periods of swing of the two pendulums the direction of current in the pendulum coils is automatically reversed every 10 min., there being two alternative trains of driving wheels so that the drive from the differential gear can be thrown over at the same time (otherwise the meter would read backwards when the current direction was reversed).

By this means error due to inequalities in the two natural periods is avoided.

The mainspring which drives the pendulums is wound electrically twice a minute by an automatic arrangement. This is accomplished by an electromagnet which attracts a small armature attached to the end of the spring. This electromagnet only comes into operation, for an instant, when the spring is run down, when a contact is made which energizes the magnet. The use of such an arrangement avoids a continuous waste of power.

Remote Indication. The complexity of present-day electricity supply systems, as well as the extensive use of electrical methods for the measurement of non-electrical quantities such, for example, as the pressure and rate of flow of gas, water or steam, the water level in reservoirs, etc., have encouraged the development of systems for the transmission of information, concerning various measured values, to a distance. Some examples of these follow.

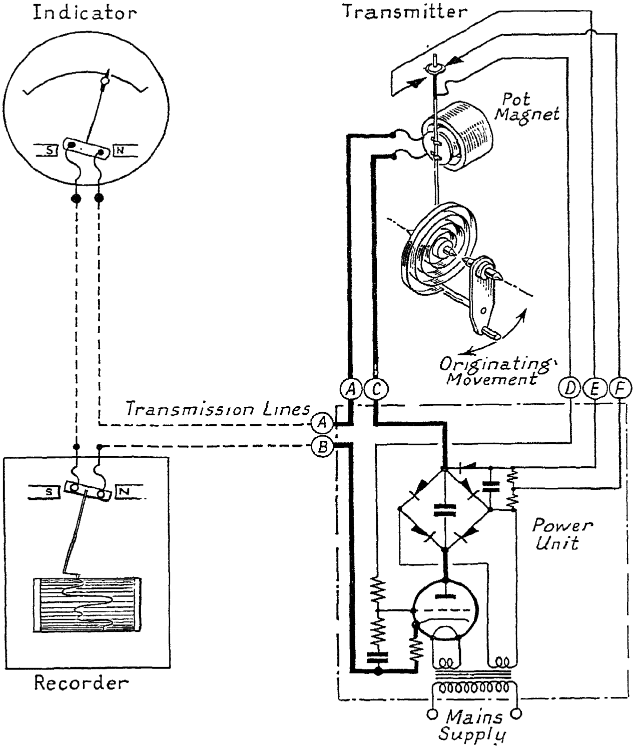
The Electronic Repeater (*Evershed and Vignoles*). The principle of this apparatus is shown in Fig. 480 (which shows a wattmeter transmitter).

The pointer of the instrument, the indication of which is to be transmitted by pilot lines carrying direct current, carries a coil working inside a pot magnet and also a disc contact which is connected to the grid of a triode valve. The coil is in the cathode circuit of the valve with the circuit completed through distant indicators or recorders. The disc lies between two fixed contacts carrying positive and negative biasing voltages respectively. The voltages alter the anode current until the torque exerted by the coil balances that due to the instrument originating the movement. The capacitor connected across the grid circuit of the valve provides storage of the required biasing voltage.

Variations in line resistance or in mains voltage are automatically compensated so that the current transmitted remains proportional to the quantity being measured. The instrument operates off a.c. mains.

Elliott Transmission System (*Elliott Brothers* (*London*), *Ltd.*). In this sytem, also operated from a.c. mains, the transmitter has an excitation field system, a moving element, and an output field system. The arrangement is shown in Fig. 481. The exciting field system has a laminated iron core and a magnetizing coil and it has a narrow, specially-shaped, air gap. A static voltage regulator maintains a constant voltage across the coil. The moving element is a metal loop which is caused to rotate by the originating instrument movement. When the loop enters the air gap of the mains-excited magnet an e.m.f. is induced in it and this is proportional to the amount of gap flux embraced by the loop. The loop is so shaped that the induced current in it is closely proportional to its angular movement.

The output field system is a closed, laminated-iron core with a single coil on it. The rotated loop is inductively coupled with this coil so that the latter has, induced in it by the loop current, an e.m.f. proportional to this current and so proportional to the angular movement of the loop and of the moving system of the originating instrument. The loop and output system of the transmitter act as a current transformer and thus the output current is, within limits, independent of the impedance of the output circuit.



(*Evershed & Vignoles, Ltd.*)

FIG. 480. MECHANICAL TRANSMITTER OPERATING INDICATOR AND RECORDER

Combined Instrument Transformer Metering Units (*Chamberlain and Hookham, Ltd.*). These units have been developed, for pole or platform mounting, to provide accurate remote metering at medium and high voltages particularly for overhead branch lines in rural areas.

The units consist of a three-phase voltage transformer mounted, with the necessary current transformers (5 VA, Class CM), all oil immersed, in a welded sheet steel tank. The arrangement is shown in Fig. 482.

Meter Testing. The testing of meters for commercial work has been standardized by the provisions of the Electricity Supply (Meters) Act, 1936. The Electricity Commissioners issued a series of papers* which contain details of the methods of testing which should be employed and the approved apparatus which should be installed in meter-testing stations for the purpose of carrying out

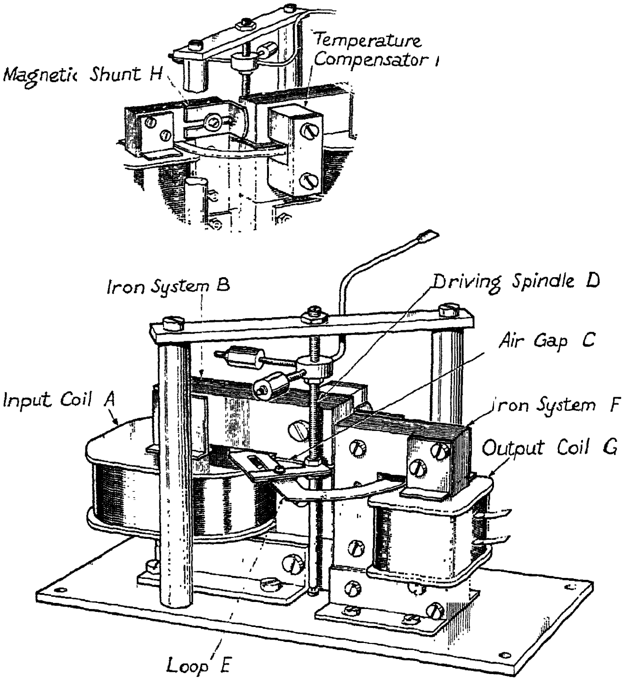


FIG. 481. ELLIOTT-SHOTTER TRANSMITTING UNIT

such tests. These papers should be studied by the reader who is interested in the subject of meter testing. Only a few of the more important provisions can be given here.

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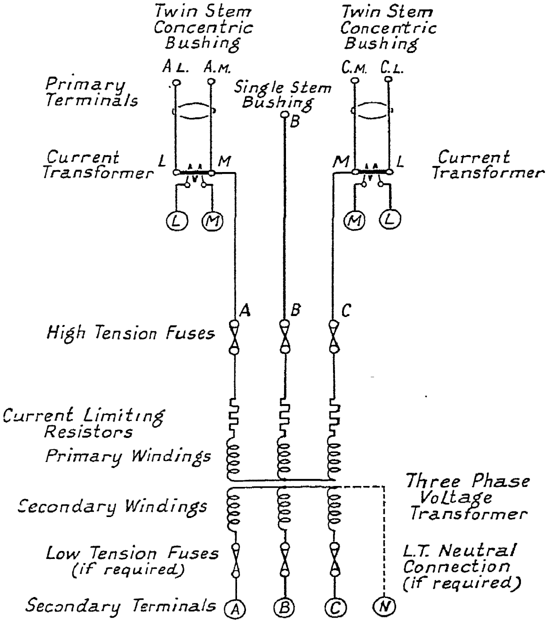
* E.g. Electricity Supply (Meters) Act, 1936: Approved Apparatus for Testing Stations, etc.; Explanatory Memorandum concerning the Testing of Electricity Meters; The Apparatus Approved for Use in Meter-testing Stations, etc.; Further Explanatory Notes concerning the Testing and Certification of Electricity Meters, etc.; Supplementary Notes in Amplification of the Explanatory Memorandum, etc. (Published by H.M. Stationery Office.)


Twin Stem Concentric Bushing
Twin Stem Concentric Bushing
A L.
A.M.
C.M.
C.L.
Primary Terminals
Single Stem Bushing
B
Current Transformer
L
M
M
L
Current Transformer
L
M
M
L
A
B
C
High Tension Fuses
Current Limiting Resistors
Primary Windings
Three Phase Voltage Transformer
Secondary Windings
Low Tension Fuses (if required)
L.T. Neutral Connection (if required)
Secondary Terminals
A
B
C
N

(*Chamberlain & Hookham, Ltd.*)

FIG. 482. CONNECTIONS OF 3-PHASE, 3-WIRE COMBINED METERING UNIT

Dealing with the approved apparatus first, this is listed as follows—

Standard Apparatus

D.C. potentiometer.

Substandard Apparatus

(*a*) Indicating wattmeter.
(*b*) Rotating meter.
(*c*) Ammeter.
(*d*) Voltmeter.
(*e*) Electrolytic ampere-hour meter.

Time Standard

Ship's chronometer or pendulum-type clock.

Time Substandard

Stop-watch or other suitable timing device.

It is laid down that the potentiometer shall be tested at intervals at the National Physical Laboratory, and that the other apparatus shall be tested against a standard d.c. potentiometer or against substandard indicating instruments at stated intervals.

Specifications are given for the approved apparatus, and these include constructional requirements and allowable accuracy limits.

Methods of Test. For motor meters, three methods of test are given—

Method A

Long-period dial tests, using substandard rotating meters.

Method B

(*a*) Tests (other than long-period dial tests) using substandard rotating meters.

(*b*) One long-period dial test.

Method C

(*a*) Tests by substandard indicating instruments and stop-watch.

(*b*) One long-period dial test.

It is laid down that "Method C alone is to be used for testing direct-current motor meters."

All meters are to be tested (*a*) at the lowest percentage of their marked current specified in the limits of error for meters of their class under the Electricity (Supply) Acts; (*b*) at one intermediate load; and (*c*) at the highest percentage of marked current specified in the limits of error.

In the case of a.c. meters, they are also to be tested at marked current and marked voltage at 0·5 power factor lagging.

Watt-hour meters must also be tested for "creep" by applying 10 per cent overvoltage to the pressure coil, the main circuit (current-coil circuit) being open. The meter must not run under these conditions.

It will be observed that the testing engineer has the liberty of choice between the above three methods of test (in a.c. meters), but all three methods involve at least one dial test.

s. All supply meters of the motor type have a meter-
d on them. This constant is expressed in ampere-seconds per revolution or in revolutions per kilowatt-hour. The full load current and line voltage for which the meter is intended are also stated. From these data the number of revolutions per minute which the meter *should make* when tested with a certain

fraction of its full load can be calculated. The number of revolutions per minute which it *actually does make* when tested at this load is then observed and the error (if any) deduced therefrom.

Thus, if K is the constant of the meter, the number of revolutions per minute which it should make when tested at some fraction "a" of its full load is given by

$$\frac{K \, . \, aW}{60}$$

where W is its full load in kilowatts.

An example illustrating such a test is given on page 786.

As an alternative to the above method of testing, a standard watt-hour meter is often used instead of timing a definite number of revolutions of the meter under test with a stop watch. In such cases the current coil of the standard instrument is connected in series with that of the meter under test, and their pressure coils are connected in parallel. This ensures that the power operating the two meters is exactly the same. The standard meter records single revolutions and is readable to a small fraction of a revolution. An arrangement is also fitted so that the register of the standard can be started and stopped instantaneously.

Specifications for such rotating standards are given in the Electricity Commission's paper on "Approved Apparatus."

The procedure, after adjusting the load to the required value, is to allow the meter under test to make a certain number of revolutions, and to observe the number of revolutions made by the standard in the same time.

If the constants of both meters are the same, the error in the meter under test can then be obtained directly.

If the constants are unequal,

Let K_x = No. of revolutions per kilowatt-hour for the meter under test (nominal value)
K_s = No. of revolutions per kilowatt-hour for the standard
N_x = No. of revolutions made in a certain time during the test by the meter under test
N_s = No. of revolutions made by the standard in the same time

Then, the energy indicated by the test meter in the time of the test

$$= \frac{N_x}{K_x} \text{ (kWh.)}$$

and that indicated by the standard

$$= \frac{N_s}{K_s} \text{ (kWh.)}$$

These should obviously be equal if the meter under test is correct. If not, the error is given by

$$\left(\frac{\dfrac{N_x}{K_x}-\dfrac{N_s}{K_s}}{\dfrac{N_s}{K_s}}\right) \times 100 \text{ per cent}$$

or

$$\left(\frac{N_x K_s}{K_x N_s}-1\right) \times 100 \text{ per cent}$$

In order to ensure that the meters have attained a steady temperature by the time the test is made, it is necessary that they shall be connected in circuit, with the load on, for some 15 to 30 min. before making the test.

The advantages of this method of test, as compared with the method using a stop-watch, are that errors introduced by the latter are avoided and also that variations in the load during the time of test affect both meters equally and are therefore unimportant.

Dial Tests. In addition to the short-time tests described above, it is usual (and is required by the Electricity Commissioners in commercial testing) to carry out a "dial" test extending over an hour or more with the load maintained constant at its full value, in order to check the gearing and ensure that the readings given by the recording train are correct.

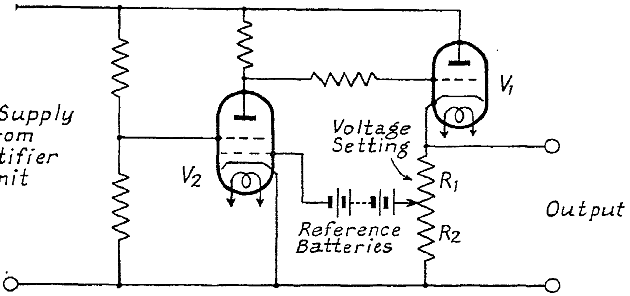
This test gives an opportunity of discovering errors due to self-heating, and it is usual, at the end of such a test, to repeat the short-time test at full load to investigate the effect of this heating upon the rate of revolution of the disc.

Other Tests. The British Standard Specification states that a meter should start at one-hundredth full load and also that it should *not* run when a pressure of 10 per cent over the normal is applied, the load current being zero. Tests must be made, extending for at least ten minutes, to ensure that a meter complies with these requirements in addition to those regarding errors for which the foregoing tests are carried out.

'Phantom" or "Fictitious" Loads. When the capacity of a meter der test is high, a test with the ordinary loading arrangements would involve a considerable waste of power. To avoid this "phantom," or "fictitious," loading an arrangement is employed which consists in supplying the pressure circuit from a circuit of the required (normal) voltage, and the current circuit from a separate low-voltage supply. This means that the total power supplied for the test is that due to the small pressure-coil current at normal voltage, plus that due to the load current at a low voltage; and the total power supplied is therefore only a comparatively small amount.

If a high-capacity meter is to be tested whilst in service, the pressure coil is supplied from the line in the normal way, but the current coil is removed from the consumer's load circuit and replaced by a short-circuiting connection. The current coil is then supplied from a battery or low voltage source for purposes of testing.

The Glynne Electronic Stabilizer. H. Tinsley and Co. make this instrument for use in meter stations where a fairly high voltage d.c. supply, of good stability, is needed; it replaces high voltage accumulators. The voltage supplied can be any value between 25 and 600 V at up to 120 mA with a short-term stability of 1 in 50,000 and a long-term one of 1 in 10,000.



(*H. Tinsley & Co., Ltd.*)

FIG. 483. GLYNNE ELECTRONIC STABILIZER

A schematic diagram is shown in Fig. 483 in which V_1 is a low impedance valve. A second valve V_2 (a tetrode) provides the control grid of V_1 with a potential such as to maintain the voltage across the load constant. The unstabilized input is derived from a full-wave mercury vapour rectifier circuit with choke input filter fed from a h.v. transformer with a tapped secondary. The instrument operates from 200 to 250 V a.c., 50 c/s supply.

[illegible] s. In testing a.c. meters, it is not only necessary [illegible] y the voltag and load current, but, in addition, the relative phases of the two quantities must be altered; in other words the power factor of the load must be variable also. Variation of power factor may be brought about by making up the load circuit with variable resistances, inductances, and capacities, and adjusting these as required. This method is, however, inconvenient in most cases, since it involves the waste of the power in the load and necessitates several fairly expensive pieces of apparatus, these objections being particularly apparent if the capacity of the meters is high.

For this reason the load employed for testing purposes is usually a fictitious one, as in the case of d.c. meter testing. Provision must, of course, be made for varying the phase of the current in the current circuit relative to that of the voltage circuit in addition to the variation of the magnitude of the current.

This phase of the current is usually altered in one of two ways: either by using, as supplies, two similar alternators coupled together, and driven by a motor, with the stator of one of them capable of rotation through any desired angle relative to the position of the stator of the other, or by using a phase-shifting transformer. In the former case one alternator supplies the pressure-coil circuits of the meters under test, and the other supplies the current-coil circuits, a step-down transformer being used to obtain the (possibly) heavy currents required for the current coils. By moving round the stator of the alternator supplying the current, using a worm-wheel and a graduated circular scale, the phase can be altered relative to that of the voltage. The angle through which the alternator stator is moved gives a measure of the phase angle between current and voltage. The frequency of both current and voltage is, of course, identical, since the two alternators are direct coupled.

When a phase-shifting transformer is used, instead of a motor-alternator set, the stator must be supplied with two- or three-phase current in order to produce a rotating field, or, if the supply is single-phase, a phase-splitting device must be used. The phase of the current induced in the rotor of the phase-shifter can then be altered as required by rotating the latter through the appropriate angle.

The magnitude and power-factor of the load having been adjusted according to the requirements of the test, the errors of the meters under test can be obtained either by timing a certain number of revolutions—the load being measured by an indicating wattmeter—or by comparing their rates of revolution with that of a standard watt-hour meter connected so as to measure the same load.

If the power-factor adjustment of a meter has been made, at some high value of the power factor, so that the error of the meter is zero, no error, due to power-factor variation, will be observed at the lower power factor. If, however, the adjustment has been made so that the error is (say) 0·5 per cent at 0·8 power factor, it will be found that the error due to this cause at 0·1 power factor is more than 6 per cent. This will be explained by a reference to the correction curves given in Fig. 434 in the preceding chapter.

Single-phase Meters. Fig. 484 shows the connections for the ng of a single-phase meter X, using two coupled alternators A_1 and A_2 for the supply to the pressure and current-coil circuits respectively. The stator of A_1 is capable of rotation for phase variation. A_2 is connected first of all to a step-down transformer T whose secondary then supplies the current coils of the two meters

X and S, the latter being either a sub-standard indicating wattmeter, or a standard watt-hour meter, according to the method of test employed. The variable resistance R is for adjustment of the current in the current-coil circuit. V and A are a voltmeter and ammeter for the measurement of the load voltage and current. A phase-shifting transformer may, of course, be used instead of the two alternators. The "equalizing lead" shown is necessary when a fictitious load is used, in order to ensure that the potential difference between the current and pressure coils of the meters is zero, as it is when they are in service.

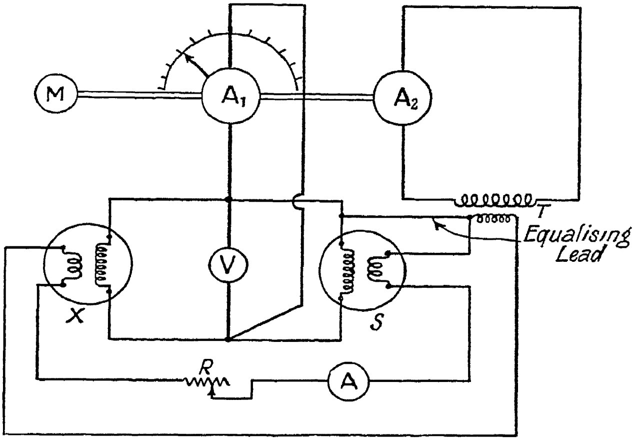


FIG. 484. CONNECTIONS FOR THE TESTING OF SINGLE-PHASE METERS

When several single-phase meters are being tested as above, it is necessary, first, to disconnect the links connecting one end of their current coil to one end of the shunt coil.

POLYPHASE METERS. Fig. 485 gives a diagram of connections for testing three-phase three-wire meters. In this case there is a three-phase supply. This is used for two purposes: (*a*) to supply the stator of a phase-shifting transformer, (*b*) to supply three step-down transformers, two of whose secondaries supply the current coils of the meter. A regulating choke coil is connected in the primary circuits of these three transformers. The rotor of the phase-shifting transformer supplies two auto-transformers having variable secondary tappings. These supply the pressure-coils of the meters, the latter consisting of a double standard wattmeter and one or more meters to be tested. These are connected as shown. A_1, A_2, and A_3 are ammeters and V_1 and V_2 voltmeters, for adjustment of the load.

Meters used for al Purposes. In addition to the forms of supply meters described above, there are several special forms of meter, the use of which has been made necessary by the various existing methods of charging for electrical energy. Thus, there are "prepayment" meters, which are fitted with a coin box and a prepayment mechanism, and are arranged so that only a limited amount of energy can be obtained after the insertion of a coin. Such meters are used for the metering of supplies to premises of which the tenants change frequently—when the settlement of accounts may become a matter of some difficulty—or when a consumer prefers this method of payment to the more usual quarterly account system.

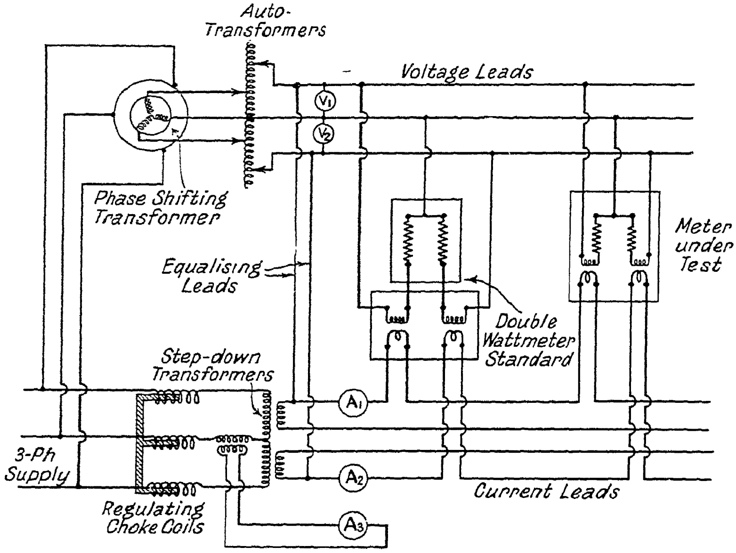


FIG. 485. CONNECTIONS FOR THE TESTING OF THREE-PHASE, THREE-WIRE, METERS

The adoption, by supply companies, of tariffs which penalize a consumer who makes periodic heavy demands of power on the system, or whose load has a low power factor, has led to the use of "maximum demand indicators," and of meters for the measurement of kilovolt-amperes. A supply company is bound to install, in its power station, plant of a considerably greater capacity than that required for the supply of the normal load on the station, on account of the sudden heavy demands which may be made by some consumers at certain periods. This means that there are considerable interest charges on the capital invested in such "reserve" plant, and it is only just that consumers, whose heavy demands of power have to be met by it, should be charged an extra amount in addition to the charge made on a lower rate of so much per unit, the latter being equivalent to the generation cost, plus a small profit. Maximum demand indicators are for the purpose of assessing this extra charge.

Again, consumers whose load has a low power factor require a much larger current, for a given amount of real power required, than would be necessary if their power factor were high. These increased current requirements necessitate the use, in the power station, of generators of a correspondingly high current-capacity, and also cause the copper losses in machines and distribution lines to be high. Thus, low power factor results in increased running costs to the supply authority, and again, it is reasonable that a consumer should be charged extra to compensate for the increase in costs for which his low power factor is responsible.

Wattless-power meters and kilovolt-ampere meters are both used in connection with tariffs which take the consumer's power factor into account. In the first case a measure of the consumer's departure from unity power factor is obtained directly. In the case of kilovolt-ampere meters, a record is obtained of the product of volts and amperes in the circuit and the charge, in the maximum-demand system of charging, is often based upon this product rather than upon the actual maximum power demand. In such cases a consumer's bill is increased if his power factor is low, since the volt-amperes taken by his load will be higher for a given amount of power than if his power factor had been high.

PREPAYMENT METERS. Essentially, these are not a special type of meter from the point of view of principle of operation, but consist of meters of one of the types described above, in which is incorporated a device which causes the load and meter circuit to remain open until a coin is inserted in the mechanism, thus closing a switch in the meter, which automatically opens the circuit again after a certain definite amount of energy has been supplied subsequent to this insertion.

Many different types of mechanism are employed by the various manufacturers in order to produce the operation required. These do not, however, merit individual description here. The principal requirements of such meters are—

(*a*) The switch regulating the opening and closing of the supply circuit must make good contact and must operate correctly after an amount of energy corresponding to that prepaid for has been supplied.

(*b*) The meter must be proof against interference by any but authorized persons.

(*c*) The coins inserted must not jam.

(*d*) The meter must be capable of easy adjustment in the case of alteration of the price per unit.

(*e*) The manipulation by the consumer must be of a simple nature.

(*f*) Allowance must be made in the operating mechanism for the fact that all the coins inserted will not be of the same weight and thickness.

(*g*) The number of coins which have been inserted should be indicated.

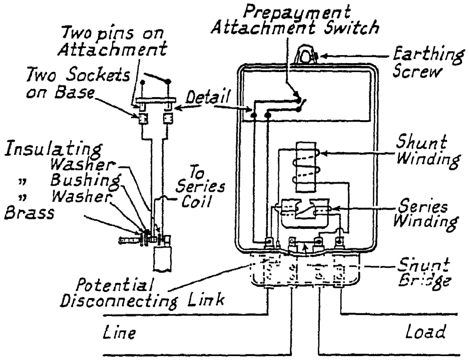


FIG. 486. DIAGRAM OF CONNECTIONS OF SINGLE-PHASE PREPAYMENT METER

Fig. 486 shows the internal connections of one type of prepayment meter.

M-DEMAND INDICATORS. The chief requirement of these indicators is that they shall record the maximum power, or rate of using energy, taken by a consumer during any particular quarter, the charge for the quarter then being based upon this maximum demand, as well as upon the total number of units consumed. In order that momentary heavy demands shall not be taken into account, and the consumer penalized, it is also necessary that the indicator shall only record the average power over successive predetermined periods, such as half-hours. Such indicators may either take the form of a separate instrument, or may be in the form of an attachment to a motor- or clock-type meter of the ordinary kind.

Wright Maximum-demand Indicator. This was one of the earliest demand indicators to be introduced. It indicates the maximum *current* taken by a consumer, the voltage (and power factor, if used on a.c.) being neglected. The principle of this instrument is illustrated by the line drawing of Fig. 487.

The indicator is essentially a differential air thermometer. It consists of a glass U-tube whose upper ends are closed by two bulbs of similar capacity, as shown. One of these bulbs is surrounded by a metal heater strip through which the current in the supply circuit flows, heating the air inside this bulb as it does so. Near the upper end of the other limb of the tube, and just below the bulb closing it, is a side tube leading to a narrow-bore index tube alongside which is a scale. The U-tube contains a liquid with a small coefficient of expansion—which acts as an indicator. When the indicator is set, initially, the right-hand limb of the tube (Fig. 487) is filled with sulphuric acid up to the level of the branch tube. Acid is then added until its level in the index tube is at the zero mark on the scale.

Now, when current flows through the heater, the air in the bulb which the heater surrounds expands and causes the sulphuric acid in the right-hand limb of the U-tube to overflow into the index tube. The greater the current, the greater the expansion of the liquid, and, therefore, the greater the overflow to the index tube. After the heater has cooled, the current in it must subsequently reach a higher value than any previous one if any further overflow into the index tube is to be obtained. The level of the liquid in this index tube gives, therefore, a true record of the maximum current which has passed through the instrument during any given period. The constrictions and traps in the limbs of the tube are for the purpose of preventing the passage of air from one bulb to the other when the indicator is reset, the latter operation being carried out by tilting the instrument and allowing the sulphuric acid from the index tube to drain back into the main tube.

Momentary heavy currents do not cause the indicator to register, owing to the length of time taken for the temperature of the heater to rise and also on account of the low thermal conductivity of the tube which it surrounds.

Fig. 488 shows the construction of the Price and Belsham Maximum-demand Indicator. The instrument is thermally operated, its movement consisting of two matched, flat coils of bi-metal strip —the actuating coil and compensating coil—the former being surrounded by a heater element carrying the load current (or a definite fraction of it, obtained by the use of a shunt or current transformer). The two coils are coupled together at their outer ends, the inner end of the actuating coil being fixed and the inner end of the compensating coil carrying the needle. The movement of the needle depends upon the heat produced in the heater, and hence upon the load current, and the needle carries forward with

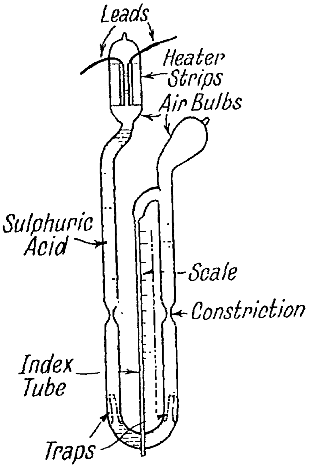
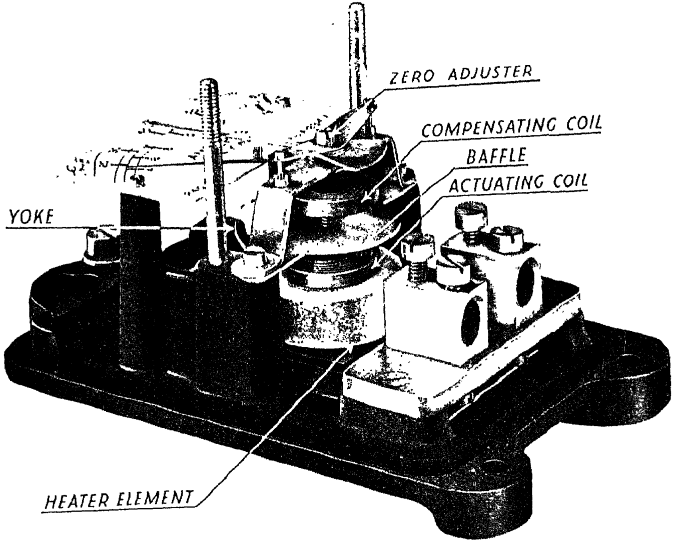


FIG. 487. PRINCIPLE OF WRIGHT MAXIMUM-DEMAND INDICATOR



(*Price and Belsham, Ltd.*)

FIG. 488. PRICE AND BELSHAM MAXIMUM-DEMAND INDICATOR

it a maximum demand pointer which remains fixed, by a special friction arrangement, at the maximum reading.

Through the use of the two matched coils the readings are unaffected by changes of ambient temperature.

G. W. Stubbings (Ref. (27)) has discussed the theory of this type of maximum-demand indicator.

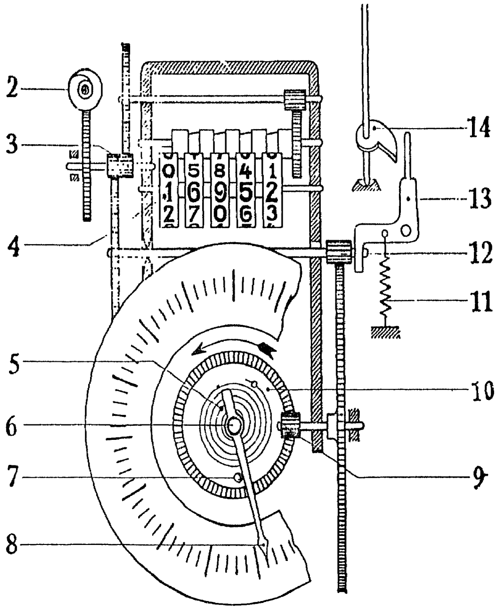
Merz-Price Demand Indicator. This indicator is not in the form of a separate instrument, but is a fitting which can be used with any type of motor or clock meter to indicate the maximum consumption during a half-hour or other period throughout a quarter (or other period between consecutive resettings of the instrument). Unlike the two preceding instruments, this indicator can be used to record either maximum amperes or maximum watts.

The principle of operation is as follows. A separate dial is fitted, inside the instrument the pointer of which is driven by the spindle of the moving system of the meter in the ordinary way. After this dial has been in gear for a certain period—usually half an hour—a device comes into operation which brings the mechanism of the disc back to the zero position. The pointer, however, does not return to zero but is lightly held by a special friction device and continues to indicate the number of units consumed in the previous half-hour period. The position of this pointer will not be moved forward unless the number of units consumed in some subsequent period exceeds the number recorded by the pointer. In this way the maximum demand, in units per half-hour. for any given period of time, is obtained.

The device for returning the driving mechanism of the demand dial to zero at the end of each half-hour period is operated by clockwork inside the meter or by a separate "time switch" external to the meter. These "time switches" consist of a clock, either electrically or hand wound, which make and break certain contacts, at predetermined intervals of time. These intervals depend upon the setting of the switch, which can be altered as required.

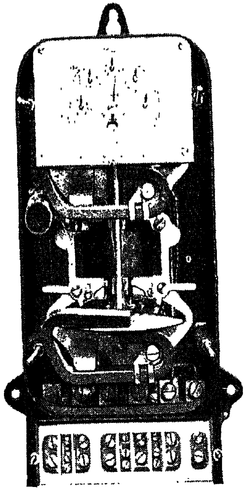
An advantage of this demand indicator is that a heavy demand must be existent for an appreciable time for it seriously to influence the number of units consumed in so long a period as half an hour, and, at the same time, short-period heavy demands are not neglected by the device.

The construction of a Merz pattern maximum-demand indicator, as manufactured by Messrs. Landis & Gyr, is shown in Fig. 489. Normally, the pin (7) drives forward the pointer (8) for (say) a half-hour period, and the energy consumed during this time is indicated on the dial. At the end of the period the cam (14) momentarily disengages the pinion (12) by means of the bell crank (13). This allows the pin (7), with its driving mechanism, to return to its zero position under the action of the spring (5). The pointer is, however, left stationary and continues to record the energy consumed during the previous half-hour period.



(*Landis & Gyr, Ltd.*)

FIG. 489. MERZ MAXIMUM-DEMAND INDICATOR



(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 490. M.V. POLYPHASE METER WITH MAXIMUM-DEMAND INDICATOR

During the next half-hour period the pin (7) is again driven forward, but the pointer is only moved forward if the energy consumed in a subsequent period exceeds that consumed in all preceding periods.

Another instrument which is now quite commonly used in connection with power factor tariffs is the Hill-Shotter maximum-demand kVA indicator, which is manufactured by Messrs. Aron Electricity Meter, Ltd. One type of this instrument is really an a.c. ampere-hour meter, and is of the induction type, having a shaded pole and a continuously driven disc. A permanent magnet is used for braking. It is used with a timing device of the Merz pattern, and records the maximum ampere-hours consumed during successive time intervals of 15 min. The dial is marked in kVA corresponding to the voltage of the circuit on which the indicator is to be used. The instrument reading is independent of power factor, and of frequency, within such variations of the latter as might be expected in the normal supply system.

The theory of this indicator is fully discussed by G. W. Stubbings (Ref. (4)).

Figs. 490 and 491 show a Metropolitan-Vickers polyphase meter with maximum-demand indicator, and the diagram of connections of such a meter, respectively.

Summation Metering. When a number of circuits, on an electricity supply system, work in parallel the total energy supplied to them is often required. This total can, however, be obtained by addition of the individual totals for the circuits so that special metering for the purpose is not essential. The measurement of the total maximum demand is, however, a different matter; it cannot, in general, be obtained by arithmetical addition of the individual maxima because these are very unlikely to occur at the same time. Arithmetical addition will give too high a value. Summation metering is commonly used, therefore, to determine the simultaneous maximum demand for a number of circuits.

There is not space here for full discussion of this subject, which is a large one, and the references at the end of the chapter should be consulted for a detailed study.

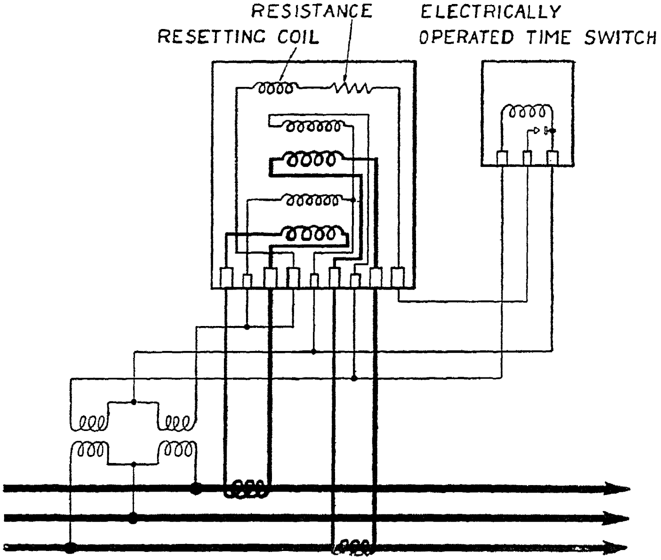
The most important of the methods used are—

(i) A meter with several current coils: this method can be used satisfactorily for summation in the case of two or three parallel circuits.

(ii) Paralleled current transformers: this method is similar to (i) except that only one current coil is needed per meter element (see Chapter XX, p. 768).

(iii) Meter with a summation current transformer: if the number of circuits to be summated is greater than can be handled by methods (i) and (ii) a transformer having a number of primary windings—one for each summated circuit—can be used. The transformer has a single secondary circuit which supplies the (single) current coil of the meter element. The primary windings are supplied from current transformers in the various load circuits.

(iv) Summation meter with multiple elements: this has a single rotor operated by a number of driving elements—one from each circuit. There is obvious difficulty in constructing such a rotor for more than a small number of circuits.

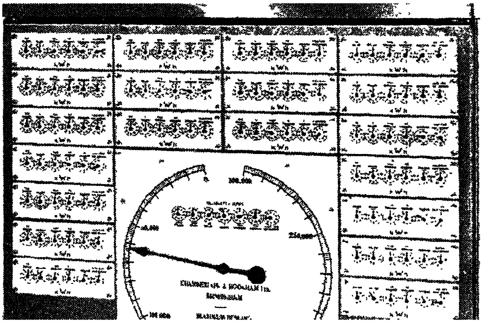


(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 491. CONNECTIONS OF M.V. POLYPHASE METER WITH MAXIMUM-DEMAND INDICATOR

(v) Summation by impulsing meters: this method can be used for a large number of circuits. The energy meter in each circuit is fitted with a contact-making device which, for a pre-determined amount of energy, sends an impulse to a mechanism in the summator which records the total number of such impulses received. This system was introduced by Chamberlain and Hookham, Ltd. in 1932 and they now make a summator for 22 circuits (maximum). This instrument, shown in Fig. 492 for 20 circuits, registers the consumption for each circuit, gives the total consumption and also the simultaneous maximum demand.

TWO-RATE METERS. These meters are required in connection with a two-rate or double tariff method of charging, which differs from the maximum-demand system in that two different rates of charge per unit are made, according to the time of the day at which the energy is consumed. By charging at a lower rate for energy consumed during periods of light load on the generating station, the supply company endeavours to distribute the demand more uniformly over the day and so improve its "load factor," this being the ratio of $\frac{\text{Average load}}{\text{Maximum load}}$. Assuming the average load to remain constant, the load factor is obviously increased by reducing the maximum load; and this can be achieved by causing some of the load to be taken at times of light load rather than when the load on the station is heavy.



(Chamberlain & Hookham, Ltd.)

FIG. 492. SWITCHBOARD PATTERN SUMMATOR FOR 20 CIRCUITS

A two-rate meter is of the ordinary type, but has two registering trains of wheels and dials, both of which are operated—but not at the same time—by the moving system of the meter. These registers are put into gear with the driving spindle alternately and are used at such times, and for such periods, as are desired by the supply company for the purpose of the double tariff method of charging. The times of operation of the high-rate and low-rate trains are usually governed by a time switch such as was mentioned in connection with the maximum demand system.

The time switches are set so that, in general, the high-rate register is in operation during the evening, when the load is heavy; the low-rate register operating during the remaining portion of the 24 hours when the load is lighter.

Measurement of Kilovolt-amperes. As has been mentioned previously, the systems of charging for energy which penalize low-power factor require the use of another meter in addition to that measuring the total real energy supplied (i.e. a watt-hour meter). Dorey (Ref. (19)) illustrates the two alternative methods of metering for such purposes by vector diagrams as in Fig. 493. In these diagrams OI represents the real power, while OV represents the kVA. In diagram (a), meter A is a watt-hour or "cosine" meter,

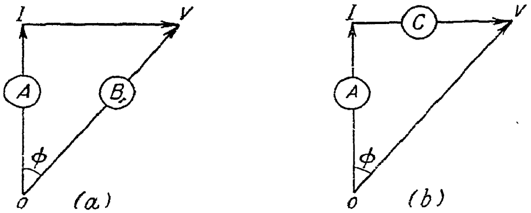


FIG. 493. METERING METHODS (DIAGRAMMATIC)

and measures the real energy supplied, while meter B records the kilovolt-ampere-hours. In other words, A records $\int_0^t \text{kVA} \cos \phi \, dt$, while B records $\int_0^t \text{kVA} \, dt$. In diagram ($b$), meter A records the true energy as before, and meter C records the wattless energy or $\int_0^t \text{kVA} \sin \phi \, dt$. The angle ϕ is the phase angle of the circuit. Obviously,

$$(\text{kVA} \sin \phi)^2 + (\text{kVA} \cos \phi)^2 = (\text{kVA})^2$$

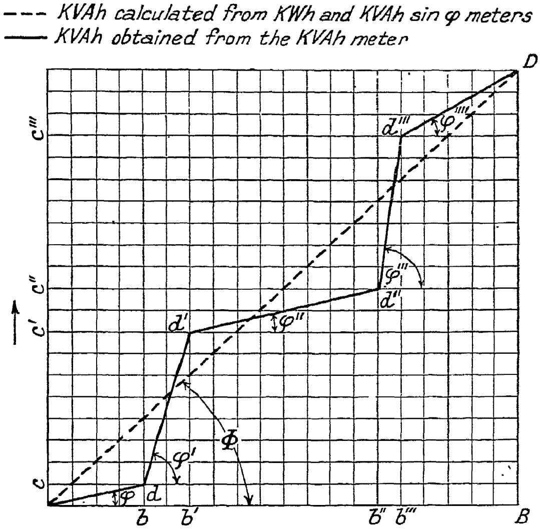
In the first diagram, meter B may be fitted with a maximum demand indicator which gives the kVA demand.

The average power factor of the load or average "energy factor" may be obtained in case (a) by the ratio $\frac{\text{Reading of meter } A}{\text{Reading of meter } B}$, the reading being for corresponding periods of time. In case (b) the average energy factor is given approximately by

$$\cos \phi = \frac{\text{Reading of meter } A}{\sqrt{(\text{Reading of meter } A)^2 + (\text{Reading of meter } C)^2}}$$

This expression is not, however, exact if the power factor is subject to considerable variations during the period of time to which the readings refer (see E. W. Hill's remarks in the discussion to the paper mentioned in Ref. (16)).

In one method of charging the total charge is made up of two parts; one at so much per unit of energy, and the other based on the quantity $\frac{\text{Kilowatt hours} \times P}{\cos \phi}$ where $\cos \phi$ is the average energy factor and P is the "lowest permissible power factor," and is laid down by the supply authority.


--- KVAh calculated from KWh and KVAh sin φ meters
—— KVAh obtained from the KVAh meter
D
c'''
d'''
φ''''
c''
c'
φ'''
d''
d'
φ''
Φ
φ'
c
φ
d
b
b'
b''
b'''
B

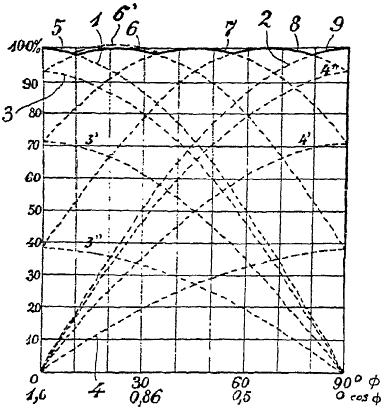
(*Landis & Gyr, Ltd.*)

FIG. 494

Obviously, if $\cos \phi$ is less than P, the consumer is charged, in the second part of the tariff, for a greater number of units than those actually consumed, whereas if his power factor is higher than P, the charge is for less units than the number actually consumed.

An induction watt-hour meter can be compensated to read kVA-hours within a limit of 2 or 3 per cent by adjusting the flux of the pressure circuit, so that it lags by $90 + \phi_a$ behind the applied voltage, instead of by merely 90°, ϕ_a being the average phase angle of the load circuit. Thus, if the phase angle of the load circuit is, at any time, exactly ϕ_a, the meter behaves as though the total applied voltage were exactly in phase with the current; i.e. it indicates volt-ampere-hours. For other power factors, the speed of the meter will not be exactly proportional to the volt-amperes, but to volt-amperes × cos $(\phi - \phi_a)$ where ϕ is the actual phase angle.

The power in an a.c. circuit—volts × amperes × cos ϕ—is very nearly the same as the volt-amperes, if the angle ϕ is small, when cos ϕ is nearly unity. As an example, if ϕ is 20°, cos ϕ is 0·9397; the power measured, in this case, is thus less—by 6·03 per cent—than the volt-amperes in the circuit. This will apply whether the power factor is lagging or leading. If, then, the meter is adjusted so that it runs 3 per cent fast at unity power factor, it will measure the volt-amperes in the circuit correct to within ± 3 per cent, provided the phase angle ϕ of the load circuit does not exceed 20° (lagging or leading).



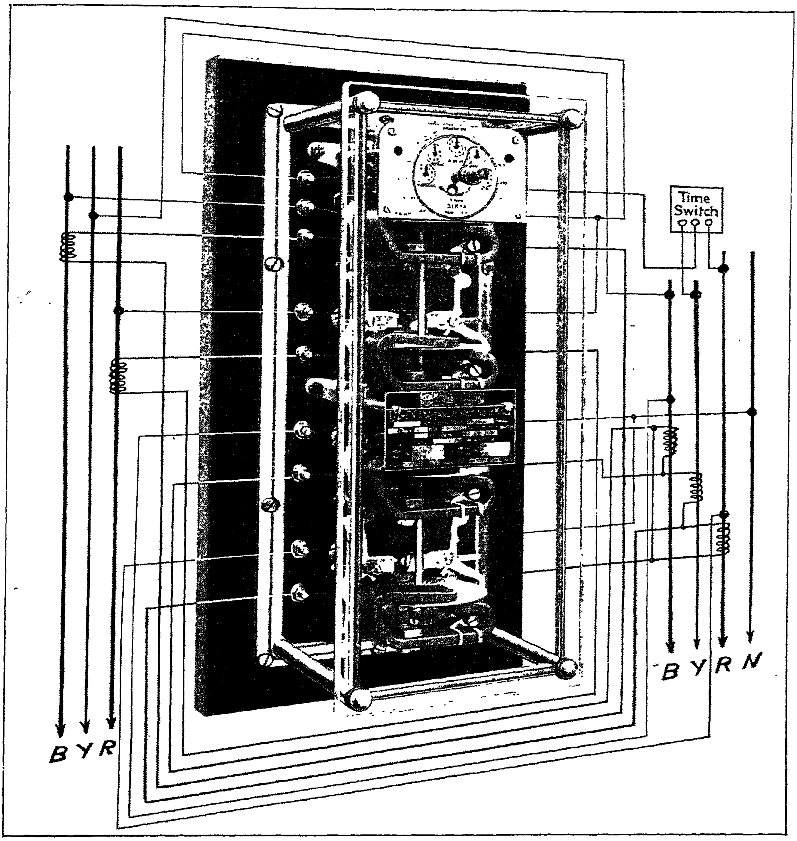
(*Landis & Gyr, Ltd.*)

FIG. 495

The required compensation of a watt-hour meter in order to cause it to indicate as above may be obtained by using quadrature loops of abnormal thickness on the shunt magnet.

Landis and Gyr Trivector Meter. This meter measures kVAh and also kVA of maximum demand.

Referring to Fig. 494, it can be seen that if the power factor of the circuit changes during a period of time from cos ϕ to cos ϕ', cos ϕ'', etc., the true total of kVAh is given by the sum of Ad, dd', $d'd''$, etc. If an attempt is made to measure this total simply by means of a kWh meter together with a reactive kVAh meter, the value obtained will be $\sqrt{AB^2 + BD^2} = AD$. This value is obviously, in general, less than the true value.



(*Metropolitan-Vickers Elec. Co., Ltd.*)

FIG. 496. METROPOLITAN-VICKERS' TYPE N FOUR-ELEMENT WATT-HOUR METER

The trivector meter registers the true value of kVAh correct to within ± 1 per cent. The meter consists of a kWh meter mounted together with a reactive kVAh meter in one case with a special summator mounted between them. Both meters drive the summator through a somewhat complicated system of gearing which arranges for the summator to register kVAh correctly at all power factors.

The general principle on which this gearing is arranged can be understood from Fig. 495. As pointed out above, for phase angles of the system between 0° and 10°, a kWh meter will register kVAh correctly within narrow limits, just as a reactive kVAh meter will register total kVAh very closely for phase angles of 80° to 90°. The gearing arranges, therefore, that the summator shall be driven principally by the kWh meter when the phase angle is small (curve 5, Fig. 495), and principally by the reactive kVAh meter when the phase angle is nearly 90° (curve 9, Fig. 495). At intermediate phase angles, both meters are responsible for the drive of the summator through different combinations of gears. The result is that the kVAh, measured at various power factors, is almost constant, as shown by the full line curve at the top of the graph.

For the fuller discussion of the question of kVA measurement, the reader is referred to Refs. (4), (7), (8), (11), (19), (24).

Metropolitan-Vickers Type N Four-element Watt-hour Meter. Fig. 496 shows the arrangement of a meter of this type fitted with a maximum demand indicator. The meter integrates the total energy, and also indicates simultaneously the combined maximum demand, in two independent circuits. The diagram of connections shown in the figure is that for a three-phase three-wire system, together with a three-phase four-wire system.

BIBLIOGRAPHY AND REFERENCES

(1) *Electricity Meters,* C. H. W. Gerhardi.

(2) *Electrical Measurements,* F. A. Laws.

(3) *Electrical Measuring Instruments,* C. V. Drysdale and A. C. Jolley.

(4) *Commercial A.C. Measurements,* G. W. Stubbings.

(5) *Electrical Measuring Instruments,* D. J. Bolton.

(6) *Dictionary of Applied Physics,* Vol. II, Section on "Watt-hour Meters."

(7) "Electricity Meters, with Notes on Meter Testing," H. A. Ratcliff and A. E. Moore, *Jour. I.E.E.*, Vol. XLVII, p. 3.

(8) "Recent Developments in Electricity Meters, with particular reference to those for Special Purposes," J. L. Carr, *Jour. I.E.E.*, Vol. LXVII, p. 859.

(9) "An Investigation of the Frequency Variations in Induction Watt-hour Meters," A. E. Moore and W. T. Slater, *Jour. I.E.E.*, Vol. LXVIII, p. 1023.

(10) "Current Transformer Summations," E. W. Hill and G. F. Shotter, *Jour. I.E.E.*, Vol. LXIX, p. 1251.

(11) "The Metering of Three-phase Supplies," O. Howarth, *Jour. I.E.E.*, Vol. LXIX, p. 381.

(12) "Apparatus and Methods for Accurate Maintenance of Large A.C. Energy Meters," E. Fawssett and G. E. Moore, *Jour. I.E.E.*, Vol. LXIX, p. 647.

(13) "The Rotor Bearings of Electricity Meters," W. Lawson, *Jour. I.E.E.*, Vol. LXVII, p. 1147.

(14) "A New Prepayment Meter," *Metropolitan-Vickers Gazette* (July, 1927), p. 159.

(15) "Load-levelling Relays and their Application in Connection with Future Metering Problems," W. Holmes, *Jour. I.E.E.*, Vol. LXVII, p. 296.

(16) "The Improvement of Power Factor," G. Kapp, *Jour. I.E.E.*, Vol. LXI, p. 89.

(17) "Some Technical Considerations concerning Power Factor in Relation to Tariffs," E. W. Hill, *Jour. I.E.E.*, Vol. LXVII, p. 1228.

(18) "Power Factor: Its Technical and Commercial Aspects," H. E. Yerbury, *Jour. I.E.E.*, Vol. LXI, p. 675.

(19) "The Improvement of Power Factor," E. W. Dorey, *Jour. I.E.E.*, Vol. LXIV, p. 633.

(20) "The Metering Arrangements for the 'Grid' Transmission System in Great Britain," C. W. Marshall, *Jour. I.E.E.*, Vol. LXVIII, p. 1497.

(21) *Handbook for Electrical Metermen*, published by the Meter Committee, Technical National Section, National Electric Light Association, 29 West 39th Street, New York.

(22) "The Mean Error of an Electricity Meter," G. W. Stubbings, *Jour. I.E.E.*, Vol. LIX, p. 335.

(23) "Integrating Electricity Meters," E. Fawssett, *Jour. I.E.E.*, Vol. LXIX, p. 545.

(24) *Meter Engineering*, J. L. Ferns.

(25) "The Use of the Grassot Fluxmeter as a Quantity Meter," E. W. Golding, *Jour. I.E.E.*, Vol. LXXVI, p. 113.

(26) "Grid Metering," J. Henderson, *Jour. I.E.E.*, Vol. LXXV, p. 185.

(27) "The Thermal Maximum Demand Indicator," G. W. Stubbings, *Distribution of Electricity*, April, 1934.

(28) "An Improved Three-phase, Three-wire, Reactive Volt-ampere-hour meter," G. A. Cheetham, *Jour. I.E.E.*, Vol. LXXIII, p. 233.

(29) "The Metering of E. H. T. Supplies on the Secondary Side of Step-down Transformers," S. H. C. Morton, *Jour. I.E.E.*, Vol. LXXI, p. 507.

(30) "Some Suggestions on the Equipment and Routine of the Meter Departments of Supply Undertakings," J. L. Ferns, *Jour. I.E.E.*, Vol. LXXVI, p. 369.

(31) "A Study of the Induction Watt-hour Meter with special reference to the Cause of Errors on Very Low Loads," T. Havekin, *Jour. I.E.E.*, Vol. LXXVII, p. 355.

(32) "The History and Development of the Integrating Electricity Meter," A. E. Moore, *Jour. I.E.E.*, Vol. LXXVII, p. 851.

(33) "The Measurement of Large Supplies of Electrical Energy for Costing Purposes," W. Casson and A. H. Gray, *Jour. I.E.E.*, Vol. LXXVIII, p. 681.

(34) "The Metering of Mercury-Arc Rectifier Supplies and Outputs," C. Dannatt, *Jour. I.E.E.*, Vol. LXXXI, p. 256.

(35) "Torque in a Bipolar Induction Meter," R. A. Morton, *Trans. A.I.E.E.*, Vol. LV, p. 354.

(36) "Development of a Modern Watt-hour Meter," I. F. Kinnard and H. E. Trekell, *Trans. A.I.E.E.*, Vol. LVI, p. 172.

(37) "The Remote Indication of Meter Readings," C. Myers, *G.E.C. Journ.*, Vol. VIII, p. 201, August, 1937.

(38) "Organization of a Meter Test Department of a Large Supply Undertaking, with Special Reference to the Electricity Supply (Meters) Act, 1936," C. W. Hughes, *Jour. I.E.E.*, Vol. LXXXII, p. 410.

(39) "The Use of Auxiliary Current Transformers for Extending the Range of Metering Equipment," G. F. Shotter, *Jour. I.E.E.*, Vol. LXXXIV, p. 128.

(40) "Reactive Metering," V. B. Shah, *Elec. Rev.*, June 13th, 1947.

(41) "Utilizing a Watt-hour Meter," A. Salzmann, *Distribution of Electricity*, July, 1951, p. 9.

(42) "The Electrical Transmission of Flow and Level Records," A. Linford, *B.E.A.M.A. Journal,* October, November and December, 1942, pp. 285, 335 and 367.

(43) *Electricity Supply Meters,* A. E. B. Perrigo.

(44) *Electricity Meters and Meter Testing,* G. W. Stubbings.

(45) *Electricity Meters and Instrument Transformers,* S. James.

CHAPTER XXII

MEASUREMENT OF SPEED, FREQUENCY AND PHASE DIFFERENCE

Rotational Speed Measurements. Electrical methods, or methods involving electrically-operated measuring equipment, can be used for the measurement of the rotational speed of a machine. If this machine is an alternator the speed is a measure of the frequency of the alternating voltage generated, the number of poles of the alternator being known.

One simple and direct method is to use a small generator, with a permanent-magnet field, and to measure, with a voltmeter, the generated voltage. This is proportional to the speed and the moving-coil voltmeter, with a uniform scale, can be calibrated directly in rotational speed (r.p.m. or r.p.s.).

Fig. 497 shows the construction of a small generator for this purpose, manufactured by the Record Electrical Co., Ltd., and used with Record "Cirscale" Voltmeters. This generator has cobalt steel magnets; the armature runs on ball bearings and is wound to give 100 volts at 2,000 r.p.m. A magnetic shunt is fitted for calibration purposes. The armature and field magnets are enclosed within a cast-iron body which both provides screening from external magnetic fields, and also protects the generator from dust and moisture—an important point if the device is to be capable of being used universally for speed measurements.

The armature resistance of the generator is low in order that the voltage drop shall be small, even when the generator is used to supply several indicators in parallel. A resistance is provided for the purpose of limiting the current from the generator in the event of a short circuit on the wiring to the indicators.

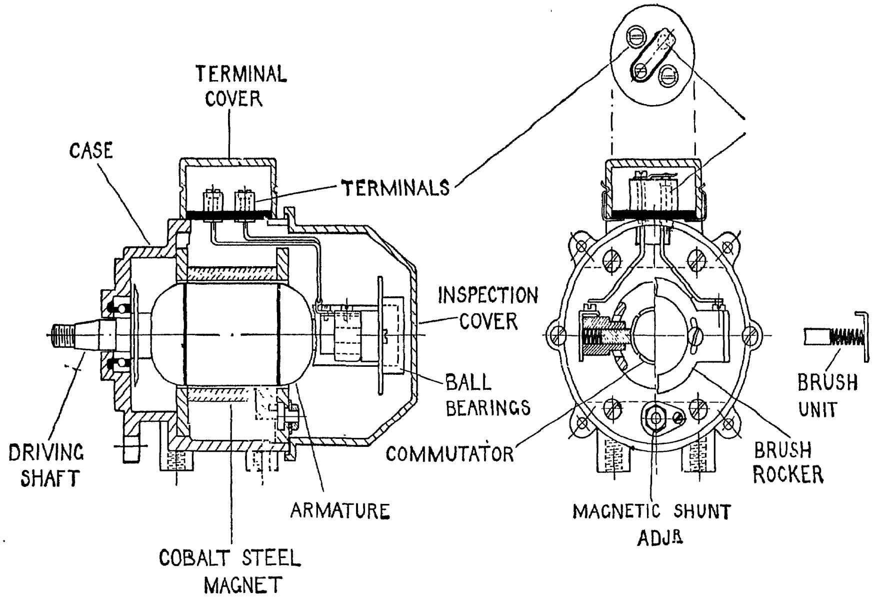
The combined accuracy of the generator and indicator at the calibrated temperature is given by the makers as follows.

From full scale to middle point, 1 per cent of the indication. From middle point to the lower limit of the scale, a possible fixed error not exceeding 1 per cent of the middle point reading.

Temperature errors are negligible over a wide range.

The generator may be mounted directly on the machine shaft, or may be driven through gearing.

An electronic tachometer, described by H. G. Jerrard and S. W. Punnett (Ref. (9)), is capable of measuring speeds of revolution of a shaft to an accuracy of ± 0·05 per cent. The rotation of the shaft is made to generate an alternating voltage in one of two ways—by an electromagnetic generator or a photo-electric generator. This voltage is amplified and then connected to the Y plates of a



(*Record Electrical Co., Ltd.*)

FIG. 497. GENERATOR FOR USE WITH A VOLTMETER FOR SPEED MEASUREMENTS

cathode-ray tube. To the X plates is applied the output of a variable-frequency oscillator calibrated to ± 0·05 per cent. The trace on the tube is a Lissajous figure and, if the frequency from the oscillator is adjusted to produce a stationary and easily identifiable Lissajous figure—most conveniently an ellipse—the frequency from the shaft generator, and therefore the speed of the shaft, become known to the same accuracy as that of the oscillator. Any change in speed is immediately detected because it produces a precession of the Lissajous figure.

For the electromagnetic generation of the alternating voltage from the rotating shaft a brass disc, carrying sixty soft iron pegs equally spaced round its circumference, is mounted on the end of the shaft. This runs in the air gap of a pick-up on the core of which are two coils (actually the earpiece of a head-phone was used to provide the coils and main part of the core of the pick-up). The change in magnetic flux caused by the passage of the soft iron pegs induces an e.m.f. in the coils and this is passed on to the amplifier.

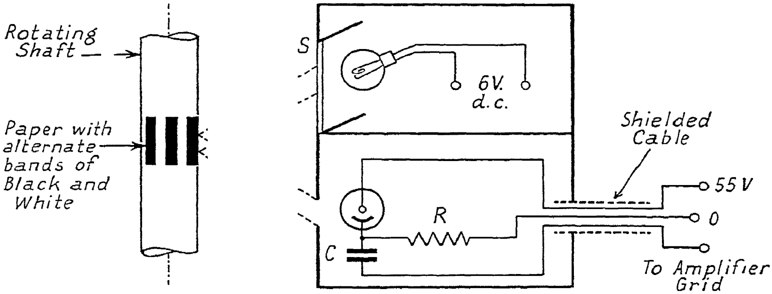


FIG. 498. ELECTRONIC TACHOMETER WITH PHOTO-ELECTRIC GENERATOR

The arrangement of the photo-electric generator is shown in Fig. 498. There is a small light source in the upper compartment and a midget photo-cell in the lower one. The resistance R and capacitance C are 2 MΩ and 0·01 μF respectively. As the shaft rotates the light is reflected from a strip of paper, 1 in. wide and having alternate equal bands of black and white, pasted on the shaft. An alternating e.m.f. is generated in the photo-cell circuit. A diffusing screen S is used because the light is not focused on the photo-cell. If it were the wave-form of the output would be square topped, not sinusoidal.

A calibration unit consisting of a standard frequency generator (having a crystal-controlled oscillator) and a selective amplifier is used with the equipment.

Stroboscopic Methods. A stroboscopic device consists essentially of a disc carrying a geometrical pattern, the disc being illuminated by a series of rapidly-recurring flashes, each of which is of very short duration compared with the time interval between successive flashes.

Fig. 499 shows a disc on which the pattern is a twelve-point star. If this disc is rigidly attached to the shaft of a rotating machine, of which the speed is such that each point of the star moves forward a distance of one point-pitch p during the time interval between successive illuminating flashes, the pattern will appear to be stationary. This effect is caused by the fact that the point adjacent to that viewed in the preceding flash, or glimpse, has now taken up the exact position of the latter, and, since all the points are identical, the eye cannot distinguish between them. Thus the impression is formed that no movement has taken place in the interval between the two illuminating flashes. The same effect is obtained if the speed of rotation is exactly twice, three times, or any other multiple of this speed—which is called the "primary" speed. If, for example, the speed of the rotating machine is twice the primary speed, the pattern will actually move forward through a distance equal to twice a point-pitch—i.e. $2p$—and the impression that it is stationary will be formed by an observer.

The series of flashes may be obtained by placing a disc, driven at a steady speed, between a source of illumination and the stroboscopic disc, the former disc containing a slit so that the latter is illuminated once for every revolution of the former. Another method of using the device is to employ an electrically-driven tuning fork, the prongs of which carry thin aluminium shutters, each with a slit in it. These shutters overlap, and give two "glimpses" of the stroboscopic disc, for every complete vibration of the tuning fork, if the observer's eye is placed behind them. Alternatively, the stroboscopic disc may be illuminated through the slits, when a series of flashes will produce the same effect as that obtained with glimpses.

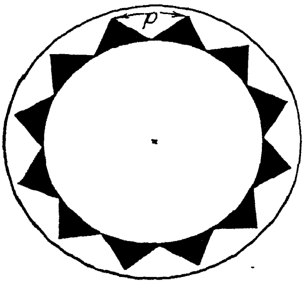


FIG. 499

The advantage of such a tuning-fork is its remarkably constant frequency of vibration.

Theory of the Method. When the pattern of the disc appears to be stationary, the speed of the rotating machine is given by

$$N = \frac{F}{n} \quad \text{rev. per min.} \qquad . \quad . \quad . \quad (450)$$

where F = No. of flashes, or glimpses, per minute

and n = No. of points on the pattern

In this case, N is the "primary" speed. The fact that the pattern will be stationary for speeds which are whole multiples of N will, in practice, cause no confusion, since the speed can easily be measured by a simple, and less accurate, method, with sufficient accuracy to distinguish between N and its multiples.

If the machine rotates at a speed slightly in excess of the "primary" speed, the pattern will appear to rotate slowly forwards; if the speed is slightly less than the primary speed, the apparent rotation of the pattern is backwards. The number of apparent revolutions per minute of the pattern gives the difference—above or below—between the actual speed of the machine and the primary speed.

Again, if the speed of the machine is exactly half the primary speed, the pattern will still appear stationary; but the number of points will appear to be twice the actual number of points; if the speed is one-third the primary, the apparent number of points is three times the actual number, and so on.

Let N = the primary speed in revolutions per minute
n = No. of points on the pattern
F = No. of flashes, or glimpses, per minute
p = the pitch of the points in inches
d = diameter of circular pattern in inches

Then,
$$n = \frac{\pi d}{p}$$

Distance moved forward by the pattern, when the speed is N, during the interval between glimpses

$$= \frac{\pi d \,.\, N}{F} \text{ inches}$$

But this is equal to the pitch p,

$$\therefore \qquad p = \frac{\pi d N}{F}$$

Also
$$p = \frac{\pi d}{n}$$

$$\therefore \qquad n = \frac{F}{N} \text{ or } N = \frac{F}{n} \text{ as stated above.}$$

Example. If the speed of a machine is about 500 r.p.m., and its variation from this speed is to be measured by stroboscopic means, the number of points on the pattern should be twelve (as in Fig. 499) if the tuning fork vibrates at 50 cycles per second—giving 6,000 glimpses per minute.

This number is obtained from

$$n = \frac{6{,}000}{500} = 12$$

Again, if instead of being exactly 500 r.p.m., the speed of the machine is 510 r.p.m., the pattern actually moves forward a distance $\frac{\pi d \times 510}{6{,}000}$ in., between successive glimpses, and thus the *apparent* movement forward in the interval between glimpses is

$$\frac{\pi d \times 510}{6{,}000} - \frac{\pi d \times 500}{6{,}000} = \frac{\pi d \times 10}{6{,}000} \text{ in.}$$

Now, this interval between glimpses is $\frac{1}{6{,}000}$ min., so that in one minute the apparent movement forward is $10\pi d$ in., which corresponds to a rotation of the pattern of 10 r.p.m., thus giving the excess of the actual speed of the machine over the primary speed (500 r.p.m. in the case considered). If the apparent motion of the pattern had been backwards, the actual speed of the machine would have been 490 r.p.m.

The general expression for the *apparent* speed of rotation of the pattern is

$$R = \pm (a - 1) N \text{ rev. per min.} \qquad . \qquad . \quad (451)$$

the positive sign applying when the apparent motion is forwards and the negative when backwards. a represents the ratio

$$\frac{\text{Actual speed of machine}}{\text{Primary speed}}$$

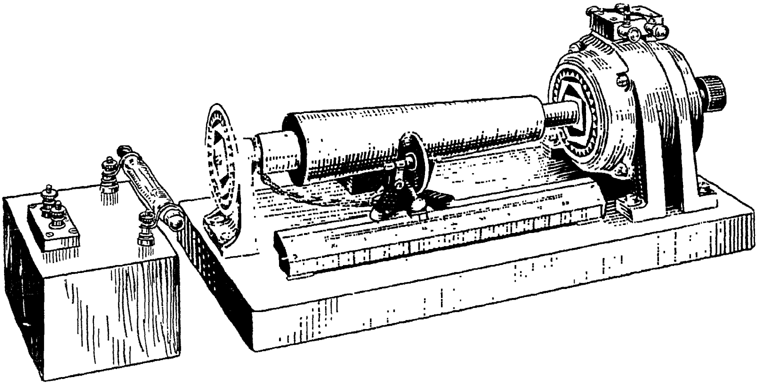
It is obvious from the above that the speed of any machine can be obtained by such means in terms of its difference from some fixed speed, the actual standard used being the vibrating tuning fork, whose frequency is known and is constant. Since the fixing of the stroboscopic disc to the shaft of the machine produces no appreciable increase in the load, the speed is not affected by the device used for its measurement, and in this fact lies one of the principal advantages of stroboscopic methods.

The disc used, instead of bearing only one pattern, as in Fig. 499, usually has several patterns with different numbers of points, the patterns being drawn concentrically. Several primary speeds can then be dealt with by using the different patterns.

CONE-ROLLER STROBOSCOPE. For the measurement of speeds which lie between the primary speeds for the various patterns on the disc, a "cone-roller stroboscope" can be employed in conjunction with a standard tuning fork. In this instrument there is a gunmetal cone which is driven by a small synchronous motor, the frequency of the supply to the latter being regulated by a contact on the vibrating tuning fork. In contact with the surface of the cone is a small pivoted disc, having six slots cut in it. This disc is driven from the cone through a rubbing contact, and its speed of rotation can be adjusted by moving its position along the axis of the cone. If the stroboscopic disc on the machine shaft is illuminated through the six slits of the small disc so driven, the number of illuminating flashes per minute can be adjusted by moving the latter along the axis of the cone.

A scale, parallel to the axis of the cone, gives the speed corresponding to any axial position of the small disc. An instrument of this type, manufactured by Messrs. H. Tinsley & Co., is shown in Fig. 500.*

Stroboscopic Observations. Mullard Electronic Products, Ltd., make a stroboscope designed especially for the stroboscopic examination of moving objects. This apparatus, which is very fully described in the makers' catalogue, produces light flashes, of duration between 3 and 10 microseconds, over a frequency range of 0·5 to 250 per second. It includes a frequency meter to measure directly the flashes per second.



(H. Tinsley & Co., Ltd.

500. CONE-ROLLER STROBOSCOPE

The Measurement of Linear Speed. The measurement of the speed of railway trains is important in maintaining precise running schedules and several electrical instruments have been developed for this measurement.

Elliott Brothers (London), Ltd. have developed an equipment consisting of an a.c. generator, directly coupled to the axle, and a speed indicating or recording instrument. The generator is designed to withstand the very severe vibration and shocks which must occur with such direct mounting. It is a 12-pole, inductor type, machine with a rotor of 35 per cent cobalt magnet steel.

Temperature compensation is provided by the use of a magnetic shunt, with a negative coefficient of permeability, fitted to the rotor. There is also a compensator unit to take into account reduction of the wheel diameter through wear.

Co-operation between Everett Edgcumbe and Co., Ltd., Evershed and Vignoles, Ltd. and the London Passenger Transport Board has led to the development of an inductor pattern Railway Speedmeter which dispenses with a mechanical drive. It consists of a transmitter (shown in Fig. 501) and an indicator which is, in fact, a frequency meter of the moving-coil ratiometer type with two coils. One coil has an inductance in series with it and the other a capacitor. Each has a rectifier in series with it as shown in Fig. 502.

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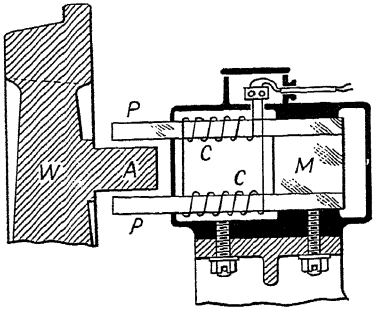
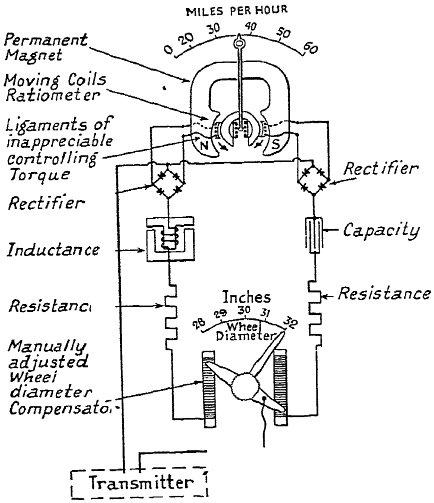
* For further information regarding stroboscopic methods, the reader is referred to the works mentioned in Refs. (1) and (6).



(*Everett Edgcumbe & Co., Ltd.*)

FIG. 501. INDUCTOR RAILWAY SPEEDMETER: SECTION OF TRANSMITTER



(*Everett Edgcumbe & Co., Ltd.*)

FIG. 502. PRINCIPLE OF OPERATION OF RAILWAY SPEEDMETER

The transmitter has a permanent magnet *M* and two coils *CC* the flux threading which varies as the armature A (of which there are 8 or 10 on the wheel concerned in the measurement) passes through the air gap of the magnet system. An e.m.f. is induced in the coils. The frequency of this e.m.f. is proportional to the speed of the train. Although the induced e.m.f. also varies with train speed the readings of the frequency meter are independent of this variation.

The principle of operation of the indicator is that, as the frequency varies, the relative values of the two opposing torques produced by currents in the two coils change, e.g. an increase in frequency reduces the current in the inductive circuit and increases that in the capacitative branch.

The Measurement of Frequency. To consider first, frequency standards: for calibration these must be referred ultimately to the fundamental frequency which is that of the rotation of the earth. This can be very accurately measured by astronomical methods.

Thus, a primary standard of frequency takes the form of an oscillator generating a frequency which is constant over long periods and which is checked against that of the earth's rotation.

Very stable oscillators used as secondary standards are checked against the primary standards.

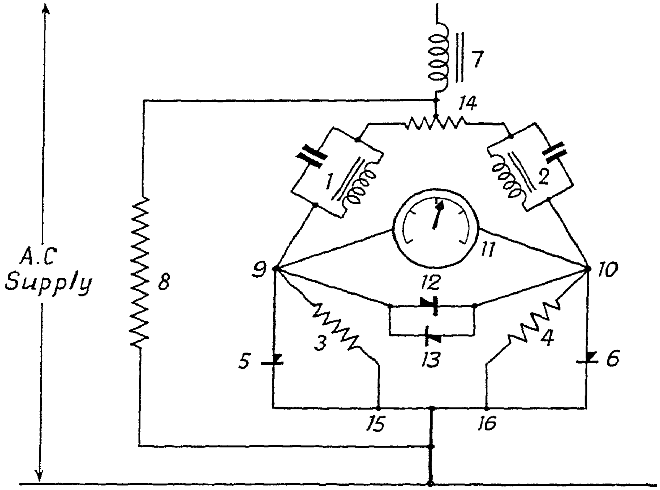
The primary standards have as a basis a quartz oscillator operating at 100 kc/s and having a low temperature coefficient of frequency with thermostatic control of temperature. It has been found possible with such methods (Ref. (12)), by careful design, to obtain an accuracy approaching 1 part in 10^8 over a period of several months, and an even greater accuracy for shorter periods. L. Hartshorn (Ref. (13)) says that "the quartz frequency standard has reached a degree of precision that is far ahead of any other electrical standard."

The range of application of quartz-controlled frequency standards has been extended, due to recent developments in microwave spectroscopy, into the region where the lowest atomic resonances appear. There is thus the possibility of using spectrum lines as frequency standards. Ammonia gas, for example, has an absorption line at 23,870 Mc/s and it has been found possible to use this spectrum line as a reference for the control of an oscillator. The research on this subject promises interesting developments, in the future, for primary frequency standards.

H. W. Sullivan, Ltd. make several valve oscillators which are sufficiently stable in frequency for them to be used as frequency standards.

A special Sullivan-Griffiths oscillator of 5 to 220 kc/s, with a direct-reading frequency scale of 0·03 per cent accuracy and a frequency stability of 0·003 per cent, has been developed especially for telecommunication testing within the "carrier" frequency band.

The same makers also produce two heterodyne oscillators of such stability that they are useful also as frequency standards of 0·2 per



(*Crompton Parkinson, Ltd.*)

FIG. 503. OPERATION OF THE BRIDGE NETWORK FREQUENCY METER

With normal frequency the magnitude of the impedances 1 and 2 are equal, and since the impedance of the resistor/rectifier circuits 3/5 and 4/6 are also equal the bridge network is balanced and no effective current will flow between points 9 and 10.

The permanent-magnet moving-coil microammeter will thus indicate zero microamperes, which point on the scale is figured to correspond with the normal frequency.

With frequency lower than normal, the magnitude of the impedance of the reactive bridge element 1 increases, and that of the reactive bridge element 2 decreases and, since the impedance of the resistor/rectifier elements 3/5 and 4/6 remains constant, the bridge is unbalanced.

The rectifiers 5 and 6 offer a path of very low impedance to one half of the a.c. current wave, so that during this half wave the points 9 and 10 of the bridge network are brought substantially to the same potential as points 15 and 16 which, being solidly connected together, are at the same potential.

Under these circumstances a difference of potential only exists between points 9 and 10 during the other half of the wave, and the pulsating current due to this half wave flows through the microammeter from point 10 to point 9 causing it to deflect in a counter-clockwise direction to a position corresponding to the magnitude of the unbalance of the bridge network, and consequently to the frequency of the supply.

Conversely, with a frequency higher than normal the magnitude of the impedance of the reactive bridge element 1 decreases and that of the reactive bridge element 2 increases, and the bridge network is unbalanced. A pulsating current then flows from point 9 to point 10 of the bridge network.

This pulsating current passes through the microammeter and causes it to deflect in a clockwise direction to a position corresponding to the magnitude of the unbalance of the bridge network, and consequently to the frequency of the supply.

In a microammeter indicating the degree of unbalance of a bridge network, the magnitude of its deflection, in addition to being a function of the degree of unbalance of the bridge, is also dependent on the voltage applied to the bridge network.

In order to render the indication of the microammeter relatively insensitive to variations of the supply voltage within the limits of ± 20 per cent departure from normal, the half wave rectifiers 12 and 13 are connected to the instrument to form shunts of non-linear impedance.

cent accuracy. They cover a very wide frequency range and are continuously variable in a single sweep of a dial which is direct reading in frequency. Two frequency ranges are made; 0–16,000 c/s and 50–170,000 c/s.

For most laboratory purposes a tuning fork which is capable of giving an accuracy of one part in a million is adequate.

An electrically-maintained tuning fork, made by H. Tinsley and Co. as a standard frequency source, is run from a 2-V accumulator. The fork has two electrical contacts; one is used for maintaining the fork in vibration and the other can be used in the drive of a chronoscope or phonic motor, or for flashing a stroboscopic lamp. The fork is usually of frequency 50 c/s or 25 c/s but can be made for frequencies up to 500 c/s. It is adjusted to within 2 parts in 10,000 at 20° C with an amplitude of 2 mm. and has a temperature coefficient of 1 part in 10,000 per 1° C rise in temperature.

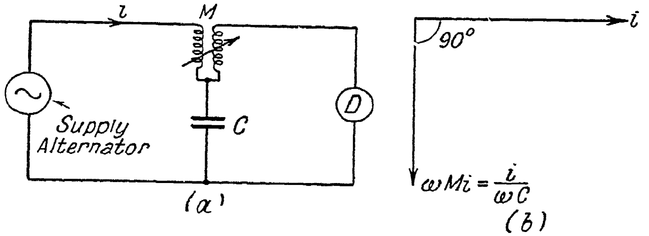


FIG. 504. CAMPBELL FREQUENCY BRIDGE

Frequency Bridges. These are bridge networks similar to those discussed in Chapter VI; indeed, in some cases they are exactly the same networks. The same combination of impedances, and other apparatus constituting the network, can be used for more than one purpose; e.g. the same bridge can be used for the measurement of inductance in terms of impedances and the frequency or, alternatively, it can be used to measure the frequency in terms of impedances.

The primary requirement of such bridge networks is that the balancing of the network shall be dependent upon the supply frequency which, when the bridge is so used, is the frequency to be measured.

The choice of a frequency bridge is influenced by the magnitude of the frequency to be measured, upon the apparatus available, and upon the ease with which a complete balance can be obtained.

CAMPBELL'S FREQUENCY BRIDGE. This bridge, which is commonly used, has the advantages of simplicity and of a fairly large range of frequency, provided the capacitor employed is loss-free, and the mutual inductance is free from impurity.

The connections are as shown in Fig. 504(*a*), in which the alternator whose frequency is to be measured is connected to the primary of a mutual inductance M, in series with a loss-free capacitor C. A detector D, which may be either telephones or a vibration galvanometer, according to the frequency under test, is connected in series with the secondary of M, as shown.

Then, neglecting impurities in both M and C, we can say that, when a current i flows in the primary of M, and when M has been adjusted so that no current flows through the detector, the voltage induced in the secondary of the mutual inductance is both equal to, and in phase with, the voltage drop across C. This is shown in the simple vector diagram Fig. 504 (*b*). Thus, the induced voltage in the secondary of M lags 90° behind the primary current i, and is equal to $\omega M i$, where $\omega = 2\pi \times$ frequency.

Again, the voltage drop across C lags 90° behind i, and equals $\frac{i}{\omega C}$.

Hence, at balance, $$\omega M i = \frac{i}{\omega C}$$

or $$\omega^2 = \frac{1}{MC}$$

from which the frequency is

$$f = \frac{1}{2\pi\sqrt{MC}} \quad . \quad . \quad . \quad . \quad (452)$$

If M and C are not free from impurity, either the impurities must be known and taken into account, or modifications to the simple circuit shown in the figure must be made, in order to render their effect small (see Refs. (5) and (7)). It is also necessary, if a perfect balance is to be obtained, that the frequency shall be constant and that the wave-form shall be free from harmonics.

To overcome the difficulty inherent in the use of the Campbell bridge at low frequencies (below 250 c/s), namely, that large values of M and C are required, the Kennelly and Velander bridge (see Ref. (5), 4th Edition, p. 476) can be used.

Frequency Meters. Such meters indicate the supply frequency of a circuit directly, and are thus more convenient for most practical purposes than the foregoing, more precise, methods.

Three main classes of frequency meters are of importance—

(*a*) Instruments depending for their operation upon the phenomenon of mechanical resonance.

(*b*) Instruments employing electrical resonance.

(*c*) Instruments whose action depends upon the variation of impedance of an inductive circuit with frequency.

One example of each of these types will be described.

VIBRATING REED FREQUENCY METER. This instrument is of the first type, and consists of a number of thin steel strips, or "reeds," arranged alongside, and close to, an electromagnet, as shown in the sketch of Fig. 505(*a*). The electromagnet is laminated and its winding is connected, in series with a resistance, across the supply whose frequency is to be measured, the external connection being thus the same as those of a voltmeter.

The reeds, which are about 4 mm. wide and $\frac{1}{2}$ mm. thick, are not exactly similar to each other, but are either of slightly different lengths and breadths, or carry slightly different loads or flags at their upper ends. These differences cause the natural frequencies of vibration of the reeds to differ, and they are arranged in ascending order of natural frequency. Thus, the natural frequency of the first may be 47 cycles per second; of the one next to it $47\frac{1}{2}$ cycles; of the next 48 cycles, and so on.

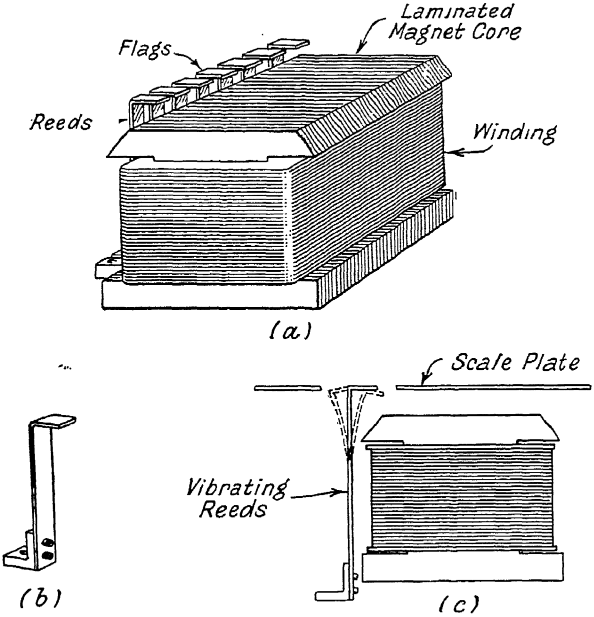


FIG. 505. VIBRATING REED FREQUENCY METER

The natural frequency of a reed can be determined from the formula

$$f = \frac{1}{2\pi}\frac{x}{l^2}\sqrt{\frac{E_K}{\Delta_K(1 + 4\cdot 1K)}} \quad \text{(see Ref. (1))} \quad . \quad . \quad (453)$$

where l is the length of the reed and x its breadth, both in centimetres. E_K is the modulus of elasticity of the material of the reed in dynes per square centimetre, Δ_K is the density of the material in grammes per cubic centimetre. K is the ratio of the mass of load or flag at the end of the reed to the mass of the reed itself.

$$E_K = 1{\cdot}93 \times 10^{12} \text{ dynes per sq. cm.}$$

$$\Delta_K = 7{\cdot}8 \text{ grammes per cu. cm.}$$

When the frequency meter is in use the magnetism of the electromagnet alternates with the frequency of the supply, and exerts an attracting force upon the reeds once every half-cycle. All the reeds thus tend to vibrate, but only the one whose natural frequency is double that of the supply will vibrate appreciably. Mechanical resonance is obtained in the case of this reed.

The flags at the top of the reeds are painted white, and the frequency is read directly from the instrument by observing the scale mark opposite to the reed which is vibrating most. Usually the vibration of the other reeds is so slight as to be unobservable. If, however, the supply frequency lies half-way between the natural frequencies of two adjacent reeds, both of these will vibrate equally, but the amplitude of the vibration will be less than when the supply frequency exactly coincides with that of one of the reeds.

Range. The usual range of frequency in a frequency meter of this type is about six cycles—say 47 cycles to 53 cycles—although it may be either greater or less than this range.

The range of such an instrument may be doubled by polarizing the reeds, i.e. by magnetizing them with direct current, so that a unidirectional magnetization is superimposed upon the alternating magnetization due to the current whose frequency is to be measured. The reeds are then attracted to the core only once per cycle instead of twice, so that, for example, the 50 cycle reed now has 100 attracting impulses per second when the supply frequency is 100 cycles per second, and it therefore vibrates.

A great advantage of this type of frequency meter is that its indications are independent of the applied voltage and wave-form, provided the voltage is not so low that the amplitude of the vibrations—which depends upon the applied voltage—is too small to give reliable readings.

ELECTRICAL RESONANCE FREQUENCY METER. A very simple form of indicating frequency meter, whose action is dependent upon electrical resonance is illustrated in Fig. 506 (*a*). A magnetizing coil, which is connected across the supply circuit, and therefore carries a current of the frequency to be measured, is wound on one end of a laminated iron core. Over this core is a moving coil, pivoted as shown, and having a pointer attached to it. The terminals of this coil are connected to a suitable capacitor C.

The principle of operation can be understood from the three vector diagrams, Fig. 506, (*b*), (*c*), (*d*), in which I is the current in the magnetizing coil and ϕ the flux in the iron core, this being assumed to be in phase with I. The voltage induced in the pivoted coil will lag 90° in phase behind ϕ in all cases. i is the current in the pivoted coil.

Now, in diagram (*b*), the circuit of the pivoted coil is assumed to be largely inductive: the current in it thus lags behind the induced voltage e, and the torque acting upon the coil is proportional to $Ii \cos (90 + \alpha)$. In diagram (*c*), the circuit is supposed to be largely capacitative, so that the current i leads e in phase. The torque is now proportional to $Ii \cos (90 - \beta)$, and is obviously in the opposite direction to what it was in the case represented by diagram (*b*). In diagram (*d*), the inductive reactance is equal to the capacitance reactance, so that i is in phase with e, and the torque upon the pivoted coil is proportional to $Ii \cos 90$, which is zero.

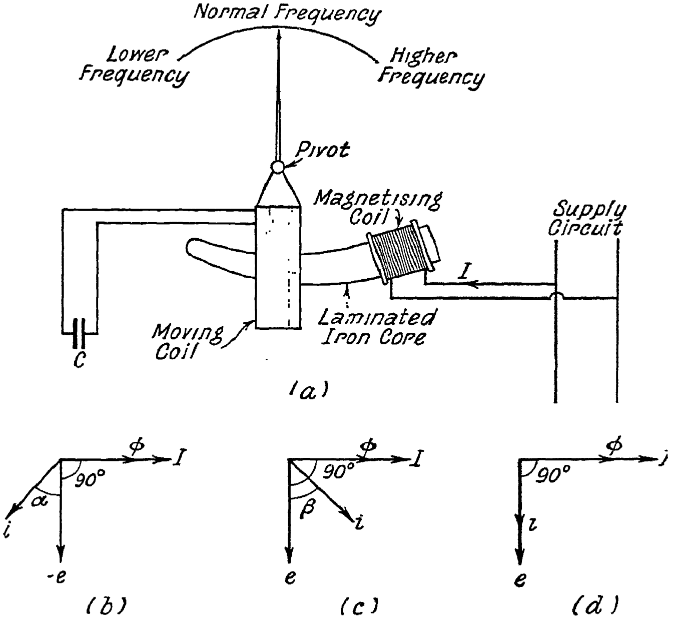


FIG. 506. B.T.-H. RESONANCE FREQUENCY METER

Operation. Referring now to the actual operation of the instrument, the capacitance reactance $\frac{1}{\omega C}$ $\left(\text{or } \frac{1}{2\pi f C}\right)$ is constant for any given frequency. The inductive reactance ωL depends upon the position of the pivoted coil on the core. The nearer this coil approaches the magnetizing coil the greater its inductance. The pivoted coil is pulled towards the magnetizing coil, therefore, until $\omega L = \frac{1}{\omega C}$, when, as already seen, the torque is zero. Obviously the circuit of the moving coil is in resonance when the torque is zero.

The value of the capacitor C is so chosen that the pivoted coil takes up a convenient mean position when the frequency is at its normal value. If the frequency increases, the capacitance reactance $\frac{1}{\omega C}$ decreases, and the coil moves *off* the core so that its inductance is reduced to the point when ωL again equals $\frac{1}{\omega C}$. The coil moves farther *on* to the core if the frequency falls.

An advantage of this type of instrument is that, if the inductance of the moving coil changes slowly with variation of its position on the core, great sensitivity can be obtained.

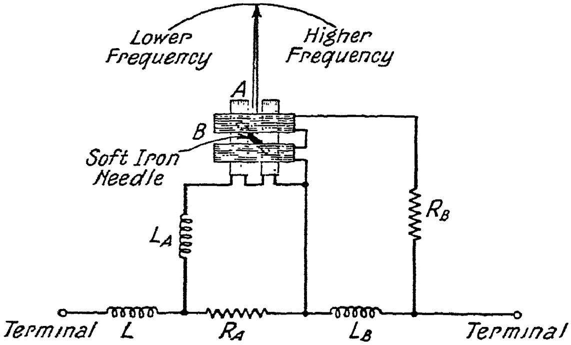


FIG. 507. WESTON FREQUENCY METER

WESTON FREQUENCY METER. This is a moving-iron instrument [illegible] pe[illegible] variation in current distribution between two parallel circuits—one inductive and the other non-inductive—when the frequency changes.

The construction and internal connections are shown in Fig. 507. There are two fixed coils, A and B, each in two equal parts. These coils are fixed so that their magnetic axes are perpendicular and at their centre is a pivoted soft-iron needle which is long and thin. The spindle bearing the needle also carries a pointer and damping vanes, but there is *no controlling device.*

Coil A is connected, in series with an inductance L_A, across a non-inductive resistance R_A, and coil B is connected, in series with a resistance R_B, across an inductance L_B, as shown. The inductance L is for the purpose of damping out harmonics in the wave-form of the current through the instrument, and so eliminating errors due to such harmonics.

The soft-iron needle takes up a definite position which is dependent upon the currents through the coils *A* and *B*. If the frequency is increased the current through coil *A* is reduced, since the reactance of L_A increases while the voltage drop across R_A remains the same. On the other hand, the current in coil *B* increases. The total effect is, therefore that of turning the pivoted needle so that it lies more nearly parallel to the axis of coil *B*.

Decrease of frequency obviously has the opposite effect, and these variations of frequency are followed by the pointer, as shown.

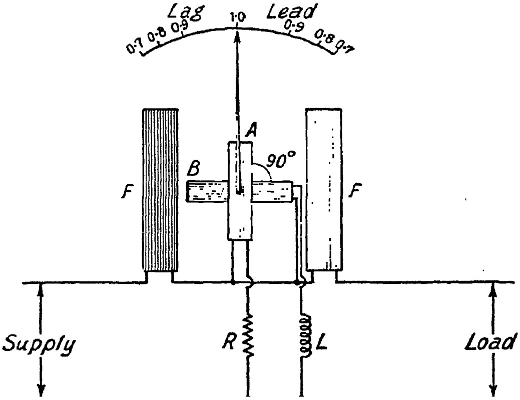


Fig. 508. Single-phase Power-factor Meter, Dynamometer Type

, or Power-factor, Meters. These instruments indicate the power-factor of a circuit directly, as distinct from instruments from whose readings the power factor may be obtained by dividing the watts supplied, by the volt-amperes in the circuit.

Power-factor meters—like wattmeters—have a current circuit and a pressure circuit. The former carries the current in the circuit whose power-factor is to be measured, or a definite fraction of this current. The pressure circuit is usually split into two parallel paths —one inductive and one non-inductive—and the deflection of the instrument depends upon the phase differences between the main current and the currents in the two branches of the pressure circuit, i.e. upon the power factor of the circuit.

Dynamometer Type Single-phase Power-factor Meter. principle of this instrument is illustrated in Fig. 508. The two fixed coils, *FF*, carry the current in the circuit under test, and the magnetic field of these coils is thus proportional to the main current.

Pivoted within this field, between the fixed coils, are two coils A and B, rigidly fixed at an angle of 90° apart. These coils move together and carry a pointer which indicates the power factor of the circuit directly, on the scale above. The numbers of turns on these two coils, as well as their dimensions, are the same, so that they produce equally strong magnetic fields (90° displaced in space and also in time phase) when equal currents, having a time phase difference of 90°, are passed through them. The values of the resistance R and inductance L, in series with coils A and B respectively, are adjusted so that these currents are equal when the frequency is at its normal value. At other values of frequency the two currents will be different, since that in coil B is dependent upon frequency—owing to the reactance of L being so dependent—while the current in A is practically independent of frequency.

In discussing the action of the instrument it will be assumed that the time-phase difference between the two currents in A and B is exactly 90°. Actually this is not quite correct, and the angle between the two coils is made rather less than 90°—i.e. equal to the actual time-phase displacement between the currents.

A capacitor may be used instead of the inductance L, when a phase displacement more nearly 90° may be obtained.

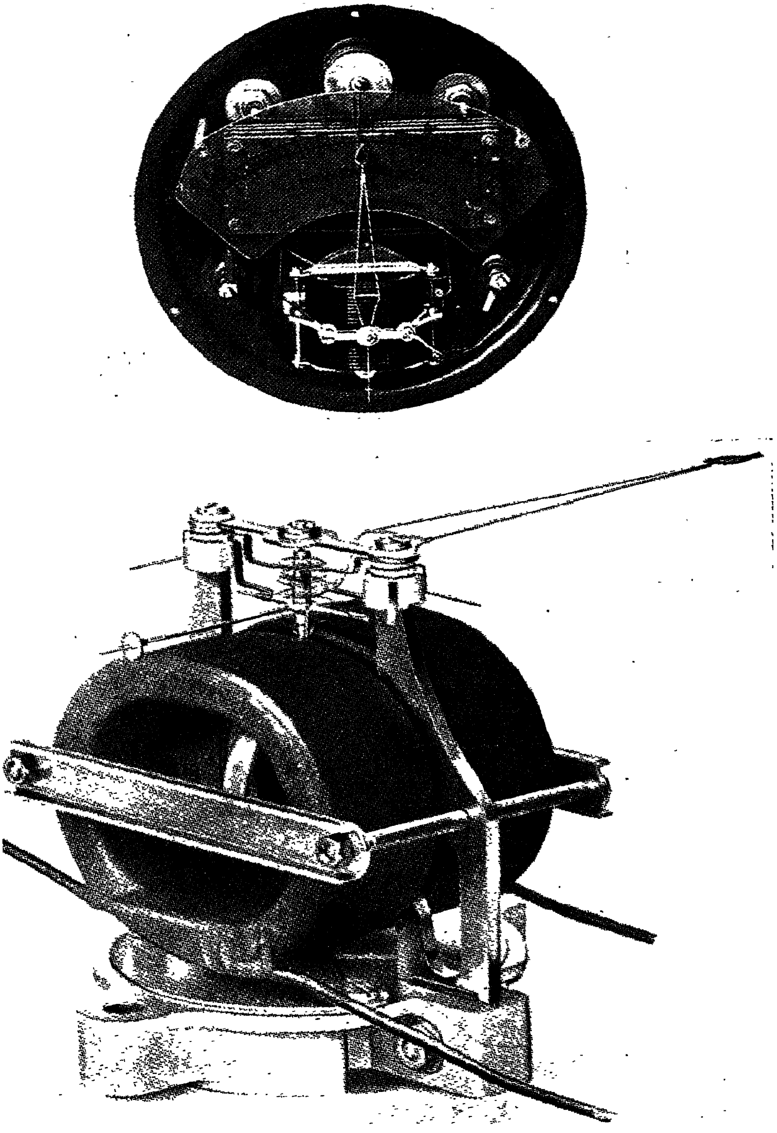
There is no controlling torque acting upon the moving system, the currents being led into the coils by fine ligaments which exert no control.

Action. Suppose that the current of the circuit is in phase with the voltage (i.e. that the power-factor is unity). Then, the current in coil A will be in phase with that in coils FF, since it is in phase with the voltage. At the same time, the current in coil B—which is displaced in phase by 90° behind the volts—will lag 90° in phase behind the current in FF. There will thus be a torque acting on coil A which will turn it into a position perpendicular to the axis of the coils FF, while the torque upon coil B will be zero. The moving system, therefore, sets as shown in the figure, and the pointer indicates unity power factor.

If the power factor of the circuit is zero, the current in coil B is in phase with the main current in coils FF, while that in coil A is 90° out of phase with the latter. The moving system then turns so that coil B is perpendicular to the axis of FF. For intermediate power factors the moving system takes up intermediate positions, the angle which the pointer makes with the axis of the fixed coils being $(90 - \phi)$ where ϕ is the phase angle of the circuit and $\cos \phi$ the power factor.

The full theory of the instrument is given by Dover (Ref. (3)) and by Laws (Ref. (2)). It has the advantage of being unaffected by the values of current and voltage unless these are very small, but its readings are affected by frequency and wave-form.

Fig. 509 shows the construction of an instrument of this type.



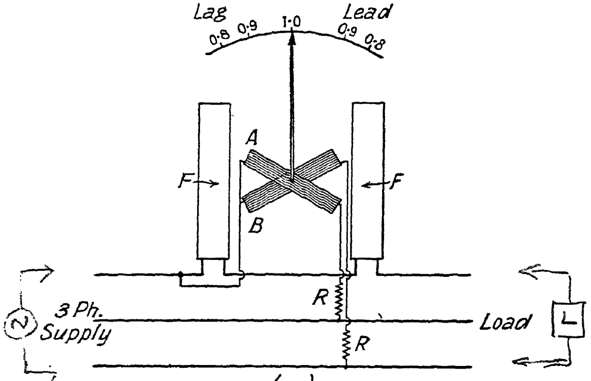
(Sangamo Weston, Ltd.)

FIG. 509. WESTON POWER-FACTOR METER

POWER-FACTOR METER FOR BALANCED THREE-PHASE LOAD. Fig. 510 shows the connections of a three-phase power-factor meter, the readings of which are correct only when the load is balanced.

In this instrument the two moving coils are fixed with their planes 120° apart, and are connected across two different phases of the supply circuit, the fixed coils being connected in the third phase and carrying the current in the line. There is now no necessity for phase-splitting by artificial means, since the required phase displacement between the currents in the two moving coils can be obtained from the supply itself, as shown.

Provided the two moving coils are 120° apart, the angle through which the pointer is deflected from the unity power factor position is equal to the phase-angle of the circuit.



(a)

FIG. 510. DYNAMOMETER POWER-FACTOR METER FOR A BALANCED THREE-PHASE LOAD

The three-phase instrument gives indications which are independent of frequency and wave-form, since the currents in the two moving coils are both affected in the same way by any change of ency.

MOVING-IRON POWER-FACTOR METERS. Instruments of the moving-iron type have certain advantages over power-factor meters of the dynamometer type which have caused them to become more generally used than those of the latter type. These advantages are—

(a) Larger working forces—these forces being rather small in the dynamometer type.

(b) The absence of ligaments to lead in current to moving coils, all the coils in the moving-iron types being fixed.

(c) A scale which extends over 360°.

Errors are, however, introduced owing to losses in the iron parts of the instrument, these losses being dependent upon the load and frequency.

In general, moving-iron power-factor meters are less accurate than those of the dynamometer type.

Rotating Field Type for Balanced Three-phase Loads. Fig. 511 illustrates the principle of the Westinghouse instrument of this type. A rotating field is set up by the three coils A_1, A_2, and A_3, which are supplied from the three-phase mains through current transformers. The coil B is placed at the centre of this system of coils, and is connected in series with a resistance, across two lines of the supply. Inside coil B is a short pivoted iron rod, with sector-shaped iron pieces I_1 and I_2, at its ends. Damping vanes and a pointer are also carried on the same spindle. The moving system is drawn separately in Fig. 511 (*b*). There are no controlling forces.

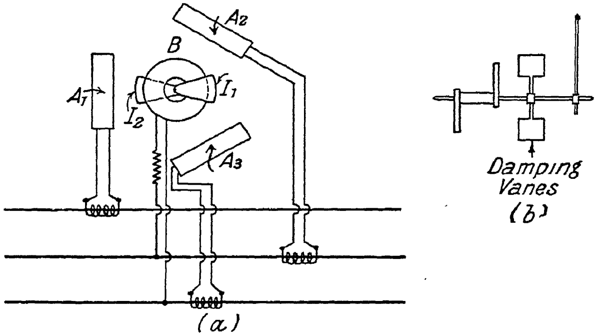


Fig. 511. Westinghouse Power-factor Meter for a Balanced Three-phase Load

The moving iron system carries an alternating flux. It does not tend to rotate continuously, but sets in a definite position which depends upon the relative phases of the current in coil B—which is practically in phase with the voltage—and of the currents in coil A. The deflection of the moving system is approximately equal to the angle of phase displacement between current and voltage in the three-phase circuit.

The presence of iron in the instrument and the dependence of the reactance of coil B upon frequency, render calibration of the instrument at the normal frequency necessary.

Nalder-Lipman Power-factor Meter. This instrument, due to Lipman and manufactured by Nalder Bros. & Thompson, Ltd., has, in the case of the three-phase instrument, three moving irons like those used in the instrument described above. These are all mounted on a single spindle, one above the other, as shown in Fig. 512. They are separated from one another by non-magnetic distance pieces, and the three pairs of sectors are displaced in space by 120° relative to one another. Each iron is magnetized by a pressure coil, these coils being connected across the three phases of the supply. There is one current coil, divided into two equal parts, parallel to one another, one on each side of the moving system, this coil being connected in one line of the supply.

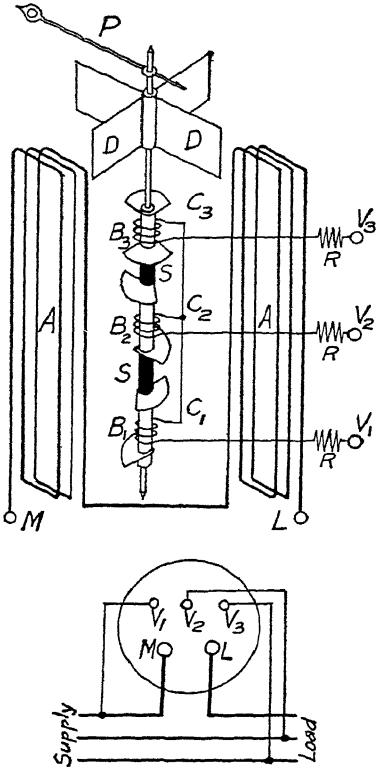


FIG. 512. MOVING-IRON POWER-FACTOR METER

(From *Theory and Practice of Alternating Currents*, Dover)

The spindle of the moving system also carries damping vanes and a pointer, but there is no controlling torque.

As arranged in the figure, the instrument is for use on a balanced three-phase load, but it can also be modified for use on an unbalanced three-phase circuit and for two-phase and single-phase circuits.

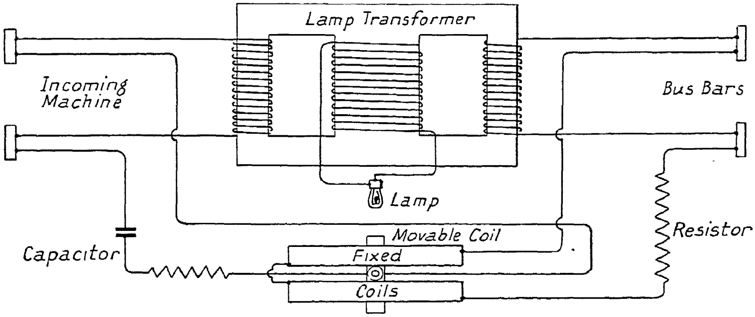
When in action, the moving system of the instrument turns into such a position that the mean torque upon one of the iron pieces is neutralized by the other two torques, so that the resultant torque is zero. In this steady position, the deflection of that iron piece which is magnetized by the same phase as the current coil, is equal to the phase angle between the currents and voltages of the three-phase circuit, provided that the effects of iron losses and of the inductance of the pressure coils are negligible.

As the three pressure coils are at different levels, no resultant *rotating* magnetic field is produced by them, so that any tendency for the moving system to rotate continuously, following the "drag" of such a field, is eliminated.

This type of meter is not appreciably affected by such variations of frequency, voltage, and wave-form, as might be expected in the ordinary supply system.

Synchroscopes. When two alternators are to be operated in parallel it is necessary, before closing the paralleling switch, to ensure that the voltages of the two machines are both equal in magnitude and exactly opposite in phase, the continuous fulfilment of the latter condition implying also that the two machines are of exactly the same frequency.

Synchroscopes are used to indicate when these conditions have been fulfilled. These instruments may be either of the dynamometer or moving-iron type, the latter being the commoner, owing to its 360° scale. Both types are really special forms of phase, or power-factor meters.



(*Sangamo Weston, Ltd.*

FIG. 513. WESTON SYNCHROSCOPE

WESTON SYNCHROSCOPE. This is an instrument of the dynamometer type, and is spring-controlled. It is of similar construction to a single-phase wattmeter of the dynamometer type, except that the fixed coil, as well as the moving coil, is designed for small currents, and is connected, in series with a resistance, across the terminals of the incoming alternator. The moving coil is connected, in series with a capacitor, across the supply with which the alternator is to be paralleled. It may be necessary also, to include a small inductance in the fixed-coil circuit, so that exact quadrature between the currents in the two coils may be obtained when the two supplies to be paralleled are in exact synchronism.

When these two currents are in quadrature, there is no deflecting torque acting on the moving system, and the instrument indicates that synchronism has been attained. Since the currents in the two coils will also be in quadrature—giving zero torque—when the two supplies are 180° out of synchronism, a synchronizing lamp is also included in the instrument, and this lamp is at its maximum brightness when synchronism has been obtained. If the supplies are 180° out of synchronism the lamp does not glow at all. This lamp is fed from the secondary winding of a special transformer, which has two primary windings—one supplied from the incoming alternator and the other from the supply with which the machine is to run in parallel. The connections of both the lamp and the coils of the instrument are shown in Fig. 513.

The scale of the instrument is of opal glass, and is marked "Slow" and "Fast" either side of the centre position. The lamp is placed behind the pointer and scale, so that there can be no confusion regarding the correct instant for synchronizing, this instant being when the pointer is steady at the centre zero position on the scale and the lamp is at full brightness.

If the frequencies of two supplies are not the same, the pointer will oscillate about the centre position.

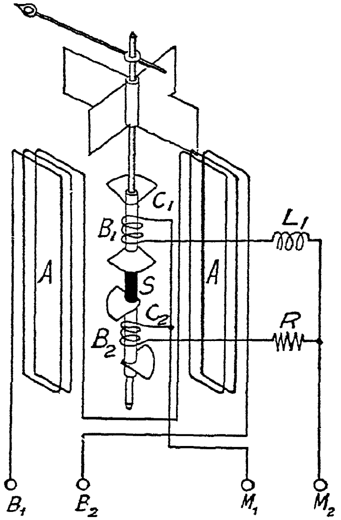


FIG. 514. PRINCIPLE OF MOVING-IRON ALTERNATING-FIELD SYNCHROSCOPE

(From *Theory and Practice of Alternating Currents*, Dover.)

MOVING-IRON TYPE. A common form of moving-iron synchroscope is the Lipman instrument, which is of similar construction to the single-phase Lipman power-factor meter. It has two iron pieces mounted one above the other on a common spindle, as shown in Fig. 514. The coil which, in the power-factor instrument, was the current coil, and was divided into two equal parts on opposite sides of the spindle, is now connected, in series with a non-inductive resistance, across the existing supply. The two pressure coils are connected across the terminals of the incoming alternator, one of the coils having a non-inductive resistance in series with it, while the other is connected in series with an inductive coil.

If the frequency of the alternating field of the moving irons—that is, of the incoming alternator—is not exactly the same as that of the existing supply, the spindle rotates continuously at a speed which, in revolutions per second, is equal to the difference between the two frequencies. The direction of the rotation depends upon whether the incoming alternator is running too fast or too slow.

The instrument has a 360° scale, and the displacement from the zero position gives the difference in phase between the two e.m.f's. The instant for synchronizing is when the pointer of the instrument is stationary at the central, or zero, position.

BIBLIOGRAPHY AND REFERENCES

(1) *Electrical Measuring Instruments*, C. V. Drysdale and A. C. Jolley.

(2) *Electrical Measurements*, F. A. Laws.

(3) *Theory and Practice of Alternating Currents*, A. T. Dover.

(4) *Electrical Measuring Instruments*, D. J. Bolton.

(5) *A.C. Bridge Methods*, B. Hague.

(6) "Separation of the No-load Stray Losses in a Continuous-current Machine by Stroboscopic Running-down Methods," D. Robertson, *Jour. I.E.E.* Vol. LIII, p. 308.

(7) *Dictionary of Applied Physics*, Vol. II.

(8) "A Thyratron Stroboscope," R. S. J. Spilsbury, *Jour. I.E.E.*, Vol. LXXIX, p. 585.

(9) "An Electronic Tachometer," H. G. Jerrard and S. W. Punnett, *Jour. Sci. Insts.*, Vol. XXVII, No. 9, September, 1950, p. 244.

(10) "A Rectifier-type Phase-Sequence Indicator," E. A. Livingston, *Jour. Sci. Insts.*, Vol. XXVII, No. 9, September, 1950, p. 246.

(11) "Components for Instruments," R. E. Hall and E. Coop., *Proc. I.E.E.*, 1951.

(12) "Frequency Standardization," L. Essen, *Proc. I.E.E.*, Vol. XCVIII, p. 154.

(13) "Units and Standards of Electrical Measurement," L. Hartshorn, *Proc. I.E.E.*, Vol. XCIX, Pt. I, p. 271.

(14) "Atomic Definition of Primary Standards," R. D. Huntoon and U. Fano, *Nature*, 1950, Vol. CLXVI, p. 167.

(15) *Electronic Measurements*, F. E. Terman and J. M. Pettit, 2nd Edition, 1952.

(16) "The Determination of Time and Frequency," H. M. Smith, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, p. 143.

(17) "Some Applications of the Phase-sensitive Rectifier Bridge to Electrical Measurements," J. de Gruchy, *Trans. Soc. Inst. Tech.*, Vol. II, No. 3, 1950.

(18) "Utilizing a Watt-hour Meter: The Instrument applied for Polarity, Power, Phase-angle and Phase Sequence Tests," A. Salzmann, *Distribution of Electricity*, July, 1951, p. 9.

(19) "The Direct Reading of the Frequency of Resonant Circuits," *The Wireless Engineer*, Vol. XX, No. 242, pp. 524–38.

(20) "Precision Heterodyne Oscillators," *The Wireless Engineer*, Vol. XI, No. 128, pp. 234–44.

(21) "Further Notes on Precision Heterodyne Oscillators," *The Wireless Engineer*, Vol. XII, No. 142, pp. 357–62.

(22) "Electrical Standards for Research and Industry," a catalogue published by Messrs. H. W. Sullivan, Ltd., in 1954.

CHAPTER XXIII

ELECTRONICS AND ELECTRONIC MEASURING DEVICES

During the last few years electronic devices of various kinds have been used increasingly for measurement purposes. Some of the most important of the methods employed and their application to the measurement of certain non-electrical quantities are described in this chapter which must, however, begin by giving a brief résumé of the underlying physical principles.

The Physics of the Thermionic Valve. The behaviour of the valve can be explained by means of the model of the atom given in 1913 by Bohr (Ref. (1)). Modern physics requires a rather more complicated model, but Bohr's atom is sufficient for the present purpose.

According to this the atom consists of a heavy positive *nucleus* surrounded by a number of unit negative charges or *electrons*. In the normal state of the atom the total negative charge of the electrons is equal to the total positive charge of the nucleus. The number of electrons determines the chemical properties of the atom, and is called the *atomic number*. The mass of the atom is mainly concentrated in the nucleus, and the relative mass (with oxygen as 16) is called the *atomic weight*. Two atoms may have the same atomic number, and therefore the same chemical properties, but different atomic weights: they are then called *isotopes*.

The electrons rotate around the nucleus in orbits which are not arbitrary but must have one of a given range of sizes. The orbits fall into shells, each of which can accommodate only a certain number of electrons, e.g. the first can accommodate 2, the second 8, the third 18, etc. The first 30 elements are listed, with the arrangement of their electrons, in Table XXIV (Ref. (2)). The atomic weights given are those of the element as it is found in nature. In many cases the element so found is a mixture of isotopes: each isotope has an atomic weight nearly equal to a whole number, e.g. ordinary chlorine consists of 75·4 per cent of an isotope having atomic weight 34·9803, and 24·6 per cent of another having atomic weight 36·9779.

In a solid the atoms are arranged in some sort of pattern. If the outer shell of each atom has only a few electrons, these will be only slightly attracted by the nucleus, and will be almost equally attracted by adjacent nuclei. They can thus move freely through the solid, which is therefore a conductor. This is the state of affairs in metals. In an insulator the electrons are all bound to their own nucleus, and considerable energy is needed to tear them away.

The electrons in a metal move about continually at random, somewhat like the atoms of a gas. Modern theory (Ref. (1)) shows that the energy will be distributed among the electrons as shown in Fig. 515. Even at a temperature of absolute zero the electrons have considerable energy. This is a consequence of the quantum theory (Ref. (3)), according to which the energy of an electron can only change by steps, and not continuously.

TABLE XXIV

Element	Atomic Number	Atomic Weight	No. of electrons in shell			
			1st	2nd	3rd	4th
Hydrogen	1	1·008	1			
Helium	2	4·003	2			
Lithium	3	6·940	2	1		
Beryllium	4	9·02	2	2		
Boron	5	10·82	2	3		
Carbon	6	12·010	2	4		
Nitrogen	7	14·008	2	5		
Oxygen	8	16·000*	2	6		
Fluorine	9	19·00	2	7		
Neon	10	20·183	2	8		
Sodium	11	22·997	2	8	1	
Magnesium	12	24·32	2	8	2	
Aluminium	13	26·97	2	8	3	
Silicon	14	28·06	2	8	4	
Phosphorus	15	30·98	2	8	5	
Sulphur	16	32·06	2	8	6	
Chlorine	17	35·457	2	8	7	
Argon	18	39·944	2	8	8	
Potassium	19	39·096	2	8	8	1
Calcium	20	40·08	2	8	8	2
Scandium	21	45·10	2	8	9	2
Titanium	22	47·90	2	8	10	2
Vanadium	23	50·95	2	8	11	2
Chromium	24	52·01	2	8	13	1
Manganese	25	54·93	2	8	13	2
Iron	26	55·85	2	8	14	2
Cobalt	27	58·94	2	8	15	2
Nickel	28	58·69	2	8	16	2
Copper	29	63·57	2	8	18	1
Zinc	30	65·38	2	8	18	2

* By definition.

An electron can move freely inside a metal because as it passes out of the attracting field of one nucleus it comes within that of another. If an electron tries to leave the metal by passing through the surface, however, it is no longer subject to nearly equal attractions on all sides, and work must be done on it in order to remove it from the metal. The amount of this work depends upon the metal. It is measured by the least amount of energy which must be used to remove an electron from the metal at 0° K. This is called the *work-function.* The state of affairs is shown diagrammatically in Fig. 516.

It is evident from the figure showing the energy distribution among electrons that if the temperature is sufficiently high, some electrons will have sufficient energy to escape. If such an electron approaches the surface it will then pass through it. (Some electrons are in fact reflected back into the metal again although they have sufficient energy to escape.) There will thus be an electron current flowing out of the metal, i.e. a conventional current flowing into the metal, and by *Richardson's equation* this current is proportional to

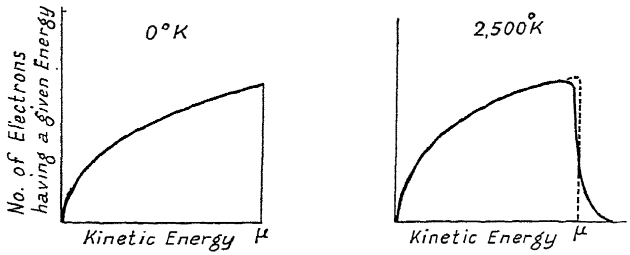


FIG. 515. DISTRIBUTION OF ENERGY AMONG ELECTRONS IN A METAL

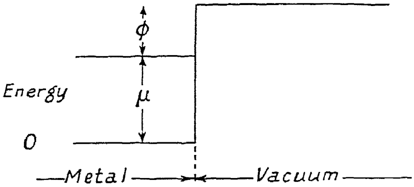


FIG. 516

$$T^2 \varepsilon^{-a\frac{\phi}{T}}$$

where T is the absolute temperature and a is a constant.

$$a = \frac{e}{k}$$

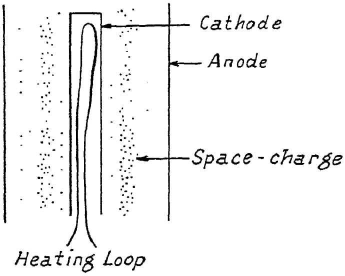
where e is the charge of an electron and
k is Boltzmann's constant.

The value of ϕ depends on the metal, and it can be changed very considerably by covering the surface with a layer of another metal only a few molecules thick. The cathodes of many valves are coated with barium and strontium oxides which, after suitable treatment.

give an emitting surface having a small value of ϕ. These are the "oxide-coated cathodes."

Electrons inside a metal can acquire sufficient energy to escape by absorbing energy from incident radiation. The result is known as *photo-electric emission.* The energy of the emitted electrons depends on the wave length of the incident radiation, while their number depends on the intensity of the radiation. This effect is used in *photo-emissive cells.*

It has been said that the electrons in an insulator require a large amount of energy before they can leave the nucleus with which they are associated. In some solids, called *semi-conductors*, the electrons cannot move freely, but require only a moderate amount of energy to enable them to move. This energy can be acquired thermally—semi-conductors have a lower resistance when hot—or through the absorption of radiation. The *photo-conductive cell* makes use of a semi-conductor which has a lower resistance when illuminated than when in the dark.



The Diode. The simplest type of thermionic valve is the *diode.* It consists of an electrically-heated *cathode* from which electrons are emitted, and an *anode* to which the electrons can pass, and the whole is enclosed in an evacuated envelope, usually of glass. The cathode of battery-operated valves is generally *directly-heated,* i.e. it consists of a loop of wire through which the heating-current is passed, while the surface (which is specially treated) is the actual emitting surface. This design is economical of heating-power but does not allow the heating circuit to be electrically isolated from the cathode. In valves designed for mains-operation the cathode is *indirectly-heated.* A loop of wire carries the (alternating) heating current, but the actual cathode is a cylinder surrounding this loop and insulated from it electrically. The cathode surface is specially-treated and when heated by the heating-loop it emits electrons.

When the cathode is first heated, some of the electrons inside the metal acquire sufficient energy to escape. If the escaped electrons are not removed they will accumulate in a cloud around the cathode and form the *space-charge* (see Fig. 517). Equilibrium will be reached when the number of electrons leaving the cathode is equal to the number which return from the space-charge. (The latter affects the number of electrons emitted, but only to a small extent, by setting up an adverse field near the cathode.) If the anode of the diode is made positive, all the electrons between the anode and cathode will have an acceleration superimposed on their previous motion, and a current of electrons will flow from cathode to anode. The space-charge will therefore be modified, becoming less dense where the electrons are moving faster, i.e. near the anode. The field produced by the anode has little effect at the surface of the cathode, which is screened by the space-charge.

It is evident that the electron current flowing from the space-charge to the anode will increase as the anode potential is raised. The current under these conditions is much less than that which the cathode is continually emitting, since many electrons return from the space-charge to the cathode. The current is said to be *space-charge limited.* If the anode is made very positive, however, all the emitted electrons will go to it, but the field of the anode can only slightly increase the emission of electrons from the cathode. The current will therefore become nearly constant and is said to be *temperature limited,* or to reach *saturation.* If the anode is made negative electrons will be repelled from it and no current will flow. (Actually a small current flows if the anode is slightly negative, as some electrons are moving fast enough to reach it against the adverse field.)

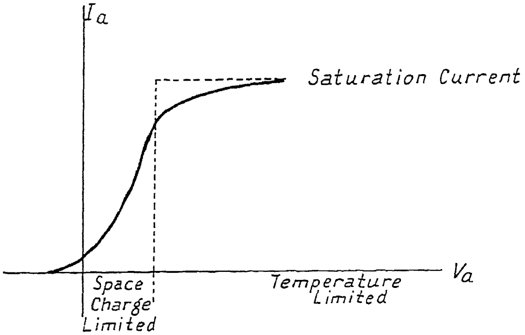


FIG. 518. CURRENT/VOLTAGE CHARACTERISTIC FOR A DIODE VALVE

Thus the graph of current against anode voltage is as shown in Fig. 518. The cold anode does not emit electrons, so that no current can flow from cathode to anode. It does, however, emit some secondary electrons (see under Tetrode) but only when anode current is flowing. The secondary electrons are then recaptured by the anode and have no effect on the operation of the valve. The diode valve can evidently be used as a rectifier. It is operated within the space-charge limited region.

The Triode. Between the cathode and anode of a triode valve is a third electrode, the *grid.* This consists of an open coil of thin wire surrounding the cathode (see Fig. 519). As in the diode, a space-charge is produced near the cathode, but it is now subject to the combined field of the anode and the grid. The grid, being nearer to the space-charge, produces a greater change of current for a given change of voltage than the anode. The characteristics of a triode valve are expressed by two sets of graphs as in Fig. 520. It will be noticed that the curves are given for negative values of V_g. Under these conditions the grid does not take any current as the electrons are repelled from it. (Minute currents due to secondary effects are found in actual valves.) Thus it serves to alter the anode current but requires no power to do so. Triodes are normally used with the grid negative, and the grid in most triodes will be damaged if made to carry any appreciable current.

The anode current can be stopped, for any value of V_a, by a sufficiently negative value of V_g. The valve then is said to be *cut off*.

Notice that the current is space-charge limited. If V_a and V_g are sufficiently increased, the current becomes constant, as in the diode. Few valves can be operated in the temperature-limited range without damage.

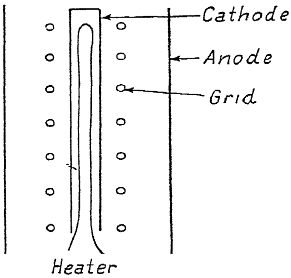


FIG. 519. DIAGRAMMATIC SECTION OF A TRIODE VALVE

The two sets of graphs for a valve are not independent, and one can be derived from the other as shown in Fig. 521.

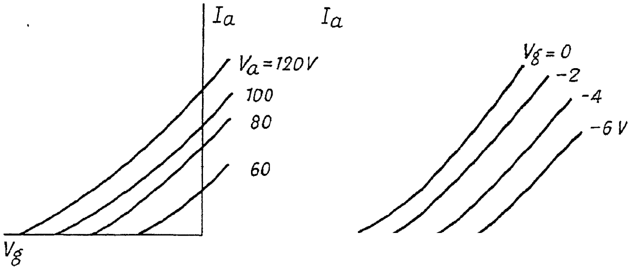


FIG. 520. CHARACTERISTICS OF A TRIODE VALVE

A triode valve is generally required to operate with small alternating voltages superimposed on the steady anode and grid voltages. Three quantities (Ref. (1)) are used to define its performance under these conditions—

The anode resistance $$R_a = \left(\frac{\partial V_a}{\partial I_a}\right)_{V_g \text{ constant}}$$

$$\text{The mutual conductance } g_m = \left(\frac{\partial I_a}{\partial V_g}\right)_{V_a \text{ constant}}$$

$$\text{The amplification factor } \mu = \left(\frac{-\partial V_a}{\partial V_g}\right)_{I_a \text{ constant}}$$

It is evident that R_a and g_m are, respectively, the reciprocal of the slope of the right-hand graph and of the slope of the left-hand graph in Fig. 520. It can be shown that

$$\mu = g_m \cdot R_a$$

The Tetrode. In the tetrode a second grid—the *screen grid*—is added between the first grid (called the *control grid* to distinguish it from the screen grid) and the anode. The screen grid is maintained at a positive voltage with respect to the cathode, but in most applications the anode is made slightly more positive than the screen. The field at the space-charge is now produced by the screen grid and the control grid; the anode is effectively screened from the space-charge by the screen grid and has very little effect on the field there.

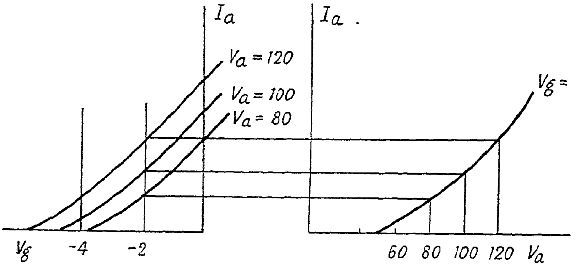


FIG. 521. RELATION BETWEEN CHARACTERISTIC CURVES OF A TRIODE VALVE

When a valve is used to amplify a signal, the change of current through it is generally detected by passing the current through a *load resistance* in series with the valve (see Fig. 522). As the input signal is made more negative, the current through the valve decreases, and the voltage of the anode (which is the supply voltage less $R_L I_a$) rises. In the triode amplifier shown the increase of the anode voltage tends to increase the current flowing through the valve by attracting more electrons from the space-charge. The screen-grid in a tetrode prevents this effect, and so increases the amplification which can be obtained. It also reduces capacitative coupling between anode and grid.

A tetrode amplifier is shown in Fig. 523 although the simple tetrode is not often so used owing to a secondary effect which occurs. Since the screen grid is positive some electrons are attracted to it and current flows in the screen-grid circuit. Most of the electrons, however, acquire sufficient speed to take them through the meshes of the screen grid and into the attracting field of the anode. When they strike the anode they impart some of their energy to the electrons in the metal and some of these electrons may have sufficient energy to escape from the anode. They are termed *secondary electrons* to distinguish them from the *primary electrons* arriving from the cathode.

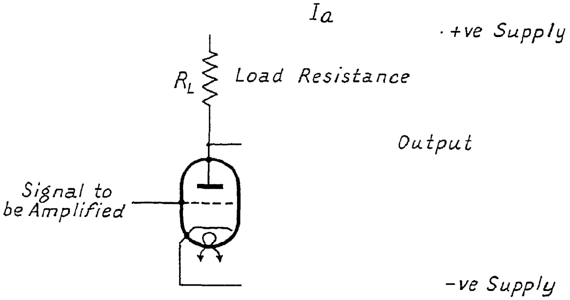


FIG. 522. SIMPLE TRIODE AMPLIFIER

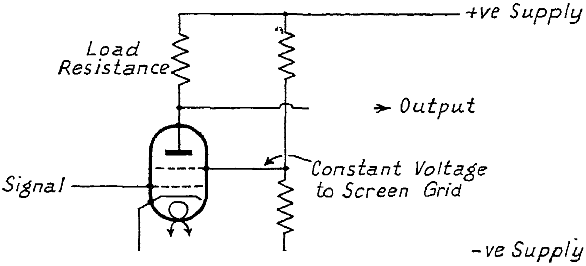


FIG. 523. TETRODE VALVE IN AMPLIFIER CIRCUIT

The path of the secondary electrons is determined by the relative potentials of anode and screen. If the anode is more positive than the screen, the majority will fall back into the anode and will not affect the operation of the valve. If, however, the screen is more positive than the anode many of the secondary electrons may be captured by the screen and they constitute a current between screen-grid and anode. The variation of anode current with anode voltage is as shown (Ref. (1)) in Fig. 524. In range A the primary electrons do not have sufficient energy when they reach the anode to produce a large number of secondary electrons. In range B, each primary electron is producing several secondary electrons, and many of the secondary electrons are captured by the screen which is more positive than the anode. The anode current falls as its voltage rises, so that the valve presents a negative resistance at its anode. In range *C* secondary electrons are still being produced but most of them are recaptured by the anode.

The negative resistance or *dynatron* range of a tetrode can be used to produce an oscillator. The valve is connected to a tuned circuit, and its negative resistance makes good the losses of this circuit (Fig. 525).

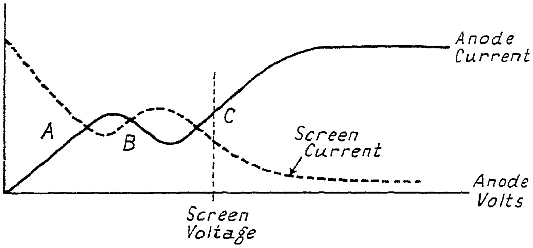


FIG. 524. ANODE AND SCREEN CURRENTS IN A TETRODE VALVE

A special form of tetrode, known as the *beam tetrode,* avoids the negative-resistance region by a special construction using auxiliary plates to concentrate the primary electrons into a beam.

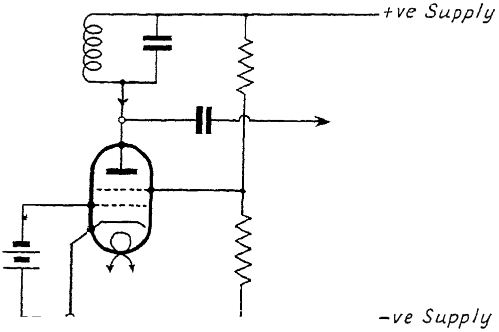


FIG. 525. DYNATRON OSCILLATOR

(Note that the screen must be positive with respect to the anode.)

The Pentode. If a further grid is introduced between the screen and the anode, and is connected to the cathode, all the secondary electrons will return to the anode, since they are protected from the field of the screen by the intervening *suppressor grid.* The anode characteristics of such a pentode are shown in Fig. 526.

The Pentode Voltage-amplifier. A very common circuit element is the *voltage-amplifier*, i.e. a valve which gives an output voltage much greater than the input voltage, and nearly proportional to it. The properties of such an amplifier will change with time, but can be stabilized by negative feed-back (see below) which, however, reduces the amplification that can be obtained.

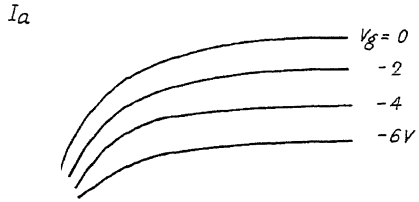


FIG. 526. ANODE CHARACTERISTICS OF A PENTODE VALVE

Amplifiers are classified as d.c. or a.c. according as they can amplify a direct voltage, or only alternating voltages above a certain frequency.

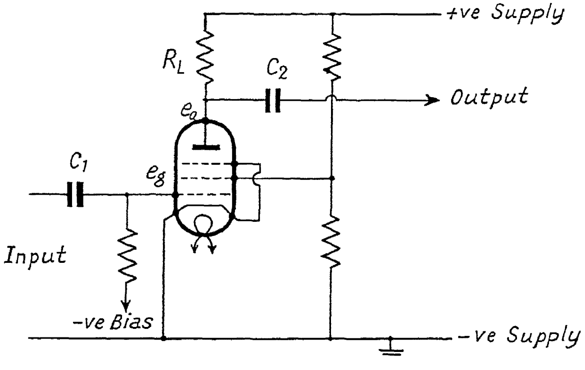


FIG. 527. PENTODE VOLTAGE AMPLIFIER (A.C.)

Fig. 527 shows a pentode used as an a.c. amplifier. The lowest frequency which can be amplified depends on the values of C_1 and C_2.

Fig. 528 shows a d.c. amplifier. The output is generally required to be near earth potential, while the anode of the valve is at about 150 V positive to earth. A negative supply is therefore provided, and R_1 and R_2 provide a potential divider between the anode and the negative line. The amplification is, of course, reduced in the ratio

$$R_1$$

In each case the operation of the amplifier is as follows. Suppose that the input becomes more negative. The grid produces a stronger negative field at the space-charge, and the current flowing to the anode through the meshes of the screen- and suppressor-grids is reduced. The current through R_L is thus reduced, and the voltage drop across it falls. The anode voltage of the valve consequently rises. The reverse occurs if the grid becomes more positive.

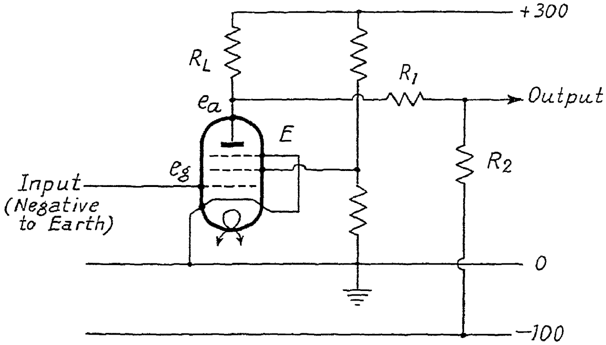


FIG. 528. PENTODE VOLTAGE AMPLIFIER (D.C.)

To obtain the amplification we have

$$g_m = \frac{\partial I_a}{\partial e_g} \qquad R_a = \frac{\partial e_a}{\partial I_a}$$

$$\delta e_a = -R_L \delta I_a$$

Now when e_g and e_a are both changed

$$\delta I_a = \frac{\partial I_a}{\partial e_g} \cdot \delta e_g + \frac{\partial I_a}{\partial e_a} \cdot \delta e_a$$

$$= g_m \delta e_g + \frac{\delta e_a}{R_a}$$

so

$$\delta e_a = -R_L\left[g_m \delta e_g + \frac{\delta e_a}{R_a}\right]$$

$$\delta e_a\left(1 + \frac{R_L}{R_a}\right) = -g_m R_L \delta e_g$$

$$A = \frac{\delta e_a}{\delta e_g} = -\frac{g_m R_L}{1 + \frac{R_L}{R_a}} = -\frac{g_m R_a R_L}{R_a + R_L} = -\frac{\mu R_L}{R_a + R_L}$$

since $\mu = g_m R_a$.

Pentodes are generally used in voltage-amplifier stages, since they have a considerably higher value of μ than triodes, and a greater amplification can be obtained with them. This is, however, not greater in the same proportion as μ, since $\frac{R_L}{R_a + R_L}$ must be less for a pentode than for a triode, to avoid the distortion which is otherwise found.

The operation of a pentode amplifier can be readily followed from the I_a/e_a curves given in Fig. 529. The operating point moves on the *load line* of slope $\frac{1}{R_L}$. The change in e_a for a given change of e_g can easily be read off, and the distortion produced when the change in e_g is large is also apparent. (*Distortion* exists when equal changes of e_g do not always produce the same change of e_a.)

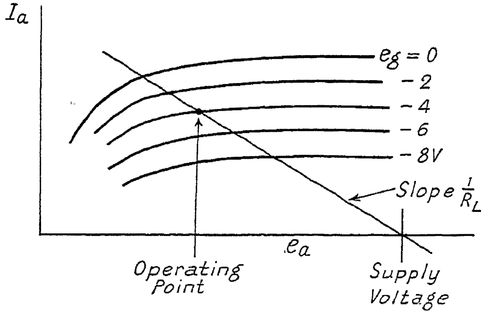


FIG. 529. OPERATION OF A PENTODE VOLTAGE AMPLIFIER

The Thyratron. All the valves so far described have been *hard valves*, i.e. the envelope is exhausted to a high vacuum. In the thyratron, gas under low pressure is purposely left within the envelope.

If an electron moving fairly fast within such a valve strikes one of the gas molecules, it imparts energy to the electrons of its constituent atoms, and one or more of these electrons may obtain sufficient energy to escape from the atomic forces. The remaining part of the molecule will have a positive charge, and is termed an *ion*.

The ions produced in this way will be accelerated by the electric field within the valve, and will move in the opposite direction to the electrons, but much more slowly owing to their greater mass. The electrons lost by the ions will join the stream of primary electrons, and may in turn ionize other molecules. The whole process will therefore be cumulative, and once started by a sufficiently strong field, will proceed until the whole of the gas is ionized.

The cathode of a thyratron has to be capable of withstanding the bombardment of the heavy ions. It is generally made of tungsten, sometimes thoriated, i.e. coated with rare-earth oxides. Thyratrons are operated at relatively high currents, and can easily be damaged by running them with the cathode below normal operating temperature. Sometimes, to avoid such damage, a time-delay is necessary to ensure that the cathode is fully-heated before the anode-voltage is applied.

A thyratron is a three-electrode gas-filled valve. The grid does not resemble that of a hard valve, but is a suitably shaped metal screen between anode and cathode. If the grid is first made very negative, and the anode voltage is applied, the field at the cathode will remain negative, and no current will flow. If then the grid is made less negative, electrons will begin to be accelerated towards the anode. At a certain grid voltage ionization will begin, and will proceed rapidly until it is complete. The valve will then represent a low resistance—the voltage across it will have a nearly constant value depending on the gas (15 to 20 V with mercury) and independent of the anode current. Once the gas is ionized the grid has no more control. If it is made negative it merely attracts the ions, and these, forming a cloud round it, neutralize its field. The discharge through the valve can only be stopped by removing the anode voltage. Thus the grid in a thyratron acts as a trigger which controls the time at which the valve becomes conducting, but thereafter has no effect.

Small thyratrons can easily carry 0·5–1 A, whereas hard valves of the same size will carry only 10–30 mA. Thyratrons are therefore useful for operating relays, etc. They can also be used as d.c. amplifiers when fed from an a.c. supply, the voltage applied to the grid determining the moment during the cycle of alternating voltage at which the valve begins to conduct, and so altering the current flowing.

Negative Feed-back. We are concerned here with the applications of valves to measurements. In most cases the purpose of the valve will be to give a quantity (current or voltage) strictly proportional to the quantity to be measured but at a much higher power level. We thus extend the range of conventional methods to powers insufficient to operate by themselves, without amplification, the conventional instruments.

The second part of our requirement—to increase the power-level—is easily fulfilled: it is provided in every radio set or gramophone amplifier. The chief difficulty is in the first requirement, that the amplified signal shall be strictly proportional to the input signal.

The valve is not in itself a stable or reproducible component, and the amplification of (say) a radio set alters with time and is affected by replacing the valves. In addition, the characteristics of a valve are only approximately linear and only over a small range.

We avoid these difficulties by *negative feed-back.* Consider an amplifier which gives an output which is $-n$ times the input and suppose that a proportion m of the output is fed back to the input, (see Fig. 530).

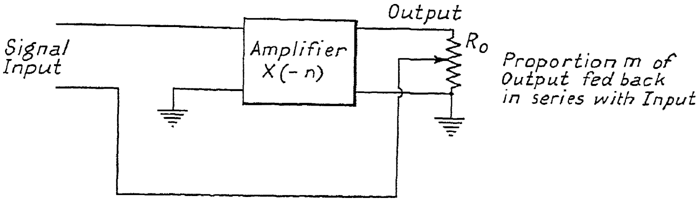


FIG. 530. ILLUSTRATION OF NEGATIVE FEED-BACK

Let the input signal be e and the output signal E. Then, the input to the amplifier is

$$e + mE$$

But the output is $-n$ times this, i.e.

$$E = -n(e + mE)$$
$$E(1 + nm) = -ne$$
$$E = -e\frac{n}{1 + nm}$$

Now suppose n is large—quite a simple amplifier will give $n = 1{,}000$ or 10,000—and that m is not very small. Then, very nearly,

$$E = \frac{-e}{m}$$

The value of m is fixed by circuit elements, i.e. resistances in the above example, and is quite stable. The effect of n, which is unstable and not independent of the amplitude of the input signal, has been eliminated. Thus if the characteristics of the valves change with time, or with changes of the supply voltages, or if a valve is replaced, the performance of the amplifier remains unchanged so long as n remains large enough.

Differentiating

$$\frac{d}{dn}\left(\frac{E}{e}\right) = \frac{d}{dn}\left(\frac{-n}{1 + nm}\right)$$
$$= -\frac{1 + nm - nm}{(1 + nm)^2} = \frac{-1}{(1 + nm)^2}$$

$$\therefore \quad \frac{\delta\left(\dfrac{E}{e}\right)}{\dfrac{E}{e}} = \frac{\delta n}{(1+nm)^2}\cdot\frac{1+nm}{n} = \frac{\delta n}{n}\cdot\frac{1}{1+nm}$$

Thus a 1 per cent change of n produces only $\frac{1}{1+nm}$ per cent change in $\frac{E}{e}$.

Servo-mechanisms. In the example shown, the feed-back voltage is obtained by an electrical network. In many cases the feed-back voltage is obtained in other ways, and we gain a more general view by adopting the language and outlook of *servo-mechanisms*. The elements of a *closed-loop* control system are shown in Fig. 531 (Ref. (4)). The input signal is compared with a feed-back signal obtained from the controlled quantity, and the difference should be zero. If it is not zero, it is amplified and applied to a servo motor which resets the controlled quantity in such a way as to reduce the error.

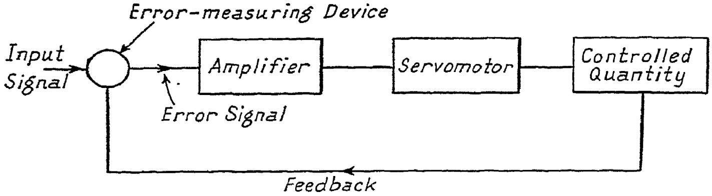


FIG. 531. CLOSED-LOOP CONTROL SYSTEM

Returning to our previous example of the feed-back amplifier (Fig. 530), we can identify each of the elements in the last example. The error-measuring device is simply the series-connected circuit at the input of the amplifier which subtracts a proportion of the output voltage from the input signal. The error signal is amplified, and the output stage of the amplifier is the "servomotor" which supplies power to the resistance R_0 and develops a voltage across it. This voltage is the controlled quantity, and a constant proportion m of it is fed back to the error-measuring device.

This identification of elements may seem rather artificial, but a second example in which some of the elements are mechanical will show how the scheme of a servo-mechanism helps in bringing very different systems under a common theory.

Fig. 532 shows a servo-system for setting a shaft to an angle proportional to an input voltage. A linear potentiometer driven by the shaft gives a voltage proportional to the angular setting of the shaft. This voltage is subtracted from the input, and any difference is amplified and drives a motor in such a direction as to reduce the error. Thus the feed-back voltage is always kept equal to the input signal, and to each value of the latter therefore corresponds a given angular setting. The potentiometer and input circuit of the amplifier here form the error-detector, and the feed-back path is the mechanical one via the shaft.

If we regard the voltage picked off from the potentiometer in the last example as the controlled quantity it is evident that the system forms a self-resetting potentiometer. The potentiometer then must be considered as part of the "servomotor," and the feed-back path is the electrical one.

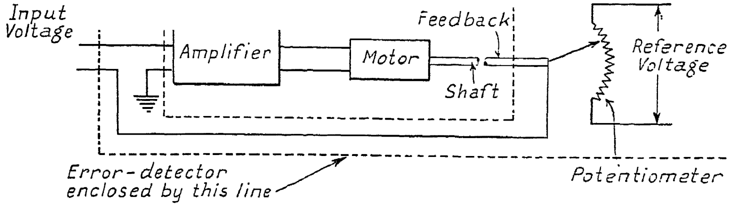


FIG. 532. DIAGRAM OF A CONTROL SYSTEM

Velocity Feed-back. There is a considerable and rapidly-growing literature of servo-mechanisms, in which their dynamical stability and response are discussed. We need not enter deeply into the subject here, but one device is so commonly used that it must be mentioned.

In the last example, suppose that the system is at rest in an equilibrium position, and that the input voltage is then suddenly changed. The amplifier drives the motor in a direction which reduces the error, and if the motor had no inertia (and the amplifier no delay) the speed of the motor would fall off as the shaft approached its final position and become zero when it reached it. The motor, however, will have inertia, and while the shaft is moving for the first time towards its final position, energy is being stored in the motor. Unless arrangements are made to remove this energy the shaft will overshoot its final position, and oscillate about it. One way of dealing with this difficulty would obviously be to fit a mechanical damping device on the shaft. A better method, however, is to make the feed-back proportional partly to position, but also partly to the rate of change of position.

Let e = the input voltage

θ = the position of the shaft

E = the output voltage of the amplifier

Then, if the inertia of the motor is large, we can neglect its back-e.m.f. and write

$$\frac{d^2\theta}{dt^2} = kE$$

where k is a constant.

But $$E = n\left(e - m\theta - \alpha\frac{d\theta}{dt}\right)$$

where n, m and α are constants.

So $$\frac{d^2\theta}{dt^2} = kn\left(e - m\theta - \alpha\frac{d\theta}{dt}\right)$$

$$\frac{d^2\theta}{dt^2} + kn\alpha\frac{d\theta}{dt} + knm\theta = kne$$

If $\alpha = 0$, as in the original diagram, the solution will be oscillatory. If $(kn\alpha)^2 = 4 \, . \, knm$ the system will be critically damped, while if $(kn\alpha)^2 > 4 \, . \, knm$ the system will be over-damped. Thus the effect of velocity feed-back is the same as that of mechanical damping. But the result is achieved without a great loss of energy, and without increasing the size of the motor in order to supply the damping-forces.

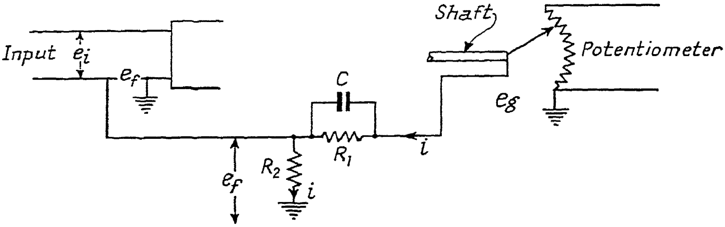


FIG. 533. CIRCUIT GIVING VELOCITY FEED-BACK

The required velocity feed-back can be achieved very simply by electrical means as in Fig. 533 which shows a slight modification of the arrangement of Fig. 532. We have, if R_2 is small, so that we may neglect the voltage across it in comparison with e_g,

$$i = \frac{e_g}{R_1} + \frac{dq}{dt}$$

$$q = Ce_g$$

so that $$e_f = R_2 i = R_2\left[\frac{e_g}{R_1} + C\frac{de_g}{dt}\right]$$

$$= \frac{R_2}{R_1}e_g + R_2 C\frac{de_g}{dt}$$

It should be noted that the above diagrams are given for example only, and are not necessarily the best means for achieving the desired performance (Ref. (8)). The actual design of such systems is difficult, and partly a matter of experiment—the diagrams are only intended to explain the operation of simple systems.

Considering the feed-back amplifier of page 857 as a servo-system it is easily seen that the amplifier resets the controlled voltage by an amount proportional to the error. We found that

$$E = -e\left(\frac{n}{1 + nm}\right)$$

so that the input to the amplifier is

$$e + mE = e\left(1 - \frac{mn}{1 + mn}\right)$$

$$- \quad 1 + mn$$

Thus, however large *mn* may be, there is a limit to the value of e which can be applied without over-loading the amplifier.

In Fig. 532, however, if the inertia of the motor is large, its acceleration will be proportional to the amplifier output: i.e. the amplifier re-sets not the controlled quantity but its acceleration. It is evident that when the system is at rest the input to the amplifier is zero, and that over-loading of the amplifier sets no limit to the input signal which can be accommodated. Furthermore, since the amplifier input is always zero when the system is at rest, the amplifier characteristics do not affect the accuracy and linearity of the servo-system, but only the behaviour while it is seeking the rest position.

Examples of Servo-systems using Velocity Feed-back. (*a*) An interesting example is given in Ref. (20). The problem is to make a frost-point hygrometer automatic in operation. Light scattered by the deposit on a cooled thimble is picked up by a photo-cell, P_1 (Fig. 534), and the resulting voltage is amplified. As the light reaching the photo-cell increases, the thimble is heated, and conversely. For equilibrium the frost deposit must be neither increasing nor decreasing.

The authors (Brewer and Dobson) state that the frost deposit has the characteristics of a large mass. The servo-system would be unstable unless precautions were taken. Velocity feed-back is used to stabilize the system and is obtained by an electrical circuit as shown. The second photo-cell, P_2, is used to eliminate the effect of changes in brightness of the lamp.

(*b*) Another example is given in Ref. (15). The direction of a wind-vane is to be indicated at a distant position, and only a mean position is required, i.e. the faster motions of the vane are to be smoothed out. This cannot easily be done by damping the vane,

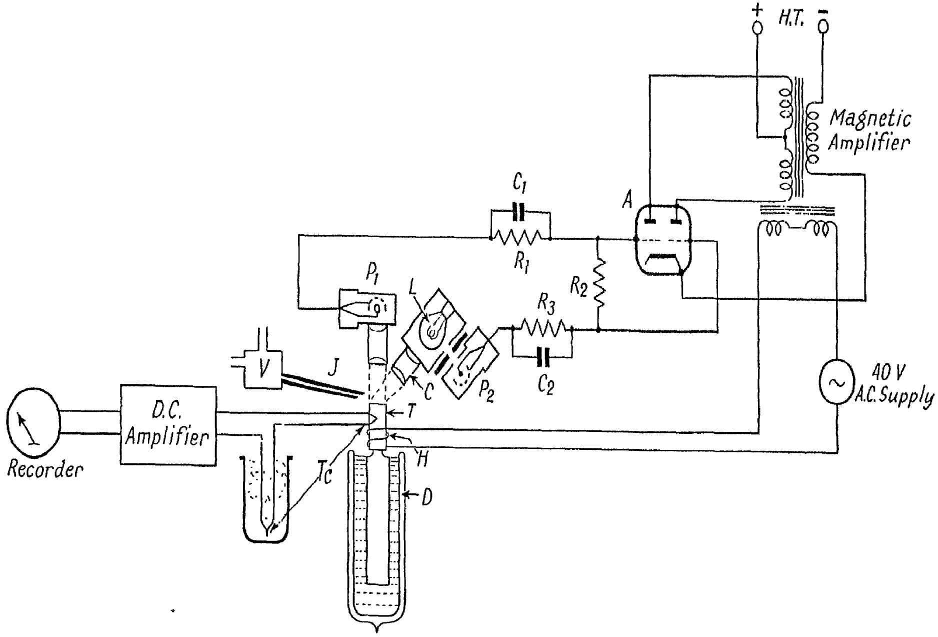


FIG. 534. AUTOMATIC FROST-POINT HYGROMETER

since the wind-forces are proportional to the square of the wind-speed; damping which is sufficient at one wind-speed will be much too great at lower speeds, and too little at higher speeds.

The information is passed on by magslip transmitter and the damping is provided by heavy velocity feed-back. As we have seen, this is equivalent to mechanical damping, and it is of course independent of wind-speed. Thus the required smoothing is obtained.

The circuit is shown diagrammatically in Fig. 535. This is very much simplified from the actual arrangement which, besides the function mentioned, also resolves the wind, and is intended to show the true wind direction on board ship by subtracting the ship's vector velocity from the wind velocity.

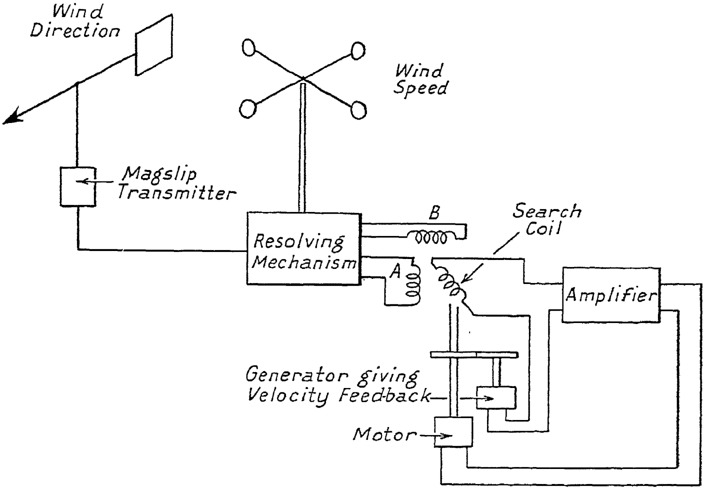


FIG. 535. WIND-DIRECTION INDICATOR

The resolving mechanism shown in the diagram gives a current in A proportional to one component of the wind velocity, and in B proportional to the component at right angles to the first. When the search coil is perpendicular to the resultant field no voltage will be induced in it, and the motor will remain at rest. If a voltage is induced in the search coil it is amplified and used to drive the motor in such a direction as to bring the search coil again perpendicular to the field. In series with the search coil is an a.c. generator which gives an output proportional to the motor speed. This provides velocity feed-back and damps the movement of the motor as already described. The position of the motor shaft is indicated by a further magslip transmitter and receiver (not shown) and shows the mean wind direction.

The Cathode-follower. A common and very simple example of a feed-back amplifier is the cathode-follower. It is used where we wish to increase the current available at a given signal voltage and is thus a device with a high input impedance and low output impedance, giving an output voltage nearly equal to the input voltage (see Fig. 536).

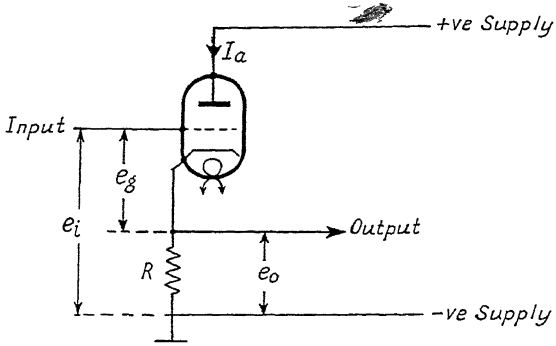


FIG. 536. CATHODE-FOLLOWER CIRCUIT

We have $$e_g = e_i - e_o$$

and $$g_m = \frac{\partial I_a}{\partial e_g}$$

(neglecting the small voltage e_o compared with the supply voltage),

while $$e_o = RI_a$$

so
$$\begin{aligned} \delta e_o &= R\delta I_a = g_m R\delta e_g \\ &= g_m R\delta(e_i - e_o) \\ \delta e_o(1 + g_m R) &= g_m R\delta e_i \\ \delta e_o &= \delta e_i \frac{g_m R}{1 + g_m R} \end{aligned}$$

or, if $g_m R \gg 1$, $$\delta e_o = \delta e_i$$

Thus a change in e_i produces an equal change in e_o.

To find the output impedance of the cathode-follower, suppose that a current I_o is drawn from the output. Then

$$e_g = e_i - e_o$$
$$g_m = \frac{\partial I_a}{\partial e_g}$$
$$e_o = R(I_a - I_o)$$

$$\begin{aligned}\delta e_o &= R\delta I_a - R\delta I_o \\ &= g_m R\delta e_g - R\delta I_o \\ &= g_m R\delta(e_i - e_o) - R\delta I_o\end{aligned}$$

If e_i is kept constant, $\delta e_i = 0$, and

$$\delta e_o = -g_m R\delta e_o - R\delta I_o$$

$$\delta e_o(1 + g_m R) = -R\delta I_o$$

so

$$R_o = -\frac{\delta e_o}{\delta I_o} = \frac{R}{1 + g_m R}$$

where R_o is the output impedance, and if $g_m R \gg 1$

$$R_o = \frac{1}{g_m}$$

For most amplifying valves (Ref. (5)) g_m will lie between 1 and 10 mA/volt, so that R_o will be between 1,000 and 100 Ω. The input impedance is extremely high, the only limitation being that the grid in most valves either collects or emits electrons, and a path to earth must be provided for this small current. Some ordinary valves (e.g. EF 37A) will operate successfully with 50–100 MΩ between grid and earth.

An Application of the Cathode-follower. *Reducing the Effect of Earth Admittances in an a.c. Bridge.* The Wagner earth, which is fitted to precision a.c. bridges at power frequencies, and which is essential at higher frequencies, has to be adjusted continually as balance is approached. Each change of the Wagner earth controls alters the balance point of the bridge, and the necessary procedure is often slow and laborious.

G. H. Rayner and R. W. Willmer (Ref. (9)) describe a method for increasing the rapidity with which balance can be reached by reducing the effect of the Wagner earth setting upon the bridge setting (see Fig. 537).

The shield surrounding the critical parts of the bridge (C, D, etc.) is driven by a cathode-follower through the capacitor C_3. The grid of the cathode-follower is connected through C_1 to point D, and if the a.c. output of the cathode follower were exactly equal to the input there would be no resultant a.c. potential between D and the screen. In fact there will be a slight a.c. voltage between them since, as we have seen, e_o does not exactly equal δe_i, but is slightly less,

$$\delta e_o = \frac{\delta e_i\, g_m R}{1 + g_m R}$$

There will also be a p.d. between C and the screen, apart from the above effect, if the bridge is not balanced. The authors state that the cathode-follower reduces the effect of a given unbalance of the Wagner arms Y_1, Y_2 to about 1/50th of what it would be with the screen earthed.

Several points in the circuit deserve notice. C_1 and R_1 form a convenient means of applying the a.c. signal to the grid while keeping the mean grid potential at a negative value determined by R_2 and R_3. R_1 is not returned to the negative supply, but to a point where the a.c. voltage is nearly equal to that of the cathode, and hence that of the grid. This keeps the a.c. input impedance high. (C_2 ensures that the a.c. voltage across R_2 is very small.) The screen-grid of the valve is similarly connected by C_4 to the cathode, and the effect of capacitance between screen-grid and control-grid is thus reduced. Finally, the grid leads of the cathode-follower are enclosed in the driven screen. The effect of all these precautions is that the shunting effect of the cathode-follower connected to D is only 50 MΩ and 1 $\mu\mu$F in parallel. Since D is at earth potential when the bridge is balanced, even this small shunting effect does not affect the balance point.

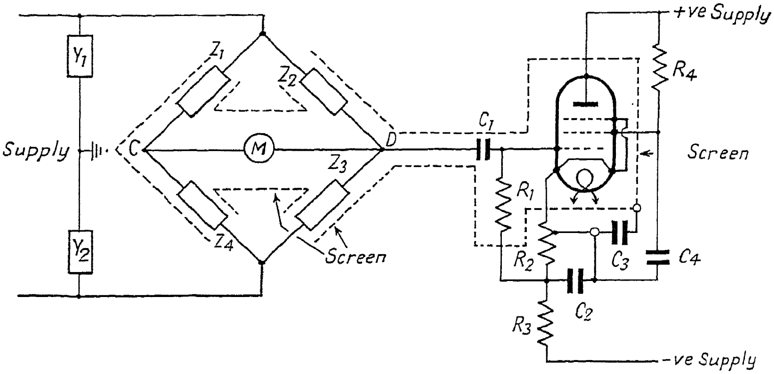


Fig. 537. Cathode-follower Driving the Screen of a Bridge

The other parts of the bridge and cathode-follower, not enclosed in the screen shown, are enclosed in an earthed screen which is omitted from the diagram.

The Differential Cathode-follower. The cathode-follower described on page 864 is essentially a device for providing a voltage output. Often it is required to operate a meter from a signal which has insufficient power to drive it directly. In this case a current output is generally required. The cathode-follower can be used, but there is a standing current when the input voltage is zero, and this is often inconvenient. The standing current can be backed-off with a separate battery but a neater solution is to use a differential cathode-follower as in Fig. 538. V_1 and V_2 can conveniently be the two halves of a double-triode containing two triodes in a single envelope. The operation of the circuit will be self-evident. R_5 sets the zero on M, while R_2 sets the sensitivity. The adjustments are independent. The circuit is a convenient and stable device for using an ordinary moving-coil meter to measure voltages at a very high impedance level—it is in fact a form of valve-voltmeter.

The Auto-repeater. The auto-repeater (Everett Edgecumbe, see

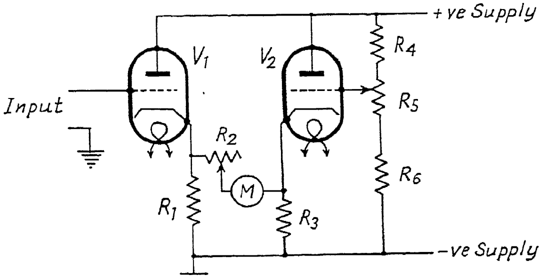


FIG. 538. CATHODE-FOLLOWER

Fig. 539) is a device for obtaining a direct current proportional to the power in an a.c. circuit. It provides an illustration of feed-back in a servo-mechanism using a d.c. amplifier.

The instrument consists of a standard three-element a.c. watt-meter movement mounted on the same shaft with a moving-coil d.c. milliammeter movement. The common shaft carries a small mirror which, as it turns, reflects the light from a lamp on to the emission photo-cell V_1.

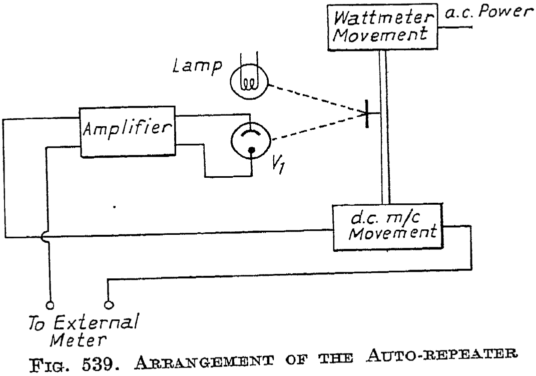


FIG. 539. ARRANGEMENT OF THE AUTO-REPEATER

The amplifier gives an output current which increases as the illumination of V_1 increases. The current from the amplifier passes through the moving-coil movement, producing a turning moment which resists that of the wattmeter movement, and tends to reduce the illumination of V_1.

For any given a.c. power, the instrument will come to rest when the moving-coil movement produces an equal and opposite torque to that of the wattmeter movement. The latter torque is proportional to the power, while the former is proportional to the current flowing through the moving coil. It follows that this current is proportional to the a.c. power, and it is passed through one or more external milliammeters which thus indicate or record the power.

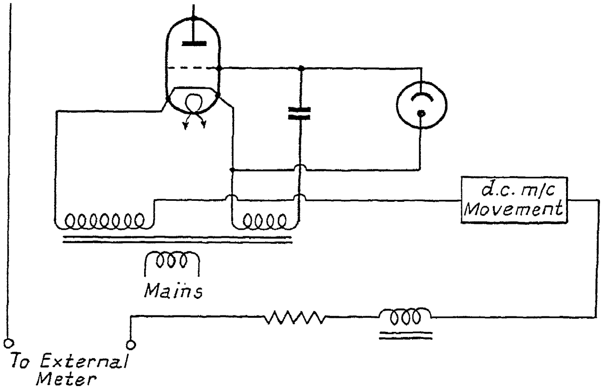


FIG. 540. AMPLIFIER IN THE AUTO-REPEATER

So long as the gain of the amplifier is large, as we have seen (see page 857), changes in the gain (i.e. amplification) have little effect on the accuracy of the instrument.

The auto-repeater enables the indicator or recorder to be placed in a distant position since only two wires need to be taken to it, and the resistance of these wires does not affect the accuracy of the indication. Also, since the movement of the shaft carrying the mirror is very small the response of the instrument is relatively fast, and it can be used to record rapid fluctuations of power.

The amplifier used in the instrument is a Tinsley d.c. amplifier using a thyratron (Fig. 540). When the photo-electric cell is not illuminated, the alternating voltage applied to the grid of the thyratron is in such a phase that it prevents the valve from conducting. Illumination of the cell changes the voltage of the grid, and the thyratron begins to conduct at some point in the a.c. cycle. The amount of illumination thus controls the proportion of each cycle during which the thyratron conducts, and so varies the mean value of the output current.

A Counting-rate Meter. Another example of the use of feed-back is provided by the A.E.R.E. counting-rate meter (Ref. (7)). The problem is to measure the average rate of pulses occurring at random—the instrument is in effect a frequency meter for measuring the mean value of a rapidly-fluctuating frequency.

The fundamental circuit which accomplishes this is shown in Fig. 541 (*a*). Each time a pulse is received it is arranged that the voltage applied to the input is changed by a constant amount (Fig. 541 (*b*)). The means by which this is done is explained later (*see* "Electronic Counters"). As the input goes positive, V_1 conducts and C_f is charged to the voltage V. As the input goes negative, V_2 conducts, and the charge on C_f is fed into C_t. It is readily shown that, if there are on the average $2n$ pulses per second, the mean voltage across R is

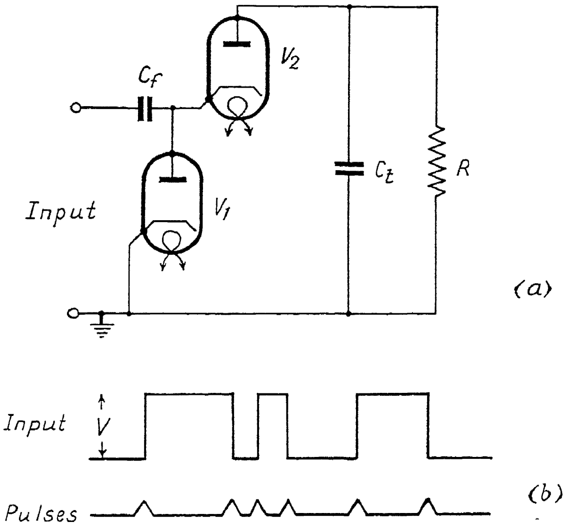


FIG. 541. ELEMENTARY CIRCUIT TO DETERMINE PULSE RATE

$$v = \frac{VnC_fR}{1 + nC_fR}$$

The quantity nC_fR in the denominator is due to the back-voltage developed across R. We require v to be proportional to n, and with the simple circuit shown this requires C_fR to be small. Unless V is very large, v itself is then small. The difficulty is due to the voltage developed across R, which reduces the charge fed into C_t. The back-voltage is avoided by making C_t and R the feed-back elements of a d.c. amplifier as in Fig. 542.

We have $E = -ae$

$$v = -E + e = (1 + a)e$$

so that $e = \dfrac{v}{1 + a}$

Thus the back-voltage is reduced in the ratio $\dfrac{1}{1 + a} : 1$, and the charge fed into C_t is therefore nearly independent of v, so that

$$v = VnC_fR \text{ very nearly.}$$

The actual circuit is shown in a simplified form in Fig. 543. The amplifier consists of a pentode to give a high amplification of the voltage, followed by a cathode-follower to provide the necessary current.

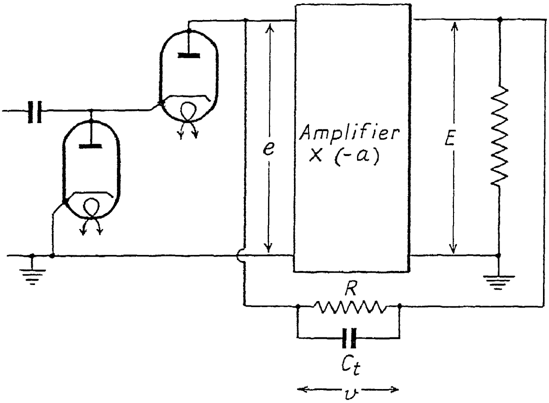


FIG. 542. CIRCUIT FOR PULSE-RATE DETERMINATION USING FEED-BACK AMPLIFIER

Evershed and Vignoles Quick-Response Recorder. This is an example of a servo-driven d.c. voltage recorder using a valve-amplifier as in Fig. 544. R_1 and R_2 are each 500,000 Ω, and the voltage at the control grid of V_1 is hence the mean of the input voltage and the voltage obtained from the potential divider R_{10}. The latter is a finely-wound wire potential divider supplied with a reference voltage, and the slider is fixed to the pen-arm of the recorder. V_1 is a pentode amplifier direct-coupled through R_7 and R_8 to V_2. R_3 provides some negative feed-back to V_1, so stabilizing its gain, and also ensures that the cathode is positive with respect to the grid. In the same way R_9 has a voltage across it, and it is not necessary to return R_8 to a negative voltage in order to make the grid of V_2 negative

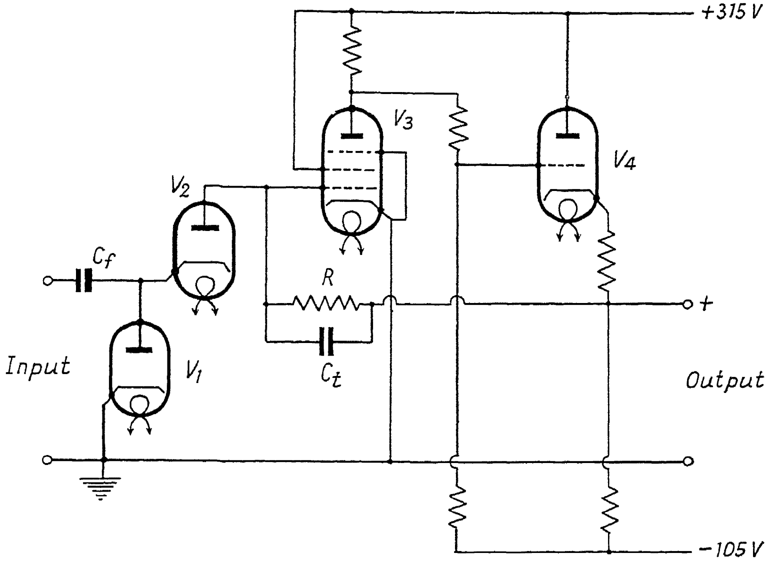


FIG. 543. SIMPLIFIED CIRCUIT OF PULSE-RATE METER

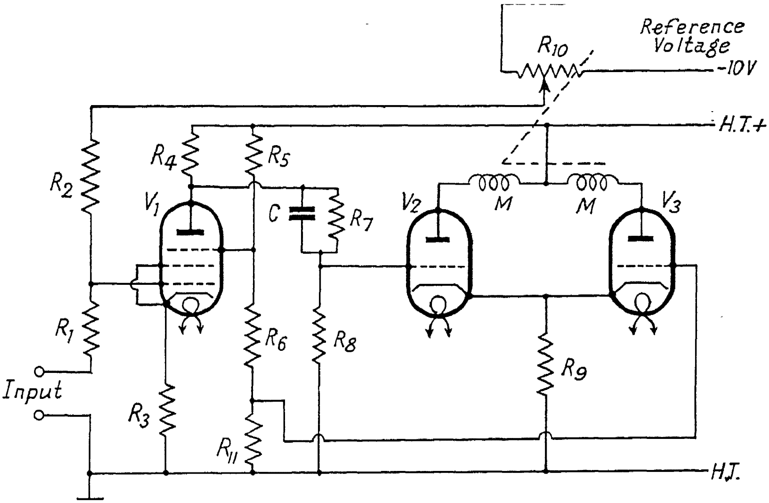


FIG. 544. SIMPLIFIED CIRCUIT OF QUICK-RESPONSE RECORDER

with respect to its cathode. V_2 and V_3 have a common grid resistor R_9. They are connected as a "long-tailed pair," their anodes being fed through the two halves M, M of the centre-tapped moving coil of the recorder.

When the voltages fed to R_1 and R_2 are equal and opposite, the grid of V_1 is at earth potential. Under these conditions V_2 and V_3 are arranged to pass equal currents, and these flowing through the two halves of the movement produce equal and opposite torques. The pen therefore remains at rest. If now the input to R_1 is made more positive the grid of V_1 becomes positive with respect to earth, and the voltage of its anode falls. The voltage at the grid of V_2 therefore falls, and V_2 conducts less current. The voltage drop across R_9 consequently falls, and the cathodes of V_2 and V_3 both become more negative. Since the grid of V_3 is fed from a nearly constant voltage, V_3 will conduct more current. The sum of the currents through V_2 and V_3 remains nearly constant, but more is now flowing through V_3 than V_2. The torques produced by the two halves of the moving coil are now unequal, and the pen begins to move. As it does so it changes the voltage fed from R_{10} to R_2, making it more negative. The voltage fed to the grid of V_1 falls, and equilibrium is again reached when the voltage picked up from R_{10} is equal and opposite to the input voltage. R_{10} is a linear potential divider, and it follows that the movement of the pen is directly proportional to the input voltage. Also, the accuracy of the recorder depends only on the accuracy of the reference voltage and of R_{10}, and not on the characteristics of the amplifier.

The pen-arm of the recorder has inertia, and the servo-system would be unstable unless arrangements were made to stabilize it. This is done by adding the capacitor C. The action of C will be clear from page 860, though it will be noticed that it is not in the same part of the servo-loop as that previously described. A further capacitor in parallel with R_2 is found in practice to help in preventing the pen from overshooting.

It will be noticed that this recording voltmeter is in fact a self-balancing potentiometer; R_{10} is the potentiometer, and the voltage from it is automatically kept equal and opposite to that at the input terminals.

"Quality Factor" Q and its Measurement. Consider the circuit shown in Fig. 545, in which R and L represent the resistance and inductance of a coil. Then, for the total parallel impedance Z we have,

$$\frac{1}{Z} = j\omega C + \frac{1}{R + j\omega L} = \frac{j\omega C(R + j\omega L) + 1}{R + j\omega L}$$

$$Z = \frac{R + j\omega L}{(1 - \omega^2 LC) + j\omega CR}$$

$$= \frac{(R + j\omega L)(1 - \omega^2 LC - j\omega CR)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

$$= \frac{R + j(\omega L - \omega^3 L^2 C - \omega C R^2)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

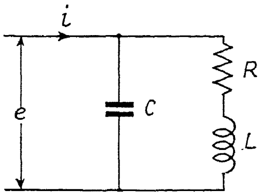


FIG. 545

If we put $\omega_0{}^2 = \frac{1}{LC}$, $Q = \frac{\omega L}{R}$, $\omega = h\omega_0$

and substitute $C = \frac{1}{\omega_0{}^2 L}$, $R = \frac{\omega L}{Q} = \frac{h\omega_0 L}{Q}$

we obtain

$$Z = Q\omega L\left[\frac{1 + j\left(Q - h^2 Q - \frac{h^2}{Q}\right)}{Q^2(1 - h^2)^2 + h^4}\right]$$

Thus Z can be expressed in terms of ωL—the reactance of L—and of Q and h. Q is termed the "quality-factor" of the coil. The resistance R of the coil varies with frequency owing to skin effect, dielectric losses, etc., and at high frequencies Q is often more constant with frequency than R. It is also generally more useful to know Q than R.

When the total reactance of the circuit is zero

$$Q - h^2\left(Q + \frac{1}{Q}\right) = 0$$

$$h^2 = \frac{Q}{Q + \frac{1}{Q}} = 1 \text{ very nearly if } Q \text{ is large.}$$

The impedance of the circuit when $h = 1$ is

$$Q\omega_0 L \frac{\sqrt{1 + \frac{1}{Q^2}}}{1} = Q\omega_0 L$$

very nearly, when Q is large. This, also, when Q is large, is very nearly the maximum value of the impedance Z (expressed as an actual value and not in symbolic j notation), and the value when the reactance is zero.

Suppose that h differs from unity by an amount α. Then

$$h = 1 \pm \alpha$$

$$Z = Q\omega L \frac{\sqrt{1 + Q^2\left(1 - h^2 - \frac{h^2}{Q^2}\right)^2}}{Q^2(1 - h^2)^2 + h^4}$$

$$\fallingdotseq Q\omega_0 Lh \frac{\sqrt{1 + Q^2(1 - h^2)^2}}{Q^2(1 - h^2)^2 + h^4}$$

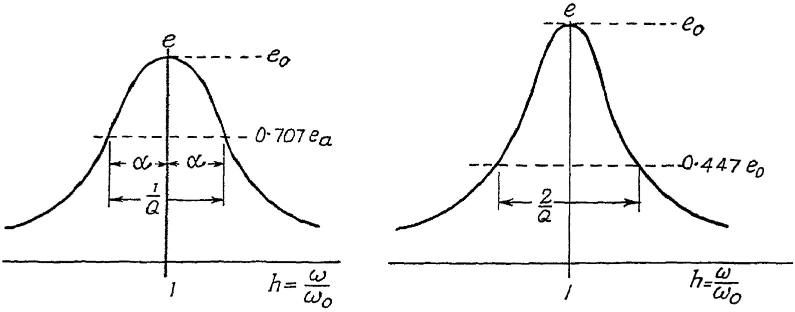
and $1 - h^2 = 1 - (1 \pm \alpha)^2 = \pm 2\alpha \ -\alpha^2 = \pm 2\alpha$ if α is small

$$\text{so } Z = Q\omega_0 L \frac{\sqrt{1 + 4\ \alpha\ ^2Q^2}}{4\alpha^2Q^2 + 1} = \frac{Q\omega_0 L}{\sqrt{1 + 4\alpha^2Q^2}}$$

$$\text{If } \alpha Q = \pm \tfrac{1}{2}, \qquad Z = \frac{Q\omega_0 h}{\sqrt{2}}$$

$$\text{and if } \alpha Q = \pm 1, \qquad Z = \frac{Q\omega_0 h}{\sqrt{5}}$$

Thus if h is altered by an amount $\pm \frac{1}{2Q}$ from resonance, the magnitude of the current remaining constant, then the magnitude of the voltage across the circuit falls to 0·707 of its value at resonance. If h is altered by $\pm \frac{1}{Q}$, the voltage falls to 0·447 of its value at resonance.



(a) (b)

FIG. 546. RESONANCE CURVES FOR A TUNED CIRCUIT

Three methods are now available for obtaining Q—

(a) We may make $h = 1$, and measure e and i. Then $\frac{e}{i} = Q\omega_0 L$, and, knowing ω_0 and L, we can calculate Q. This method is not very accurate, since e and i are difficult to measure at high frequencies.

(b) We may keep i constant, and vary h, noting the maximum value of e. We then change h by such an amount that e falls to 0·707 of its maximum value. We repeat the reading on the other side of resonance (α negative) (see Fig. 546 (a)). Then $\frac{1}{2\alpha} = Q$.

The variation of h can be achieved by varying the frequency of i, or by changing ω_0 by altering C. There are thus two variants of this method.

(c) In the same way we may reduce e to 0·447 e_0 (see Fig. 546 (b)).

There are again two ways of altering h—by varying ω or ω_0.

Methods (*b*) and (*c*) do not require measurements of the actual values of i and e; the current i has merely to be kept constant, and only a ratio of voltages ($e_0 = 0{\cdot}707e_a$, etc.) is required. These methods are consequently preferable to (*a*).

A similar analysis can be made of series resonant circuits. The analysis is simpler, and leads to similar results; it is therefore omitted.

The Q-Meter. The principle of a common device for measuring Q, the so-called "Q-meter" (Ref. (10)), is illustrated in Fig. 547. The current I is set to a standard value by the thermocouple meter M. It produces a standard voltage E, which is applied to the coil (R, L) and a variable capacitor C in series. The voltage across C is measured by a valve voltmeter. It is easily shown that the maximum voltage across C is QE, and the valve-voltmeter can be calibrated directly in values of Q. Alternatively more accurate methods corresponding to (*b*) and (*c*) on page 874 can be used.

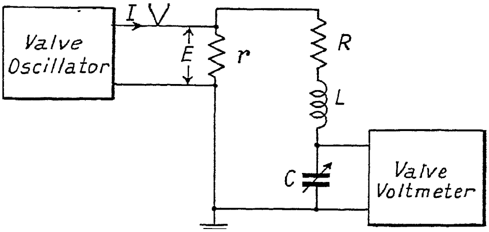


FIG. 547. DIAGRAM OF Q-METER

Many other circuit arrangements can be used for measuring Q. The possible arrangements, such as varying the oscillator frequency, or using parallel circuits, etc., will be obvious.

If C is calibrated in capacitance, the Q-meter forms a useful means for finding the effective inductance of a coil at radio-frequencies (which may not be the same as its inductance at lower frequencies). The frequency of the oscillator and the value of C at resonance are found, and $L = \dfrac{1}{\omega_0^2 C}$.

Electronic Counters (Ref. 18). Consider the circuit given in Fig. 548 (see Refs. (19) and (23)). This is the Eccles-Jordan Trigger Circuit.

Suppose V_1 is conducting. Then the voltage at its anode will be relatively low, and R_3R_6 form a potential divider such that the grid of V_2 is very negative—so negative that V_2 cannot conduct. The anode of V_2 is therefore at a high voltage, and R_4R_5 ensure that the grid of V_1 is at about earth potential, so that V_1 conducts. The arrangement is obviously stable for small disturbances.

If now a large positive pulse is applied through C_3 and C_4 to both grids, it will have no appreciable effect on V_1. V_1 is already conducting, and its grid cannot go positive, as any attempt to drive it positive will result in the flow of grid current. There will, however, be a considerable effect on V_2 which was cut off. The pulse will drive the grid positive until V_2 begins to conduct. The anode voltage of V_2 will then begin to fall, and this, through R_4R_5 reduces the voltage at the grid of V_1. This in turn increases the voltage at the anode of V_1 and hence at the grid of V_2. The process is cumulative, and it stops only when V_1 is cut off and V_2 conducting. C_1 and C_2 assist the process of changing-over.

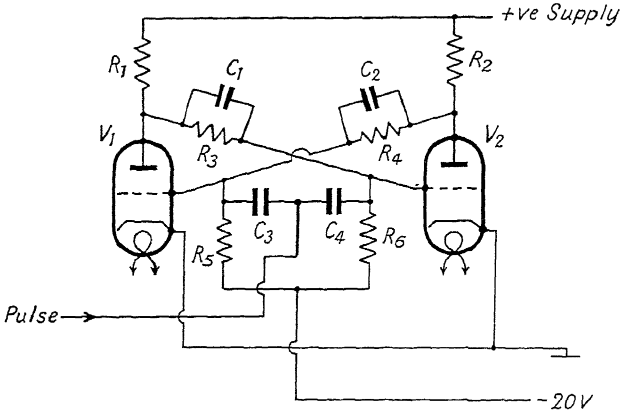


FIG. 548. TRIGGER CIRCUIT

When a second positive pulse is applied, a similar effect occurs, and V_1 again becomes conducting while V_2 is cut off. The voltages at the anodes of V_1 and V_2 are thus as shown in Fig. 549.

The lowest line shows the pulses which would be obtained by connecting a capacitor to the anode of V_2. It is easy to cut off the negative pulses, and it will be seen that there is one positive output pulse for every two input pulses.

If a number of such circuits are arranged one after the other, we can divide successively by two. If there is some device to show which of the two valves in each pair is conducting, we have a simple "scale-of-two counter" as in Fig. 550 (*a*). We count each stage as "0" if the left-hand valve is conducting, "1" if the right-hand valve is conducting. Suppose six pulses are fed in. The final result will be as in Fig. 550 (*b*). We count this (taking the stages from right to left) as 1, 1, and 0 so that the count is

$$1 \times 4 + 1 \times 2 + 0 \times 1 = 6$$

Most electronic calculators work on a scale-of-two using circuits similar to the one described. It is also possible to have electronic scale-of-ten counters, but they use more valves than the equivalent scale-of-two counter.

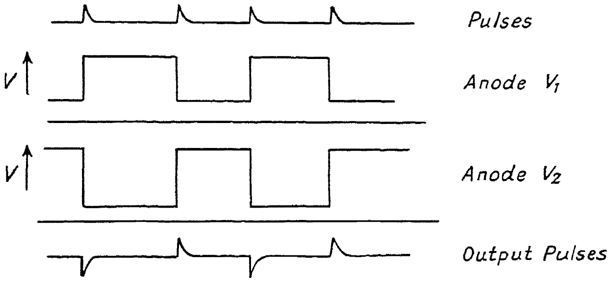


FIG. 549. VOLTAGES IN THE CIRCUIT OF FIG. 548

Application of Electronic Counters. (*a*) In the Cintel oscillator there is a built-in scale-of-ten counter. Each cycle from the oscillator produces one pulse, and the pulses are counted by the electronic counter for an accurately-determined time. The frequency is thus automatically presented. The operation is automatic—the counter counts for the standard time, presents the result, clears to zero, and counts again.

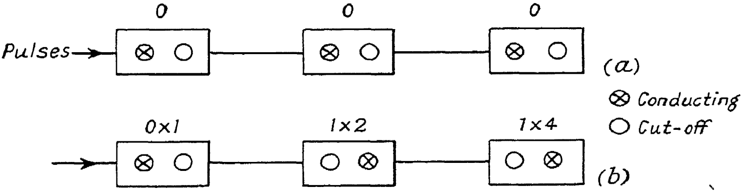


FIG. 550. PRINCIPLE OF SCALE-OF-TWO COUNTER

(*b*) A similar counter can be used to measure small time intervals. At the beginning of the interval a "gate" is opened, allowing pulses from an accurate pulse generator to be fed into the counter. At the end of the period the "gate" is closed. In the Cintel micro-second timer the pulse generator gives one million pulses per second. The interval can thus be measured in micro-seconds with an error of $\pm 1 \mu$ sec.

(*c*) As already mentioned, electronic counters form one element in electronic calculating machines (Ref. (18)).

BIBLIOGRAPHY AND REFERENCES

(1) *Electronics*, P. Parker, Edward Arnold, 1950.

(2) *Physical and Chemical Constants*, G. W. C. Kaye and T. H. Laby, Longmans Green & Co., 10th Edition, 1948.

(3) *Electronics*, Ed. Bernard Lovell, Pilot Press, 1947.

(4) *Principles of Servo-mechanisms*, G. S. Brown and D. P. Campbell, John Wiley, 1948.

(5) *Wireless World Handbook.*

(6) *High-frequency Measurements*, A. Hund, McGraw-Hill, 1951.

(7) "A Counting-Rate Meter of High Accuracy," E. H. Cooke-Yarborough and E. W. Pulseford, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, p. 191.

(8) *Theory of Servo-mechanisms*, M.I.T. Radn. Lab. Series, Vol. XXV, where filters in the feedback circuit are said to be unsatisfactory in general.

(9) "A Method of Decreasing the Effect of Earth Admittances in a.c. Bridges," G. H. Rayner and R. W. Willmer, *Jour. Sci. Inst.*, Vol. XXVII, 1950.

(10) *Radio-frequency Electrical Measurements*, H. A. Brown, McGraw-Hill, 1938.

(11) *Vacuum-tube Voltmeters*, J. F. Rider, New York, 1951.

(12) *Radio-frequency Measurements*, L. Hartshorn, Chapman and Hall, 1940.

(13) *Telemetering*, P. A. Bordue and G. M. Thynell, Reinhold, New York, 1948.

(14) "The Inductor Compass," A. Huie, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, August, 1951.

(15) "Mean-Wind-Velocity Instruments for Use on Naval Craft," J. Bell and E. M. Langham, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, August, 1951.

(16) "An Electrical Method of Resolving True Wind Velocity Aboard Ship," H. D. Hawkes and J. M. Lawes, *Proc. I.E.E.*, Pt. II, Vol. XCVIII, August, 1951.

(17) *Electricity Supply Meters*, A. E. B. Perrigo, Chapman and Hall, 1947.

(18) *High Speed Computing Devices*, Eng. Res. Ass. Inc., McGraw-Hill, 1950.

(19) *Radio Engineering*, F. E. Terman, McGraw-Hill, 1947.

(20) "An Automatic Frost-Point Hygrometer, etc.," A. W. Brewer and R. H. Dobson, *Proc. I.E.E.*, Vol. XCVIII, Pt. II, August, 1951.

(21) "Industrial Applications of Electronic Techniques," H. A. Thomas, *Jour. I.E.E.*, Vol. XCIV, Pt. I, 1947.

(22) *Time Bases*, O. S. Puckle, Chapman and Hall, 1951.

EXAMINATION QUESTIONS

(1) Show that the energy stored in a condenser is given by the expression $\frac{1}{2}CV^2$.

An air condenser of capacitance 0·005 microfarad is connected to a direct voltage of 500 volts, disconnected, and then immersed in oil with a dielectric constant of 2·5. Find the energy stored in the condenser before and after immersion, and account for the difference. (*Lond. Univ.*, 1930, *Elec. Tech.*)

Ans. Before immersion 625×10^{-6} Joules; after immersion 250×10^{-6} Joules.

(2) Show that near a charged body in air or vacuum the field strength is $4\pi\sigma$, σ being the surface density of the charge.

In a parallel plate condenser with solid dielectric the plates are a tenth of a millimetre apart, and when charged the surface density is 5×10^{-10} coulombs per square centimetre. What is the electric flux density? If the specific inductive capacity is 2·2, what is the voltage between the plates? [1 coulomb $= 3 \times 10^9$ electrostatic units.] (*Lond. Univ.*, 1925, *Elec. Tech.*)

Ans. 6π lines per sq. cm.; 25·7 volts.

(3) Calculate the strength of the magnetic field at the centre of a single-turn square coil of 1 ft. side when carrying a current of 20 amp. Prove the formula employed. (*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 0·741 dynes.

(4) Establish a formula for the energy stored per cubic centimetre in a magnetic field of given strength and deduce therefrom a formula for the lifting force of an electromagnet.

A ring of 1 in. diameter round iron has a mean diameter of 1 ft.; it is wound with 1,000 turns of wire carrying a current of 3 amp.; it is split across a diameter and an air gap of 0·01 in. is formed on each side. Calculate the approximate pull required to separate the two parts. The iron may be assumed to have a permeability of 500. (*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 220 lb. wt.

(5) A solenoid is 2 ft. long and 1 in. diameter, and is uniformly wound with 600 turns of insulated wire. Calculate the strength of the magnetic field at the centre when the current is 2 amp. If a secondary coil of 50 turns is wound round the central part of the solenoid and is connected to a ballistic galvanometer through a resistance which makes the total resistance of the circuit 10,000 ohms, calculate the quantity of electricity discharged through the galvanometer on reversing the current of 2 amp. in the primary winding. Estimate the error due to assuming the solenoid infinitely long.

(*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. 24·7 C.G.S. units.
$1{\cdot}25 \times 10^{-8}$ coulombs. Error 9 parts in 10,000.

(6) Draw diagrams showing the magnetic field around two parallel straight conductors carrying current (*a*) in the same direction, and (*b*) in opposite directions. If two conductors 3 in. apart each carry 1,000 amp. in the same direction, find the direction and magnitude per inch run of the force on each conductor. (*Lond. Univ.*, 1927, *Elec. Tech.*)

Ans. 6·78 gramme wt. attraction.

(7) A single circular turn of insulated wire, 30 cm. mean diameter, rests on a similar turn, the distance between the centres of the wires being 5 mm.

A current of 30 amp. flows through one and a current in the opposite direction of 50 amp. flows in the other. Find the direction and magnitude of the resultant force. (*Lond. Univ.*, 1924, *Elec. Tech.*)

Ans. 5,650 dynes.

(8) Two coils each of 20 cm. diameter and 100 closely wound turns of fine wire are mounted coaxially 10 cm. apart. A current of 1 amp. is passed through the coils in series, the connections being such that the fields are additive. Plot a curve showing the field strength along the axis of the coils. Prove any formula used. (*Lond. Univ.*, 1930, *Elec. Meas.*)

(9) Prove that in a specimen of iron the hysteresis loss in ergs per cubic centimetre per cycle is given by the area of the hysteresis loop divided by a constant.

The hysteresis loop for a specimen weighing 12 kg. is equivalent to 3,000 ergs per c.c. Find the loss of energy per hour at 50 cycles per sec. Density of iron, 7·5. (*C. and G. Final I*, 1930.)

Ans. 24 watt-hours.

(10) Define the unit charge of electricity in the electromagnetic and in the electrostatic systems.

Describe a direct method of measuring the ratio of the two charges. (*Lond. Univ.*, 1929, *Elec. Meas.*)

(11) What is meant by the dimensions of a quantity? Derive the dimensions of potential difference in the electrostatic system in terms of Mass, Length, and Time.

In the course of a calculation an expression of the following form was arrived at—

$$I = E\left\{\frac{1}{Z_1} + \frac{j\omega M}{Z_2}\left(\frac{1}{R} + \frac{C}{L}\right)\right\}$$

Show that there must have been an algebraical error, and point out the term or terms which require correction. (*Lond. Univ.*, 1926, *Elec. Meas.*)

Ans. The term $\frac{C}{L}$ should be multiplied by some quantity having the dimensions of resistance (or impedance).

(12) The voltage V across the coils of a telephone receiver when a current I is flowing through them may be written

$$V = I\left(Z + \frac{A^2}{z}\right)$$

where $Z = R + j\omega L;\quad A = 2B_0\frac{N}{\mathscr{R}}$

$$z = r + j\left(\omega m + \frac{s}{\omega}\right)$$

R = the resistance, L = the inductance, and
N = the number of turns of the receiver coils
B_0 = the flux density in the air gap
$\mathscr{R}$ = the reluctance of the magnetic circuit

and r, m, and s are the equivalent mechanical resistance, mass, and stiffness of the receiver diaphragm.

Suggest a suitable unit for each of the quantities involved.

Find the dimensions of each quantity, and so make a dimensional check of the equation. (*Lond. Univ.*, 1930, *Elec. Meas.*)

Ans. Dimensions—

V—$[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}]$
I—$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$
Z—$[LT^{-1}\mu]$
z—$[MT^{-1}]$
r—$[MT^{-1}]$
m—$[M]$
s—$[MT^{-2}]$
A—$[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}]$
B_o—$[L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}]$

$\mathscr{R}$—$[L^{-1}\ \mu^{-1}]$
R—$[LT^{-1}\mu]$
L—$[L\mu]$

(13) (i) The expression for the mean torque T of an electro-dynamic wattmeter may be written,

$$T \propto M^pE^qZ^t$$

where M = the mutual inductance between fixed and moving coils
E = applied voltage
Z = impedance of the load circuit

From the dimensions of quantities involved determine p, q, and t.

(ii) In the same way, determine a, b, c, and g in the following expression for the eddy current loss W per centimetre length of round wire—

$$W \propto f^a\, B^b_{max}\, d^c\, \rho^g$$

where f = frequency
B_{max} = maximum flux density
d = diameter of the wire
ρ = the specific resistance of the material

Ans. (i) $p = 1$, $q = 2$, $t = -2$; (ii) $a = 2$, $b = 2$, $c = 4$, $g = -1$.

(14) If the practical units of voltage and current were each made 100 times as large as they are at present, what would be the consequent alteration in the size of the units of resistance, capacity and inductance?

(*Lond. Univ.*, 1922, *Elec. Tech.*)

Ans. No alteration.

(15) Describe a method of determining the ohm absolutely and discuss the means taken to minimize the most probable causes of error.

(*Lond. Univ.*, 1926, *Elec. Meas.*)

(16) Give an account of the principles and construction of the current balance as used at the National Physical Laboratory for the absolute determination of the ampere.

What factors limit the accuracy and what is the accuracy actually realized?

(*Lond. Univ.*, 1925, *Elec. Meas.*)

(17) Three non-inductive resistances of 1,000 ohms are star-connected to a three-phase supply with 200 volts between lines. What will be the reading on a voltmeter connected between one of the lines and the star point thus formed, if the voltmeter also has a non-inductive resistance of 1,000 ohms?

(*Lond. Univ.*, 1926, *Elec. Meas.*)

Ans. 86·6 volts.

(18) An alternating current circuit contains a coil of inductance L_1; near this, but not connected to it, is a coil of inductance L_2, across which is connected a resistance R. If the mutual inductance between the two coils is M, calculate the effective inductance and resistance of the first circuit.

(*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. Res. $= \dfrac{\omega^2M^2R}{R^2 + \omega^2L_2{}^2}$, Ind. $= \left(L_1 - \dfrac{\omega^2M^2L_2}{R^2 + \omega^2L_2{}^2}\right)$.

(19) An alternating current passes through a non-inductive resistance R and an inductance L in series. Find the value of the non-inductive resistance which can be shunted across the inductance without altering the value of the main current. (*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. $\dfrac{\omega^2 L^2}{2R}$.

(20) The impedances of the three phases of a star-connected load (no neutral wire) are $5 + j20$, $12 + j0$, and $1 - j10$ in order. The line voltage is 400 volts. Find the line currents. (*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. $0{\cdot}5 - 29{\cdot}65j$, $16{\cdot}24 - 11{\cdot}5j$, $-16{\cdot}74 + 41{\cdot}15j$.

(21) Explain the method of representing a vector quantity by the "j" notation, and show how the method may be used in A.C. calculations. Two circuits whose impedances are given by $8 - 7j$ and $5 + 6j$ are connected in parallel across a 100 volt A.C. supply. Calculate the current passing through each circuit and the total current flowing to both of them. Find also the angle of phase difference between these currents and the applied P.D. (*Lond. Univ.*, 1923, *Elec. Tech.*)

Ans. 9·4 amp. 41° 11′ leading,
12·8 amp. 50° 12′ lagging, 15·7 amp. 13° 25′ lagging.

(22) Three branch circuits consisting respectively of (i) 5 ohms resistance and 0·025 henry inductance, (ii) 4 ohms resistance and a condenser of capacity 300 microfarads, (iii) an inductance of 0·01 henry and a condenser of 500 microfarads capacity, are connected in parallel. A resistance of 3 ohms is then connected in series with the combination. Using the symbolic "j" notation, calculate the currents in the main circuit, and in the three branches, when an alternating voltage $v = 100 \sin 314t$ is applied to the whole circuit.

Ans. $i = 20{\cdot}2 \sin (314t + 37^\circ\ 55')$,
$i_1 = 6{\cdot}9 \sin (314t - 93^\circ\ 20')$,
$i_2 = 5{\cdot}64 \sin (314t + 33^\circ\ 35')$,
$i_3 = 19{\cdot}9 \sin (314t + 54^\circ\ 10')$.

(23) Calculate the capacity of a spherical condenser if the diameter of the inner sphere is 20 cm., and that of the outer sphere is 30 cm., the space between them being filled with a liquid with a specific inductive capacity of 2. Express your answer in microfarads. (*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. $\frac{2}{3} \times 10^{-4}$ microfarads.

(24) A condenser is made up of two parallel metal discs separated by three layers of dielectric of equal thickness but having dielectric constants of 2, 3, and 4 respectively. If the metal discs are 6 in. diameter and the distance between them 0·3 in., calculate the potential gradient in each dielectric and the total energy stored in each when a potential difference of 1,000 volts is applied between the discs. (*Lond. Univ.*, 1927, *Elec. Tech.*)

Ans. 1,815, 1,210, 907·5 volts per cm.; $13{\cdot}51 \times 10^{-6}$, $8{\cdot}98 \times 10^{-6}$, $6{\cdot}755 \times 10^{-6}$ Joules.

(25) Explain the following terms as applied to dielectrics subjected to alternating electric fields: Permittivity (K), loss angle (δ), volume resistivity, surface resistivity, dielectric hysteresis.

Prove that the power loss per cubic centimetre can be expressed by the formula

$$55{\cdot}5K \,.\, \tan \delta \,.\, f \,.\, g^2 \,.\, 10^{-6} \text{ watts}$$

where f is the frequency and g the potential gradient. In what units is g expressed in this formula. (*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. Kilovolts per millimetre.

(26) Calculate the capacity of the following system—

Two conductors $\frac{3}{8}$ in. diameter, and 400 yd. long, lying parallel to each other and to an earthed plane, their height above the plane being 3 ft. The conductors are 3 ft. apart.

Ans. 1,980 micro-microfarads.

(27) Capacity measurements on a three-phase cable were made as follows—

Capacity per mile between the sheath and one conductor connected together and the other two conductors connected together = 0·45 microfarad.

Capacity per mile between the three conductors connected together and the sheath = 0·50 microfarad.

The potential between the conductors is 3,300 volts and the frequency is 50 cycles per second.

Find the charging current flowing to each conductor in a 20-mile length of the cable. (*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. 4·1 amp.

(28) Describe the use of the Schering bridge for the measurement of the dielectric loss in a length of high-voltage cable.

State all the precautions necessary for both accuracy and safety. (*Lond. Univ.*, 1931, *Elec. Meas.*)

(29) Explain how the electrostatic wattmeter or quadrant electrometer may be used for the measurement of the power lost in dielectrics at high voltages.

Prove the formula, giving the power in terms of the deflection. (*Lond. Univ.*, 1925, *Elec. Meas.*)

(30) Define electric field strength and state the practical units in which it is measured.

A concentric cable has a conductor diameter of 1 cm., and an insulation thickness of 1·5 cm. Find the maximum field strength when the cable is subjected to a test pressure of 33 kV. (*Lond. Univ.*, 1931, *Elec. Tech.*)

Ans. 47·6 kV per cm.

(31) Annular shaped iron stampings are built up to form a ring 8 in. inside diameter and 10 in. outside diameter and 1 in. thick. The square cross-section is wound with 1,000 turns to form a toroidal coil. If the resistance of the wire is 200 ohms, and the permeability of the iron is 800, what current will flow under an impressed alternating voltage of 100 volts, 50 cycles?

What would be the effect of replacing the stampings by a solid iron core? (*Lond. Univ.*, 1929, *Elec. Tech.*)

Ans. 0·288 amp.

(32) A ring 1 ft. mean diameter is made of round iron 1 in. diameter, and is uniformly wound with 500 turns of copper wire 0·05 in. diameter. A second winding of 1,000 turns of wire 0·025 in. diameter is uniformly wound over the first. Assuming the iron to have a permeability of 800, calculate the self-inductance of each winding and the mutual inductance between them; express your answers in practical units. (*Lond. Univ.*, 1927, *Elec. Tech.*)

Ans. 0·133, 0·532, 0·266 henries.

(33) Calculate the resistance and inductance of a single layer cylindrical coil consisting of 1,000 turns of copper wire 0·5 mm. diameter, wound uniformly on a cardboard tube 2 in. diameter and 30 in. long, the turns being spaced so that the coil occupies the whole length of the tube. (*Lond. Univ.*, 1925, *Elec. Tech.*)

Ans. 13·8 ohms, 3·42 millihenries.

(34) A toroidal coil is uniformly wound with 250 turns. The core is of marble with a rectangular cross-section $2\frac{1}{2}$ in. deep and 2 in. wide; the outer diameter of the toroid is 6 in., and the inner diameter is 2 in.

Find the inductance of the toroid, proving any formula used. (*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. 0·873 millihenries.

(35) Estimate the mutual inductance between two parallel and coaxial circles of 40 cm. and 5 cm. diameter respectively, and 20 cm. apart, making the assumption that a current in the larger circle produces a uniform field through the smaller.

How does the true mutual inductance differ from the value estimated in this manner, and how would it be calculated? (*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. 0·00218 microhenries.

(36) Explain why conductors have a higher resistance to alternating than to direct currents. Do the same causes affect the inductance of the conductor; if so, in what way? How can one construct conductors to minimize the variation of resistance with frequency? (*Lond. Univ.*, 1926, *Elec. Tech.*)

(37) Define the practical unit of mutual inductance.

Two coils with terminals T_1T_2 and T_3T_4 respectively are placed side by side. Measured separately, the inductance of the first coil is 1,200 microhenries, and that of the second coil is 800 microhenries.

With T_2 joined to T_3 the inductance between T_1 and T_4 is 2,500 microhenries. What is the mutual inductance between the two coils, and what would be the inductance between T_1 and T_3 with T_2 joined to T_4?

Prove any formula used. (*Lond. Univ.*, 1930, *Elec. Tech.*)

Ans. $M = 250$ microhenries, $L = 1{,}500$ microhenries.

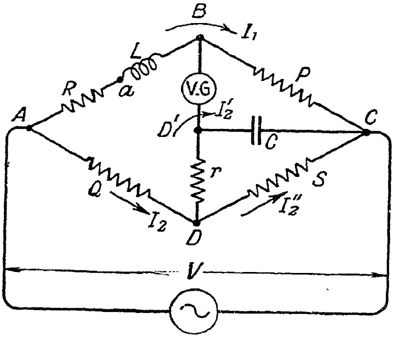
(38) An alternating current bridge is arranged as follows: The arms *AB* and *BC* consist of non-inductive resistances of 100 ohms, the arms *BE* and *CD* of non-inductive variable resistances, the arm *EC* of a condenser of 1 microfarad capacity, the arm *DA* of an inductive resistance. The alternating current source is connected to *A* and *C*, and the telephone receiver to *E* and *D*. A balance is obtained when the resistances of the arms *CD* and *BE* are 50 and 2,500 ohms respectively.

Calculate the resistance and inductance of the arm *DA*.

What would be the effect of harmonics in the wave-form of the alternating current source? (*Lond. Univ.*, 1925, *Elec. Meas.*)

Ans. 50 ohms, 0·255 henry.

(39) The diagram gives the connection of Anderson's bridge for measuring the inductance *L* and resistance *R* of an unknown impedance between the points *A* and *B*. Find *R* and *L* if balance is obtained when $Q = S = 1{,}000$ ohms, $P = 500$ ohms, $r = 200$ ohms, and $C = 2\mu F$.



Draw a vector diagram showing the voltage and current at every point of the network when the voltage across *AC* is 10 volts and the frequency is 100 cycles per second. (*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. $R = 500$ ohms, $L = 1{\cdot}4$ henries.

(40) One of the branches of a Wheatstone bridge consists of a variable inductance coil with a variable resistance, in series with a condenser shunted by a resistance. The other branches consist of non-inductive resistances. The bridge is supplied with alternating currents of known frequency. Calculate the relations which must hold between the inductance and capacity, and between the various resistances in order to obtain balance.
(*Lond. Univ.*, 1922, *Elec. Meas.*)

Ans. $R_1R_4 =$ and $L = \quad Cr^2$; where r is the resistance shunting C.

(41) When a current of 0·0001 amp. is passed through a certain ballistic galvanometer it gives a steady deflection of 50 scale divisions. A condenser is charged at a pressure of 100 volts, and discharged through the galvanometer, and gives a throw of 220 scale divisions. The complete period of swing is 4 seconds. Find the capacity of the condenser.
(*C. and G. Final*, 1918, *Elect. Eng., I.*)

Ans. 2·8 microfarads.

(42) Explain how an inductance may be measured by comparison with a standard condenser in an alternating-current bridge.

The four arms of a Wheatstone bridge arrangement are as follows: AB is an inductive resistance Lr_1, BC is a non-inductive resistance r_3, CD is a condenser of capacitance C shunted by a resistance r_4, DA is a non-inductive resistance r_2. An alternating-current supply is connected between the points A and C and a telephone receiver across the points B and D. Work out the conditions for balance and show that the result is independent of the frequency of the supply. (*C. and G. Final*, 1930, *Elect. Eng., II.*)

Ans. $r_1r_4 = r_2r_3 = \dfrac{L}{C}$.

(43) The four arms of a Wheatstone bridge have the following resistances: AB 100, BC 10, CD 4, DA 50 ohms.

A galvanometer of 20 ohms resistance is connected across BD. Calculate the current through the galvanometer when a potential difference of 10 volts is maintained across AC. (*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 0·00513 amp.

(44) Explain carefully how the conductor resistance and insulation resistance of a mile of an electric light cable would be determined experimentally by measurements on a 100-yd. sample. (*Lond. Univ.*, 1929, *Elec. Meas.*)

(45) The conductor of a single-core cable has a diameter of 0·25 in., and the diameter over the insulation is 1 in. If the insulation resistance of the cable is 10,000 ohms per mile, calculate the specific resistance of the dielectric.
(*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 2,870 megohms per inch cube.

(46) Two mains are working at a P.D. of 220 volts. A voltmeter of 4,500 ohms resistance when connected between the positive main and earth reads 148 volts, but when connected between the negative main and earth the reading is only 42 volts. Calculate the insulation resistance to earth of each main. (*Lon. Univ.*, 1926, *Elec. Tech.*)

Ans. Positive to earth 3,220 ohms, negative to earth 910 ohms.

(47) Describe with a diagram of connections the loss of charge method of determining the insulation resistance of a length of cable. Prove the formula used for this determination, and calculate the insulation resistance of a short length of cable in which the voltage falls from 100 to 80 in 20 sec., the capacity being 0·0003 microfarads. (*Lond. Univ.*, 1925, *Elec. Tech.*)

Ans. 298,000 megohms.

(48) Describe the Kelvin double bridge for the comparison of small resistances.

Give the theory of the bridge, and detail the arrangements necessary in order that the greatest precision possible may be obtained.
(*Lond. Univ.*, 1931, *Elec. Meas.*)

(49) Determine the currents to the nearest milliampere in each of the five sections of the following network, when one volt is applied between the points *a* and *d*; *ab* 2 ohms, *bd* 8 ohms, *dc* 6 ohms, *ca* 4 ohms, and *cb* 10 ohms.
(*Lond. Univ.*, 1923, *Elec. Tech.*)

Ans.

ab....	111	milliamps
bd....	97	,,
dc....	−105	,,
ca ...	−91	,,
cb....	−14	,,

(50) Describe in detail how a sub-standard direct current voltmeter of range from 0–250 volts, would be calibrated by the potentiometer method.
(*Lond. Univ.*, 1929, *Elec. Meas.*)

(51) Describe some form of accurate direct reading potentiometer. Indicate how the instrument is standardized and point out any special features in its construction. Explain carefully how to connect up and use a potentiometer to carry out the following tests: (i) the calibration of a moving-coil voltmeter reading up to 150 volts, (ii) the measurement of the resistance of a 0·01 ohm shunt. In each case state what auxiliary apparatus is required.
(*Lond. Univ.*, 1923, *Elec. Meas.*)

(52) Describe the principle, construction, and operation of an alternating current potentiometer. (*Lond. Univ.*, 1927, *Elec. Meas.*)

(53) In the measurement of a low resistance by means of a potentiometer the following readings were obtained—

Volt drop across low resistance under test	0·83942 volt
Volt drop across a 0·1 ohm standard resistance connected in series with the "unknown"	1·01575 volt

The resistance of the standard at the temperature of the test is 0·10014 ohm. Upon setting the potentiometer dials to zero and breaking the current passing through the "unknown" resistance, the thermal E.M.F. of the latter produced a galvanometer deflection equivalent to 23 microvolts, the direction of the deflection being the same as that produced by an increase of the potentiometer reading during the volt drop measurements.

Calculate the resistance of the "unknown."

Ans. 0·08276$_0$ ohm.

(54) State the theory underlying the action of the Chattock magnetic potentiometer. Describe the instrument, and show how it can be calibrated and used in conjunction with a ballistic galvanometer for the measurement of difference of magnetic potential. The constant of a given potentiometer was obtained by aid of a coil of 300 turns in which a current of 0·6 amp. was reversed. The resulting throw of the galvanometer was 157 scale divisions. It was then used to measure the magnetic potential difference between two points and the throw was 304 scale divisions. Find the magnetic potential difference, and state the units in which it is measured.
(*Lond. Univ.*, 1925, *Elec. Meas.*)

Ans. 876 C.G.S. units of magnetic potential.

(55) The demagnetization curve for a sample of permanent magnet steel after hardening and ageing is as follows—

Permanent flux density in lines per sq. cm.	6,500	5,900	5,200	4,300	3,100	1,400
Demagnetizing ampere turns per centimetre	4	12	20	28	36	44

The air gap flux density in a moving coil instrument where this steel is used is to be 900 lines per sq. cm., the length of the single gap is to be 0·12 cm., and the area of the gap 10 sq. cm. To ensure the necessary permanence, the ratio of the area of the gap divided by its length to the area of cross-section of the magnet divided by its length is to be 300. Assuming the leakage flux to be equal to the useful flux, and regarding all the leakage as being concentrated at the pole shoes, calculate the necessary length and cross-sectional area of the magnet.

To what extent is the above-mentioned ratio applicable to all permanent magnet steels? (*Lond. Univ.*, 1925, *Elec. Meas.*)

Ans. 11·5 cm., 3·2 sq. cm.

(56) Describe a method for finding the B-H curve of bar specimens.

An iron ring of 3·5 sq. cm. cross-sectional area with a mean length of 100 cm. is wound with a magnetizing winding of 100 turns. A secondary coil with 200 turns of wire is connected to a ballistic galvanometer having a constant of 1 microcoulomb per scale division, the total resistance of the secondary circuit being 2,000 ohms. On reversing a current of 10 amp. in the magnetizing coil, the galvanometer gave a throw of 100 scale divisions. Calculate the flux density in the specimen and the value of the permeability at this flux density. (*C. and G. Final*, 1927.)

Ans. $B = 14{,}280$, $\mu = 1{,}136$.

(57) Describe a standard form of apparatus for measuring the iron losses in steel sheets at specified values of flux density and frequency by means of a wattmeter.

The following test results were obtained on a sample of steel stampings at 50 frequency—

Volts	Amperes	Watts
4·9	0·20	9·5
69·3	0·30	16·8
91·8	0·46	27·5
100·5	0·52	32·5
110·5	0·64	39·0
118·0	0·77	44·8

Mean width of the plates, 3 cm.; mean thickness, 0·0489 cm.; number of plates, 51; total weight, 24·2 lb.; number of magnetizing turns in coil, 600. Allowing 2 watts copper loss in the magnetizing winding, calculate the iron loss in watts per pound at a flux density of 10,000 lines per sq. cm., and 50 frequency. (*C. and G. Final*, 1927.)

Ans. 1·18.

(58) In a test with a traction permeameter, the force required to separate the two U-shaped portions of iron forming the magnetic circuit is 15 lb. The cross-sectional area of the iron is 2 sq. cm., and the length of magnetic path

45 cm. The magnetizing coil has 250 turns and the current in the coil is 0·79 amp.

Calculate the permeability of the iron at the flux density used in the test.

Ans. 1172

• (59) Describe the wattmeter method of determining the iron losses in a sample of transformer stampings. Show how the copper loss in the magnetizing winding may be excluded from the wattmeter reading. Explain how the hysteresis and eddy current losses may be separately determined from the test results. (*Lond. Univ.*, 1925, *Elec. Meas.*)

(60) The reluctance of a magnetic circuit excited by 8,000 ampere-turns is 0·0015. A fluxmeter is used to measure the flux produced. If the fluxmeter scale has 120 divisions, and the line-turns required for a deflection of 1 division is 15,000, calculate the resistance of shunt required for use with the search coil; given—

Number of turns on search coil = 1
Resistance of search coil = 0·025 ohm

The measurement is made by switching off the excitation. It may be assumed that the resistances of the search coil and shunt are small compared with the resistance of the fluxmeter coil.

Ans. 0·00915 ohm.

(61) An open space is illuminated by four lamps each giving 300 candle-power in every direction below the horizontal. They are suspended 23 ft. above the ground at the corners of a rectangle 20 ft. by 15 ft. Calculate the illumination in foot-candles of a horizontal surface 3 ft. above the ground (*a*) at the middle of the shorter side, and (*b*) at the mid-point of the rectangle. Describe briefly a portable photometer suitable for testing the accuracy of your calculations. (*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 1·72 ft.-candles; 1·83 ft.-candles.

(62) Define the following terms: (*a*) mean spherical candle power; (*b*) lumen; (*c*) illumination; (*d*) brightness. What is the illumination at the edge of a circular table 6 ft. in diameter lit by one 100 c.p. lamp 4 ft. above the centre? (*A.M.I.E.E.*, *Oct.*, 1923.)

Ans. 3·2 foot-candles.

(63) Describe a good type of portable illumination photometer suitable for outdoor use; explain how it can be calibrated and discuss the various sources of error in its use.

If two lamps giving 500 candle-power in every direction are suspended 30 ft. high and 100 ft. apart, compare the illumination of the horizontal road surface under one lamp with that midway between them. (*Lond. Univ.*, 1925, *Elec. Meas.*)

Ans. 3·76 : 1

(64) How would you determine the constants a and β in the expression

$$\text{Candle-power} = aV^{\beta} \text{ for a glow lamp where } V \text{ is the lamp voltage?}$$

Taking the value $\beta = 4{\cdot}5$ for a tungsten filament vacuum lamp, determine the percentage variation of candle-power due to a voltage variation of ± 4 per cent from the normal value. (*C. and G. Final*, 1927.)

Ans. 19·2 per cent above normal, 16·75 below normal.

(65) A road is to be illuminated by means of lamps supported at a height of 20 ft., and arranged 100 ft. apart. Determine the necessary distribution of candle-power in a vertical plane in order that the lamps may produce a uniform horizontal illumination on the ground along a line vertically beneath a pair of lamps. The illumination due to a lamp at a greater distance than 50 ft. may be neglected. (*Lond. Univ.*, 1922, *Elec. Tech.*)

Ans. $(\text{C.P.})_\theta$, where θ is the angle between the line vertically down through the lamp and the line joining the lamp to any point on the road between the lamps.
$(\text{C.P.})_o = \text{C.P.}$ vertically under a lamp.

(66) Define luminous flux, luminous intensity, and illumination, and state and define the units in which they are measured.

Describe the construction and use of the Lummer-Brodhun photometer.
(*Lond. Univ.*, 1929, *Elec. Meas.*)

(67) Describe an apparatus by means of which the mean spherical candle-power of a 1,000 watt gas-filled lamp can be determined by a single reading. Explain the principle involved and discuss the precautions which must be taken to ensure accuracy. (*Lond. Univ.*, 1926, *Elec. Tech.*)

(68) Describe two methods of measuring peak voltages by means of two-electrode thermionic valves, one employing an electrostatic and the other a moving-coil instrument.

Explain the action in each case, and discuss the errors that may arise.
(*Lond. Univ.*, 1930, *Elec. Meas.*)

(69) Explain the various methods that are used for the measurement of very high voltage. (*Lond. Univ.*, 1929, *Elec. Meas.*)

(70) Describe the arrangements that you would adopt to determine (*a*) the root-mean-square value, (*b*) the average value per half-cycle, and (*c*) the peak value of an alternating voltage of about 50,000 volts.
(*Lond. Univ.*, 1926, *Elec. Meas.*)

(71) Describe two methods of measuring the peak value of an alternating voltage of about 15,000 volts, giving a full explanation of the principles involved in each and enumerating the precautions necessary in order to obtain accurate results. State the probable accuracy obtainable in each case.
(*Lond. Univ.*, 1925, *Elec. Meas.*)

(72) Describe with connection diagram how the peak value of a high voltage may be measured by means of a neon tube. Explain how the method is applied to the calibration of an extra high-voltage voltmeter. How would you arrange to detect with certainty the striking of the neon tube?
(*C. and G. Final*, 1930.)

(73) What methods may be used in testing cables for dielectric strength at very high voltages? Explain the difficulties that occur when tests with alternating current voltages are made on long lengths of cable, and state what difference there is (if any) in the dielectric strength of (*a*) paper (*b*) air, when tested respectively with alternating and direct voltages.

(74) Describe the Murray loop method of localizing an earth fault on a length of cable.

In a test for a fault to earth on a 520 yd. length of cable having a resistance of 1·10 ohm per 1,000 yd., the faulty cable is looped with a sound cable of the same length but having a resistance per 1,000 yd. of 2·29 ohm. The resistances of the other two arms of the testing network, at balance, are in the ratio 2·7 : 1. Calculate the distance of the fault from the testing end of the cable.
Ans. 432 yd.

(75) Describe a method by which the position of a fault to earth on a long feeder may be found approximately by a "loop" test.

What methods may be used for finding the position of the fault when a break in the conductor occurs without affecting the insulation resistance very seriously? (*Lond. Univ.*, 1923, *Elec. Tech.*)

(76) Show that if α_1 be the resistance-temperature coefficient of a conductor at t_1° C. expressed as a fraction, the coefficient at t_2° C. is given by

$$\frac{1}{\frac{1}{\alpha_1} + (t_2 - t_1)}$$

A specimen of copper wire has a specific resistance of $1{\cdot}6 \times 10^{-6}$ ohm per degree Centigrade at 0° C., and a temperature coefficient of $\frac{1}{254{\cdot}5}$ at 20° C. Find the temperature coefficient and the specific resistance at 60° C.

(*C. and G. Final*, 1930.)

Ans. $2{\cdot}009 \times 10^{-6}$ ohms per cm. cube; $\frac{1}{294{\cdot}5}$.

(77) What methods may be used for measuring the temperature rise in the field-magnet windings of a direct-current generator? Explain briefly how the final temperature rise of a machine may be estimated from the curve of temperature rise at the beginning of the heat run.

(*Lond. Univ.*, 1923, *Elec. Tech.*)

(78) Describe suitable methods for the measurement of the temperature of the following—

(*a*) The centre portion of the winding of a field coil on an electrical machine.
(*b*) A small quantity of molten metal.
(*c*) The interior of a furnace.

State why each method described is the most suitable for its purpose.

(79) A disc of copper 2 in. in diameter and 0·005 in. thick is placed inside a solenoid having a winding of 20 turns per inch length. Assuming that the flux is normal to the surface of the disc and is not appreciably disturbed by the eddy currents in the disc, calculate the loss of power in the disc when the coil carries an alternating current of 10 amp. at a frequency of 50. What is the phase of the current in the disc? Point out some method of estimating whether the assumption is justifiable. Is it justifiable in this particular case?

(*Lond. Univ.*, 1926, *Elec. Meas.*)

Ans. 0·012 watts.

(80) Under what conditions may a conducting ring be repelled from the pole of an alternating current magnet, and upon what factors does the repulsive force depend? Show how the force will vary during one cycle of the alternating flux.

If the metal in the ring were to be made up into a multiple turn coil of sensibly the same dimensions as the original ring and the ends joined together, how would the repulsive force compare with that in the case of the solid ring?

How would the repulsive force be affected by connecting a condenser of widely variable capacity across the terminals of the coil, instead of joining them together?

(*Lond. Univ.*, 1925, *Elec. Meas.*)

(81) Calculate the eddy current loss per cubic centimetre in a transformer core made up of stampings of given thickness and specific resistance when carrying a sinusoidal flux of given maximum density at a given frequency.

(*Lond. Univ.*, 1925, *Elec. Meas.*)

(82) Describe a modern form of Duddell oscillograph. Show that in a critically damped oscillograph the ratio R_n of the amplitude of the deflection due to a given harmonic to the value it would have if the instrument were without inertia and without damping is given by

$$R_n = \frac{1}{1 + n^2\left(\frac{T_0}{T}\right)^2}$$

where T_0 is the free period of the vibrating system, T the period of the current under investigation, and n the order of the harmonic.
(*Lond. Univ.*, 1929, *Elec. Meas.*)

(83) A potential difference represented by the formula

$$v = \sqrt{2} \,.\, 100 \,.\, \sin 2\pi \,.\, 50 \,.\, t + \sqrt{2} \,.\, 20 \,.\, \sin 2\pi \,.\, 150 \,.\, t$$

is applied to the terminals of a circuit made up of a resistance of 5 ohms, an inductance of 0·0318 henry, and a capacity of 12·5 microfarads all in series. Calculate the effective current and the power supplied to the circuit.
(*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 0·547 amp.; 1·51 watts.

(84) Explain why alternating voltage and current wave-forms usually contain no even harmonics. In what practical instances may even harmonics occur?

A rectifier gives a current wave which has a sinusoidal form but with the negative half-wave completely suppressed. The maximum height of the wave is 100 amp. Determine (i) the steady component and (ii) the fundamental of the wave. (*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. 31·8 amp.; 50 sin θ.

(85) A critically-damped Duddell oscillograph is required to reproduce the 13th harmonic of a 50 cycle wave with relative amplitude correct to within 2 per cent.

What should be the natural frequency of the movement?

What will the relative phase angle departure of the 13th harmonic be?
(*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. 4,590 cycles per sec.; 16° 7′.

(86) An electromotive force, $e = 2{,}000 \sin \omega t + 400 \sin 3\omega t + 100 \sin 5\omega$ is connected to a circuit consisting of a resistance of 10 ohms, a variable inductance, and a capacity of 30 microfarads arranged in series with a hot-wire ammeter. Find the value of the inductance which will give resonance with the triple frequency component of the pressure and estimate the readings on the ammeter and on a hot-wire voltmeter connected across the supply when resonant conditions exist. ($\omega = 300$.)
(*Lond. Univ.*, 1922, *Elec. Tech.*)

Ans. 0·0411 henry; 31·7 amp.; 1,442 volts.

(87) Describe the cathode ray oscillograph and explain the recent improvements made in its construction and operation. Give a diagram of connections showing its application to some definite measurement, and indicate the nature of the curve obtained. (*Lond: Univ.*, 1926, *Elec. Meas.*)

(88) Derive an expression for the current in a circuit containing resistance and inductance, due to an alternating voltage connected to the circuit at time $t = 0$.

If the resistance is 10·0 ohms, the inductance 2·5 henries, and the circuit is connected to a 200 volt, 50 cycle supply at the instant when the voltage is zero, draw the first four cycles of the current wave.
(*Lond. Univ.*, 1929, *Elec. Meas.*)

(89) A steady P.D. of 200 volts is suddenly applied to a coil of 2 ohms resistance and 5 henries inductance; calculate the time required for the current to reach 90 per cent of its final value, and prove the formula employed.
(*Lond. Univ.*, 1927, *Elec. Tech.*)

Ans. 5·76 seconds.

(90) A choking coil of relatively low resistance is suddenly switched on across the alternating current mains of given voltage and frequency. Work out an expression for the current taken by the coil during the transient period,

assuming (i) that the switch is closed when the voltage is zero, and (ii) that the switch is closed when the voltage is a maximum.

Draw diagrams illustrating the two cases.
(*Lond. Univ.*, 1925, *Elec. Meas.*)

(91) Derive an expression from which the instantaneous current flowing in a circuit of inductance L and resistance R may be calculated at any time t after applying a voltage $V \cos(\omega t + \phi)$.

From the expression find the ratio of the maximum value to which the current rises to the steady-state maximum value, when the voltage is applied at the instant when it is zero, taking $R = 20$ ohms, $L = 0{\cdot}1$ henry, and $\omega = 1{,}000\pi$. (*Lond. Univ.*, 1931, *Elec. Meas.*)

Ans. $i = \dfrac{V}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t + \phi + \alpha) + A\varepsilon^{-\frac{Rt}{L}}$, where $\alpha = \tan^{-1} \dfrac{R}{\omega L}$ and A is a constant depending on the initial conditions; 1·8 (approx.).

(92) A condenser is charged through a large non-inductive resistance by connecting it to a battery of constant E.M.F. Obtain an expression by which the rate at which the condenser receives its charge may be calculated. A condenser of 3 mfd. capacity is connected through a resistance of 1 megohm to a constant E.M.F. Find how long it will take before the condenser receives 99 per cent of its final charge. (*Lond. Univ.*, 1923, *Elec. Tech.*)

Ans. $q = EC(1 - \varepsilon^{-\frac{t}{RC}})$, 13·8 seconds.

(93) One coil of an armature winding having a circuit resistance when running, including brush contacts, of 0·01 ohm, and an inductance of 10^{-5} henry, is suddenly placed in a steady reversing field when carrying a current of 18 amp. Find the value of the field to make the current reach 20 amp. in the opposite direction in 0·001 sec. (*Lond. Univ.*, 1924, *Elec. Tech.*)

Ans. 42,100 lines.

(94) In a certain recording instrument, the electromagnetic torque corresponding to the full-scale deflection of 60°, is 10 g-cm. The control is exerted through two phosphor-bronze spiral springs. Allowing a maximum stress in the phosphor-bronze of 660 kg. per sq. cm., and taking the modulus of elasticity E as $1{\cdot}15 \times 10^6$ kg. per sq. cm., calculate suitable dimensions for the control springs. (*Lond. Univ.*, 1924, *Elec. Meas.*)

Ans. Thickness, 0·021 cm. Length, 19·4 cm. (Assuming width of strip used = 0·1 cm.)

(95) An instrument spring, constructed of phosphor-bronze strip, has the following dimensions: Length of strip, 370 mm.; thickness of strip, 0·073 mm.; breadth of strip, 0·51 mm.

If E (Young's modulus for phosphor-bronze) be taken as $1{\cdot}15 \times 10^6$ kg. per cm.^{2}, estimate the approximate torque exerted by the spring when it is turned through an angle of 90°.
(*A.M.I.E.E.*, Nov., 1930, *Meters and Meas. Insts.*)

Ans. 0·0805 g-cm.

(96) How are the temperature errors of a switchboard type D.C. ammeter compensated for? In a particular case the P.D. between the potential terminals of the shunt of a 1,000 amp. instrument is 0·03 volt. The connecting leads to the instrument are of copper and have a resistance of 0·12 ohm, the moving coil has a resistance of 1·20 ohm and requires a current of 15 milliamp. to deflect it to the 1,000 amp. point on the scale. The temperature coefficient of the alloy used for the shunt is 0·00001, and that of copper 0·004 per degree Centigrade in terms of the resistance at 15° C. To what extent could this

instrument be compensated, and if it is accurately adjusted at 15° C., what error would be expected when all parts are at 35° C.?

(*Lond. Univ.*, 1926, *Elec. Meas.*)

Ans. 4·99 per cent.

(97) A moving-coil ammeter, a hot-wire ammeter, and a resistance of 100 ohms are connected in series with a rectifying device across a sinusoidal alternating supply at 200 volts. If the device has a resistance of 100 ohms to current in one direction, and of 500 ohms to current in the opposite direction, calculate the readings on the two ammeters, the power taken from the mains, and that dissipated in the rectifying device.

(*Lond. Univ.*, 1926, *Elec. Tech.*)

Ans. 0·3 amp.; 0·74 amp.; 133·3 watts; 77·8 watts.

(98) The relationship between the inductance of a 2 amp. moving iron ammeter, the current, and the position of the movement, is as follows—

Ammeter reading in amperes	0·8	1·0	1·2	1·4	1·6	1·8	2·0
Deflection of the pointer in degrees	16	26	36·5	49·5	61·5	74·5	86·5
Inductance in millihenries.	573·2	574·2	575·2	576·6	577·8	578·8	579·5

Deduce an expression for the deflecting torque in terms of the rate of change of the inductance with position of the movement, and calculate the deflecting torque at 1 amp. and at 2 amp. (*Lond. Univ.*, 1925, *Elec. Meas.*)

Ans. At 1 amp., 29·1 g-cm. At 2 amp., 67·5 g-cm.

(99) Give a sketch showing the construction of a moving coil voltmeter.

If the moving coil consists of 100 turns wound on a square former which has a length of 3 cm. and the flux density, in the air gap, is 600 lines per sq. cm., calculate the turning moment acting on the coil when it is carrying a current of 12 milliamp. (*Lond. Univ.*, 1925, *Elec. Tech.*)

Ans. 0·66 g-cm.

(100) Find the expression for the deflection of a quadrant electrometer in terms of the potentials of the two pairs of quadrants and the needle.

Hence explain the use of the instrument as (i) a voltmeter, and (ii) a wattmeter.

Describe a gravity controlled modification suitable for measuring voltages up to 10,000. (*Lond. Univ.*, 1931, *Elec. Meas.*)

(101) An alternating current voltmeter with a maximum scale reading of 50 volts has a resistance of 500 ohms, and an inductance of 0·09 henry. The magnetizing coil is wound with 50 ohms of copper wire, and the remainder of the circuit is a non-inductive resistance in series with it. What additional apparatus is needed to make this instrument read correctly both on direct current and alternating current circuits of 50 frequency?

(*Lond. Univ.*, 1924, *Elec. Tech.*)

Ans. 0·444 microfarad in parallel with the series resistance.

(102) The moving coil of a permanent-magnet ammeter is wound with 10 turns of copper wire, and has a resistance of 0·1 ohm. The total torque exerted by the control springs is 0·02 g-cm. per degree. When the terminals are connected to a Grassot flux meter, and a scale angle of 90° is traversed by the ammeter pointer, a linkage of 3×10^5 line-turns is indicated.

Estimate the temperature coefficient of the ammeter when operating off a shunt having a drop of 0·075 volt at full load.

(*A.M.I.E.E.*, May, 1931, *Meters and Meas. Insts.*)

Ans. 0·00062 ohms/ohm/° C.

(103) A D.C. relay of the permanent magnet, moving-coil pattern, is fitted with a coil of the following dimensions: Length of coil, 4 cm.; breadth of coil, 3 cm. The coil is wound with 100 turns of insulated copper wire. The

field strength produced by the magnet is 1,000 lines per cm.^{2} If the total resistance of the relay is 2,000 ohms, estimate the pressure exerted on a pair of contacts placed at a distance of 0·5 cm. from the axis of the coil, when a voltage of 100 volts is applied to the relay.

Ans. 12·22 g. wt. (*A.M.I.E.E.*, Nov., 1930, *Meters and Meas. Insts.*)

(104) The coil of a 150 volt moving-iron voltmeter has a resistance of 400 ohms and an inductance of 0·75 henry. The current consumed by the instrument when placed on a 150 volt D.C. supply is 0·05 amp. Estimate—

(*a*) The temperature coefficient of the instrument per degree Centigrade.

(*b*) The alteration of the reading between direct current and alternating current at 100 cycles.

(*c*) The capacitance of the condenser necessary to eliminate this frequency error. Show method of connecting condenser.

(*A.M.I.E.E.*, Nov., 1930, *Meters and Meas. Insts.*)

Ans. 0·00066 ohms/ohm/°C.
1·17 per cent low at 100 cycles.
0·111 microfarad in parallel with the series resistance.

(105) A dynamometer ammeter is fitted with two field coils having a total resistance of 3·0 ohms and a total inductance of 0·12 henry, and a moving coil of resistance 30 ohms and inductance 0·003 henry. Calculate the temperature coefficient for changes of external temperature, and the error in reading when the instrument is calibrated with direct current and used on alternating current, 50 cycles, for each of the following arrangements—

(*a*) When the moving coil is shunted direct across the field coils.

(*b*) When the moving coil is shunted across a non-inductive resistance placed in series with the field coils.

(*c*) When connected as in (*b*), the non-inductive resistance having a value of 10 ohms, a suitable swamp resistance being placed in series with the moving coil. (*A.M.I.E.E.*, May, 1930, *Meters and Meas. Insts.*)

Ans. 0·004, 41·1 per cent low; 0·0024, 0·04 per cent low; 0·0006, 0·003 per cent low.

(106) A soft iron voltmeter for a maximum reading of 120 volts has an inductance of 0·6 henry and a total resistance of 2,400 ohms. It is calibrated to read correctly on a 60-cycle circuit. What series resistance would be necessary to increase its range to 600 volts? Draw up suitable workshop instructions for making up the resistance. (*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. 9,660 ohms.

(107) The capacity of an electrostatic voltmeter reading from 0 to 2,000 volts increases uniformly from 45 to 55 micro-microfarads as the pointer moves from zero to full-scale deflection. It is required to increase the range of the instrument to 20,000 volts by means of an external air condenser.

Calculate the area of a pair of condenser plates suitable for the purpose. If the condenser is adjusted to make the full-scale reading correct, what will be the error per cent at half-scale reading?

(*A.M.I.E.E.*, May, 1930, *Meters and Meas. Insts.*)

Ans. 175 sq. cm.,* 8·9 per cent high.

(108) Explain briefly the factors which must be taken into account in the design of a current transformer in order that the ratio and phase-angle errors may be as small as possible. A current transformer has a single-turn primary and a 200 turn secondary winding. The secondary supplies a current of 5 amp. to a non-inductive burden of 1 ohm resistance; the requisite flux is set up in the core by 80 ampere-turns in the primary winding. The frequency is 50 cycles per second, and the net cross-section of the core is 10 sq. cm. (i) Draw a diagram showing the currents, flux, and voltages to scale. (ii)

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* Assuming the distance between the plates to be 2·5 cm.

Calculate the ratio and phase angle of the transformer. (iii) Determine the flux density in the core. Neglect the effects of magnetic leakage and the iron and copper losses. (*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. 200·64, 4° 35′, $B_{\text{R·M·S·}} = 795$.

(109) The magnetizing current of a ring-core current transformer, ratio 1,000/5, when operating at full primary current and with a secondary burden consisting of a non-inductive resistance of 1 ohm, is 1 amp. at a power factor of 0·4. Calculate—

(*a*) The phase displacement between the primary and secondary current.

(*b*) The ratio error at full load, assuming that there has been no turn correction.

(*c*) The ratio error at one-tenth load, assuming the Steinmetz index to be 1·6.

(*A.M.I.E.E.*, May, 1931, *Meters and Meas. Insts.*)

Ans. 0° 3′, – 0·04 per cent, – 0·10 per cent.

(110) A potential transformer, ratio 1,000/100 volts, has the following constants—

Primary resistance .	94·5 ohms
Secondary resistance .	0·86 ohm
Primary reactance . .	66·2 ohms
Equivalent reactance .	66·2 ohms
Magnetizing current .	0·02 amp. at 0·4 power factor

Calculate—

(i) The phase angle at no load between primary and secondary voltages.

(ii) The load in volt-amperes at unity power factor at which the phase angle will be zero.

(*A.M.I.E.E.*, May, 1931, *Meters and Meas. Insts.*)

Ans. 0° 4′, 18·1 volt-amperes.

(111) The primary magnetizing current of a nickel-iron cored current transformer with bar primary, nominal ratio 100/1, operating on an external burden of 1·6 ohms non-inductive, the secondary winding resistance being 0·2 ohms, is 1·9 amp., lagging 40·6° to the secondary volts reversed, there being 100 secondary turns. With 1·0 amp. flowing in the secondary calculate—

(*a*) The actual ratio of primary to secondary current.

(*b*) The phase angle between them, in minutes.

(*A.M.I.E.E.*, Nov., 1931, *Meters and Meas. Insts.*)

Ans. 101·45, 42 min.

(112) What conditions must be fulfilled by a current transformer suitable for use with a precision wattmeter, and how do these conditions affect the design and construction of the transformer?

A 500/5, 50-frequency current transformer has a secondary burden which has a resistance of 0·5 ohm considered to be non-inductive. The primary winding consists of 1 turn, and 100 ampere-turns are required to produce the flux on rated full load. Calculate the flux in the core and the current-ratio at full load. (*C. and G. Final*, 1930.)

Ans. 7,950 (R.M.S.), 101·98.

(113) Describe a method of determining accurately the ratio and phase error of a pressure transformer intended for use with a wattmeter on a 6,600 volt circuit. (*Lond. Univ.*, 1927, *Elec. Meas.*)

(114) Show that in a dynamometer wattmeter of the usual type the error due to the unavoidable inductance of the moving coil is to a high degree of accuracy proportional to this inductance. What use is made of this fact to enable corrections to be made in wattmeters designed for accurate work at low power factors? (*Lond. Univ.*, 1926, *Elec. Meas.*)

(115) In a test by the three-voltmeter method, the following readings were obtained: Across the mains, 80 volts; across the non-inductive resistance of 6 ohms, 88 volts; across the load, 106 volts. Calculate the self-inductance and effective resistance of the load and the power supplied to it.
(*Lond. Univ.*, 1927, *Elec. Meas.*)

Ans. Eff. Res. = − 4·86 ohms; Reactance = 5·34 ohms;
Power = − 1,048 watts.

(116) Describe the two wattmeter method of measuring power in a three-phase circuit. If the readings of the wattmeter are 3 kW and 1 kW respectively, the latter reading being obtained after reversing the connections to the current coil of the wattmeter, calculate the power and the power factor. Prove the formulae employed. (*Lond. Univ.*, 1925, *Elec. Tech.*)

Ans. 2 kW, 0·277.

(117) A dynamometer wattmeter has a shunt coil with a resistance of 750 ohms and a series resistance of 2,250 ohms. A condenser of 1 microfarad capacity is arranged so that it can be shunted across the series resistance. If two readings of the wattmeter are taken, W_1 without the condenser shunt, and W_2 with the condenser shunt connected, determine a formula by which the power factor of the circuit in which the power is being measured can be found in terms of these readings. Frequency 50 cycles.
(*Lond. Univ.*, 1922, *Elec. Meas.*)

Ans. Power factor = $\cos \qquad \frac{W_2}{W_1} - 5{\cdot}68\Big)\Big]$.

(118) A dynamometer pattern wattmeter has a field system which may be considered as long compared with the diameter of the moving coil. The flux density B in the field coils is 100 lines per cm.^{2} The mean diameter of the moving coil is 3 cm., and it is wound with 500 turns of copper wire.

If the current through the moving coil is 0·05 amp. and the wattmeter is measuring the power flowing in a circuit having a power factor of 0·7, estimate the torque, if the axes of the field and moving coils are at (*a*) 45°, and (*b*) 90°.
(*A.M.I.E.E.*, May, 1931, *Meters and Meas. Insts.*)

Ans. 0·89 g-cm.; 1·26 g-cm.

(119) A small single-phase transformer is connected across a single-phase supply in series with a low-resistance ammeter A_1. In parallel with the transformer and across the same terminals is connected a resistance of 100 ohms in series with another ammeter A_2. A third ammeter, A_3, is placed directly in series with the supply mains. If the readings on the three ammeters are: A_1, 10·0 amp.; A_2, 1·0 amp.; A_3, 10·5 amp., find (*a*) the watts input into the transformer; (*b*) the power factor of the load due to the transformer.
(*A.M.I.E.E.*, Nov., 1930, *Meters and Meas. Insts.*)

Ans. 462·5; 0·462.

(120) The power flowing in a three-phase, three-wire, balanced-load system is measured by the two-wattmeter method.

The reading on wattmeter "A" is 5,000 watts, and on wattmeter "B" is − 1,000 watts.

(*a*) What is the power factor of the system?

(*b*) If the voltage of the circuit is 440, what is the value of capacitance which must be introduced into each phase to cause the whole of the power measured to appear on wattmeter "A"?
(*A.M.I.E.E.*, May, 1931, *Meters and Meas. Insts.*)

Ans. 0·359; 5·43 ohms.

(121) Describe briefly one type of mercury motor ampere-hour meter, and explain how the meter is compensated for the effects of fluid friction at high loads.

In a test run of 30 min. duration with a constant current of 5 amp. such a meter was found to register 0·51 kWh. If the meter is to be used in a 200 volt circuit, find its error, and state if it is running fast or slow. How can the instrument be adjusted to read correctly?

(*Lond. Univ.*, 1926, *Elec. Meas.*)

Ans. 2 per cent fast.

(122) Describe the principle and action of a shaded-pole motor. Explain how the torque is produced, show clearly in which direction it acts, and in what way it depends on the frequency.

How is the shaded-pole device applied in A.C. ammeters?

(*Lond. Univ.*, 1927, *Elec. Meas.*)

(123) Describe, with sketches, the construction of a modern type of alternating-current energy meter. Give the theory of the action of the instrument, and show how compensation is effected for temperature, friction, and power factor. Describe how such an instrument is calibrated.

(*Lond. Univ.*, 1929, *Elec. Meas.*)

(124) In a simple bipolar form of A.C. energy meter, the distance between the pole centres is 1·5 cm. and the effective radius of action is 2·5 cm. The fluxes produced by the series and shunt magnets are 350 and 275 lines (R.M.S.) respectively, their phase displacement being 82°. The aluminium driving disc is 0·06 cm. thick and its specific resistance may be taken as 3 microhms per cm. cube.

Neglecting the edge effect of the disc, calculate its speed if the brake magnet exerts a braking torque of 7·5 dyne-cm. when the speed is 1 revolution per minute. Frequency = 50 cycles per second.

Ans. 40·4 r.p.m.

(125) A large consumer has a kVA demand and a kVAh tariff, measured (by mutual agreement) by "sine" and "cosine" watt-hour type meters, each equipped with a Merz demand-indicator. The tariff is 10s. per month per kVA + $\frac{3}{8}$d. per kVAh. Render the consumer his bill for one month of 30 days, based on the following readings: "Sine" meter advance 90,000 reactive-kVAh, demand indicator 150 reactive-kVA. "Cosine" meter advance 120,000 kWh, demand indicator 200 kW.

What are his average monthly power factor and load factor, and his total cost per *unit*? (*A.M.I.E.E.*, Nov., 1931, *Meters and Meas. Insts.*)

Ans. 0·719d. per unit. Average load factor = Average power factor = 0·8.

(126) Describe the routine tests you would apply to single-phase watt-hour meters? How would you compare the merits of two such meters?

How are the transformers for such instruments calibrated?

(127) Describe any method of measuring precisely a frequency of the order of 500 cycles per second. (*Lond. Univ.*, 1931, *Elec. Meas.*)

(128) Describe with a dimensioned sketch one form of frequency meter suitable for power station use. Explain how the range of this instrument could be extended. (*C. and G. Final*, 1927.)

(129) Describe the principle and construction of a single-phase synchroscope. (*Lond. Univ.*, 1930, *Elec. Meas.*)

(130) A coil of 200 turns, wound on a rectangular former 10 cm. long and 8 cm. wide, is placed with its longer side parallel with and 5 cm. distant from a current-carrying conductor, the plane of the coil being arranged so that it contains the conductor.

Calculate the E.M.F. induced in the coil by a current

$$i = 10 \sin 314 \,.\, t + 5 \sin 942 \,.\, t$$

flowing in the conductor. What is the mutual inductance between the coil and the conductor?

Ans. $e = -[0{\cdot}012 \cos 314t + 0{\cdot}018 \cos 942t]$ $M = 3{\cdot}82$ microhenries.

(131) Give a concise account of the M.K.S. (Giorgi) system of units, and compare the advantages and disadvantages of this system with those of the C.G.S. electromagnetic system. (*Lond. Univ.*, 1939, *Elec. Meas.*)

(132) Explain the essential principle of the method of symmetrical components as applied to the solution of asymmetrical polyphase A.C. network problems.

Show how a direct measurement of the positive and negative sequence components of an unbalanced three-phase current system can be made. Give a diagram of connections and indicate clearly what the observations signify with regard to the actual currents in the three lines. (*Lond. Univ.*, 1934, *Elec. Meas.*)

(133) Two coils, of self-inductance 0·01 henry and 10 henries, and resistance 0·3 ohm and 300 ohms respectively, have a coefficient of coupling of 0·9.

Calculate the change in effective resistance of the first coil when a resistance of 100 ohms is connected to the terminals of the second, the frequency being 50 c.p.s. (*Lond. Univ.*, 1936, *Elec. Meas.*)

Ans. 0·318 ohm.

(134) It is required to measure the inductance and resistance of an iron-cored choke of about 1 henry at frequencies varying from 50 to 3,000 c.p.s. and with D.C. current flowing through it. An oscillator, standard condensers, and non-inductive resistance boxes are available.

Describe a suitable A.C. bridge and detector, mentioning any precautions necessary to ensure accuracy in the result. Derive the equation of balance of the bridge used. (*Lond. Univ.*, 1935, *Elec. Meas.*)

(135) Discuss the difficulties of constructing a standard condenser for use in the high-voltage arm of a Schering bridge working on voltages above 100 kV.

A condenser bushing forms arm AB of a Schering bridge, and a standard condenser of 500 μμF. capacitance and negligible loss forms arm AD. Arm BC consists of a non-inductive resistance of 300 ohms. When the bridge is balanced, the resistance and condenser in parallel in the remaining arm CD have values of 72·6 ohms and 0·148 μF. respectively. The frequency is 50 c.p.s. Calculate from first principles the capacitance and the dielectric loss angle of the bushing. (*Lond. Univ.*, 1939, *Elec. Meas.*)

Ans. 121 μμF.; 0° 11·6′.

(136) In a balanced bridge network, AB is a resistance of 500 ohms in series with an inductance of 0·18 henry, BC and DA are non-inductive resistances of 1,000 ohms, and CD consists of a resistance R in series with a capacitance C. A potential difference of 5 volts at a frequency of $5{,}000/2\pi$ is established between the points A and C.

Draw to scale a vector diagram showing the currents and potential differences in the bridge, and from it determine the values of R and C.

Check the result algebraically. (*Lond. Univ.*, 1938, *Elec. Meas.*)

Ans. 472 ohms; 0·235 μF.

(137) A moving-coil galvanometer has a sensitivity of 4 cm. per microampere, with a scale 1 metre distant, and the time of free oscillation is 2·8 sec. If the galvanometer is dead beat when the total circuit resistance (coil and external circuit) is 2,500 ohms, find the moment of inertia of the moving system.

Prove any formula used. (*Lond. Univ.*, 1934, *Elec. Meas.*)

Ans. 2·7 g-cm.^{2}

(138) Describe a method of using an A.C. potentiometer for measuring the loss in an iron ring made up of thin stampings. Explain how the loss may be calculated in terms of the maximum density, and state any assumptions made. (*Lond. Univ.*, 1939, *Elec. Meas.*)

(139) Describe fully how a co-ordinate type of A.C. potentiometer can be used to determine the ratio and phase angle of a current transformer. Draw

a diagram of the connections, and explain the methods of deducing the values of the required quantities from the observations. What are the most important precautions necessary to prevent errors? (*Lond. Univ.*, 1935, *Elec. Meas.*)

(140) What are the conditions to be fulfilled by a ballistic galvanometer? Describe the construction of such an instrument, and explain how to determine the constant and the logarithmic decrement.

The periodic time of an undamped reflecting ballistic galvanometer is 10 sec., and a current of 0·1 mA. gives a steady deflection of 200 scale divisions. Find the quantity of electricity which produces a swing of 100 divisions. What is the quantity of electricity corresponding to this swing if the instrument has a decrement of 1·03? (*C. and G. Final*, 1938.)

Ans. 0·0000795 coulombs; 0·000121 coulombs.

(141) Describe and explain a method by which the hysteresis loop for a given sample of iron may be delineated. Show how to calculate from the hysteresis loop the energy loss due to this phenomenon at a given frequency, and state how the accuracy of the calculation is likely to be affected by the frequency considered, assuming the maximum magnetizing force to remain unchanged.

Calculate the hysteresis loss per hour in a sample of iron weighing 10 lb. for which the hysteresis loop has an area equivalent to 1,200 ergs per cm.^{3}, when subjected to alternating magnetization of 50 frequency. Take the density of the iron as 7·5. (*Lond. Univ.*, 1934, *Elec. Meas.*)

Ans. 0·288 watt-hour.

(142) What are the criteria of the most suitable characteristics of a permanent magnet for use in a measuring instrument such as a moving-coil ammeter? Deduce the magnetic condition in which the permanent magnet should be operated in order that its volume may be a minimum, assuming given gap dimensions and flux density in the gap. (*Lond. Univ.*, 1939, *Elec. Meas.*)

(143) Explain carefully how the construction of a fluxmeter differs from that of a moving-coil ammeter or voltmeter.

A certain fluxmeter has the following constants—

Air-gap flux density, 50 lines per cm.^{2}
Turns on moving coil, 40.
Area of moving coil, 7·5 cm.^{2}

If a 10-turn search coil of 2 cm.^{2} area, which is connected to the fluxmeter, is reversed in a uniform field of flux density 500 lines per cm.^{2}, calculate the deflection of the meter.

Why is it necessary to keep the resistance of the moving coil, and of the search coil and leads, low? How may a correction for an unavoidably high resistance in the search coil be made? (*Lond. Univ.*, 1936, *Elec. Meas.*)

Ans. $76\frac{1}{2}°$.

(144) Describe an equipment for the production of high voltages for surge or impulse tests. Explain the action of the circuit described, and show precisely how the shape of the impulse wave can be controlled. (*C. and G. Final*, 1939.)

(145) Show that the eddy-current torque on a metallic disc rotating between the poles of a permanent magnet is directly proportional to the angular velocity of the disc. How would the torque be expected to vary with the position of the magnet poles relative to the axis of the disc? What use is made of this device in the construction of energy meters, and what part does it play in the operating mechanism? (*Lond. Univ.*, 1934, *Elec. Meas.*)

(146) Describe and give the theory underlying the operation of some form of electric harmonic analyser. A certain periodic wave that repeats itself every half-cycle is found to have ordinates y corresponding to angles θ degrees as follows—

θ	0	15	30	45	60	75	90	105	120	135	150	165	180
y	1·4	3·9	5·2	5·5	5·3	5·0	5·2	5·7	6·4	6·3	4·9	2·0	−2·7

Determine the amplitude and phase of the third harmonic present in the wave. (*Lond. Univ.*, 1934, *Elec. Meas.*)

Ans. 1·83, leading $22\frac{1}{2}°$ relative to the fundamental.

(147) A voltage, represented by 300 sin ωt volts, is applied to a circuit consisting of a non-inductive resistance of 20 ohms in series with a luminous discharge lamp and produces a current represented by

$$(5 \sin \omega t - 2 \sin 3\,\omega t) \text{ amp.}$$

Calculate the power absorbed by the resistance and by the lamp; also the power factor of the lamp and of the complete circuit. (*Lond. Univ.*, 1939, *Elec. Meas.*)

Ans. 290 watts; 460 watts; 0·838; 0·93.

(148) Describe, with a sketch, one form of modern precision moving-iron ammeter for A.C. circuits. Prove that no frequency error is introduced when using a shunt having the same time-constant as the instrument. (*C. and G. Final*, 1939.)

(149) Explain the action of a shaded-pole ammeter, showing by means of a sketch the direction of movement of the disc relative to the magnet poles. How is adequate damping secured, and how is the spread of the scale controlled? What are the advantages of this type of instrument for switchboard use, and to what errors is it subject? (*Lond. Univ.*, 1936, *Elec. Meas.*)

(150) What is meant by critical damping, and to what extent should this condition be approached in the case of an ordinary moving-coil type of instrument?

The coil of a moving-coil galvanometer has 300 turns and is suspended in a uniform magnetic field of 1,000 lines per cm.^{2} by a phosphor-bronze strip, of which the torsion constant is 2 dyne-cm. per radian. The coil is 2 cm. wide and $2\frac{1}{2}$ cm. high, with a moment of inertia of 1·5 g-cm.^{2}

If the galvanometer resistance is 200 ohms, calculate the value of the resistance which, when connected across the galvanometer terminals, will give critical damping. Assume the damping to be entirely electromagnetic. (*Lond. Univ.*, 1935, *Elec. Meas.*)

Ans. 450 ohms.

(151) An 8/1 current transformer has an accurate current ratio when the secondary is short-circuited. The inductance of the secondary is 60 millihenries and its resistance is 0·5 ohm, and the frequency is 50 c.p.s. Estimate the current ratio and phase-angle error when the instrument load has a resistance of 0·4 ohm and an inductance of 0·7 millihenry. State the assumptions made. (*Lond. Univ.*, 1934, *Elec. Meas.*)

Ans. 8·001; 0° 2′ assuming no iron loss, constant permeability, and magnetizing current = 1 per cent of primary current.

(152) Give an account of the precautions to be observed in the design and construction of precision current transformers for use with a standard wattmeter.

Deduce an expression for the correction factor of a wattmeter which has a negligible inherent error, when used with a current transformer having a ratio error of p per cent and a phase-angle error of q deg., and a potential transformer which has a ratio error of x per cent and a phase-angle error of y deg. (*C. and G. Final*, 1938.)

Ans. $\frac{100}{p} \times \frac{100}{x + 100} \times \frac{\cos \phi}{\cos \phi \pm q \pm y}$ where ϕ is the phase angle of the load.

(153) What is meant by the "dimensions" of an electrical or magnetic quantity?

Obtain the dimensions of resistance and magnetic flux in the c.g.s. electromagnetic system, and of charge and energy in the c.g.s. electrostatic system.

Briefly explain what is meant by a "rationalized" system of dimensions.

(*Lond. Univ.*, 1950, *Elec. Meas.*)

(154) Explain the advantages of using a mutual inductance as a primary standard of inductance.

Estimate the mutual inductance between two parallel coaxial single-turn circular coils of diameter 40 cm. and 4 cm. respectively. The planes of the coils are 25 cm. apart.

Justify the expressions used in this calculation.

(*Lond. Univ.*, 1950, *Elec. Meas.*)

Ans. 0·00096 μH.

(155) Explain with the aid of a diagram of connections, a method of measuring the symmetrical components of the currents in an unbalanced 3-phase, 3-wire system.

If in such a system the line currents, in amperes, are

$$I_r = 10 - j2; \; I_y = -2 - j4; \; I_b = -8 + j6$$

calculate their symmetrical components.

(*Lond. Univ.*, 1949, *Elec. Meas.*)

Ans. $I_{Y0} = \;_{B0} \; 0$
$_{R1} - 7{\cdot}89 + 0{\cdot}732j; \; I_{Y1} = a \quad I_{B1} = ($
$I_{R2} = 2{\cdot}113 - 2{\cdot}732j; \; I_{Y2} =$

(156) Describe a suitable a.c. bridge method for measuring, at a frequency of 500 cycles per second, the self-inductance and effective resistance of a coil of approximately 0·2-henry inductance and 5 ohms resistance. Draw the vector diagram for the balance conditions and give the equations for balance.

(*Univ. Lond.*, 1947, *Elec. Meas.*)

(157) Describe briefly the construction and method of adjustment of one type of vibration galvanometer. Compare its advantages and disadvantages with those of a telephone receiver for use in a.c. measurements at various frequencies.

Derive an expression for the sensitivity of the tuned galvanometer, showing how it is dependent on frequency and damping.

(*Lond. Univ.*, 1948, *Elec. Meas.*)

(158) Explain the principle of one type of co-ordinate a.c. potentiometer. Draw a diagram of the scheme of connections and describe how the potentiometer is standardized.

Measurements for the determination of the impedance of a coil were made on a co-ordinate potentiometer as follows—

Voltage across a 1-Ω standard resistance in series with the coil, + 0·952 V on in-phase dial, − 0·34 V on quadrature dial.

Voltage across a 10 : 1 potential divider connected to the terminals of the coil, + 1·35 V on in-phase dial, + 1·128 V on quadrature dial.

Calculate the resistance and the reactance of the coil.

(*Lond. Univ.*, 1949, *Elec. Meas.*)

Ans. $R = 8{\cdot}82 \; \Omega$
$X = 15 \; \Omega$.

(159) Describe a method by which the insulation resistance to earth of each of a pair of live mains can be measured by a voltmeter of known resistance. Discuss the limitations of the method.

The following readings were taken with a 250-volt, 1,000-ohms-per-volt, voltmeter—

Between two mains . . .	218 volts
Positive main to earth . .	188 volts
Negative main to earth . .	10 volts

Calculate the insulation resistance of each main.

(*Lond. Univ.*, 1947, *Elec. Meas.*)

Ans. Positive 500,000 Ω; Negative 26,600 Ω.

(160) Compare and contrast a wire (Crompton-type) potentiometer and a vernier-type potentiometer, each being a precision instrument arranged for direct reading.

What features characterize a potentiometer for use with thermocouples in the precision measurement of temperatures of the order of 1,000° C.? What kind of empirical relation normally exists between voltage and temperature in the case of thermocouples suitable for this purpose?

(*Lond. Univ.*, 1950, *Elec. Meas.*)

(161) Derive an expression, in terms of the angular movement of the suspended coil of a fluxmeter, for the change of magnetic flux through a search coil of T turns connected to the meter. The total resistance of the whole circuit, moving coil, and search coil is R, and the total self-inductance L. The moving coil consists of N turns each of area A suspended in a uniform field B. (*Lond. Univ.*, 1947, *Elec. Meas.*)

(162) Derive an expression for the r.m.s. value of a complex wave-form of voltage.

Analysis of the wave-form of the voltage generated by an alternator shows that there are third and fifth harmonics having amplitudes equal, respectively, to 30 per cent and 20 per cent of that of the fundamental. The third harmonic lags behind the fundamental by 20 degrees and the fifth harmonic by 10 degrees. A dynamometer voltmeter connected to the alternator terminals gives the same reading as is produced by 150 volts d.c. Derive an expression for the instantaneous value of the voltage in the circuit.

(*Lond. Univ.*, 1948, *Elec. Meas.*)

Ans. $v = 200 \sin \omega t + 60 \sin \left(3\,\omega t - \frac{\pi}{3}\right) + 40 \sin \qquad \left. - \frac{5}{18}\pi\right)$

(163) Sketch and describe briefly the essential features of a cathode-ray tube and suggest a simple method of providing a time-base that is practically linear.

Derive an expression for the electrostatic deflection in cm. per volt. State any approximations used. Assume that the screen is flat and perpendicular to the centre line of the tube and neglect fringing effects at the edges of the deflecting plates. (*Lond. Univ.*, 1949, *Elec. Meas.*)

(164) A sinusoidal voltage of 1,000 V and frequency 50 cycles per second is suddenly switched on to an inductive circuit of reactance 40 ohms and resistance 2·5 ohms. If the switch is closed when the instantaneous value of the voltage is zero, what will be the value of the current half a cycle later in time?

(*Lond. Univ.*, 1947, *Elec. Meas.*)

Ans. 4·01 A.

(165) Why is it advisable to short-circuit the terminals of a sensitive moving-coil instrument during transport?

The coil of a moving-coil meter is wound on a non-conducting former whose height and width are both 2 cm. It moves in a constant field of 1,200 lines per cm.^{2} The moment of inertia of its moving parts is 2·5 g-cm., and the control spring exerts a torque of 300 dyne-cm. per radian. Calculate (*a*) how many turns must be wound on the coil to produce a deflection of 150 degrees with a current of 10 mA, and (*b*) the resistance of the coil to produce critical damping, all damping being taken as electromagnetic.

(*Lond. Univ.*, 1948, *Elec. Meas.*)

Ans. 164; 2·81 Ω.

(166) State the causes of change of accuracy in moving-coil instruments with change of temperature. Explain how compensation is made in ammeters for change of electrical resistance of the moving coil with change of temperature.

It is required to construct a resistance of 5·0 ohms with a resistance-temperature coefficient of 8×10^{-5} per ° C. Platinoid and manganin wire of cross-sectional area 0·4 mm^2 are available and are to be connected in series. Calculate the lengths required if the resistivity of platinoid is 34·4 microhm-cm. and its resistance-temperature coefficient is $2{\cdot}5 \times 10^{-4}$ per ° C.; corresponding figures for manganin are 48 microhm-cm. and 2×10^{-5} per ° C. All figures refer to 0° C. (*Lond. Univ.*, 1950, *Elec. Meas.*)

Ans. 151·3 cm. platinoid.
308 cm. manganin.

(167) A metal rectifier is used for the measurement of a sinusoidal alternating current. The rectifier consists of four units arranged in bridge, each unit having approximate forward and reverse resistances of 5 ohms and 500 ohms respectively. The moving-coil indicator has a resistance of 20 ohms. Determine the indication when the r.m.s. input current is 4 milliamperes.

(*Lond. Univ.*, 1950, *Elec. Meas.*)

Ans. 3·26 mA.

(168) Discuss the reasons why the errors of a current transformer are usually greater with relatively small loads than at rated full load.

At its rated load of 25VA, a 100/5-ampere current transformer has an iron loss of 0·2 W and a magnetizing current of 1·5 A. Calculate its ratio error and phase angle when supplying rated output to a meter having a ratio of resistance to reactance of 5. (*Lond. Univ.*, 1948, *Elec. Meas.*)

Ans. − 1·07 per cent; 0° 45′.

(169) Why is it important to use high-grade magnetic material for the laminations of current transformers? Discuss some of the advantages gained by substituting nickel-iron for silicon-iron in current transformer construction.

(*Lond. Univ.* 1947, *Elec. Meas.*)

(170) Explain how the amplitude of the harmonics in a supply voltage of complex wave-form may be determined experimentally with the aid of a dynamometer wattmeter and a beat-frequency oscillator of pure wave-form.

A single-phase load takes a current of—

$$4 \sin\left(\omega t + \frac{\pi}{6}\right) + 1{\cdot}5 \sin\left(3\omega t + \frac{\pi}{3}\right) \text{amperes}$$

from a source represented by 360 sin ωt volts. Calculate the power dissipated by the circuit and the circuit power-factor.

(*Lond. Univ.*, 1950, *Elec. Meas.*)

Ans. 623·5 W; 0·837.

(171) Describe a direct-reading frequency meter for measuring a frequency of the order of *either* (*a*) 50 cycles per second *or* (*b*) 500 cycles per second. Suggest a suitable method for calibrating the instrument.

(*Lond. Univ.*, 1948, *Elec. Meas.*)

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